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A FIXED POINT THEOREM FOR POSITIVE OPERATORS ON KB SPACES

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The existence of nonzero fixed points of positive contractions in L_1 spaces has received considerable attention in recent years. In 1966, Dean and Sucheston [**1**] and independently, Neveu [**5**] showed that a positive contraction has a strictly positive invariant function if and only if $\inf_n \int_A T^n 1 dm > 0$ for any measurable subset A with positive measure, where 1 is the constant function of value one.

The condition of being a contraction has been reduced to more general conditions by some authors later, for example, Fong [**3**] and Sato [**6**]. In Fong's paper he considered the case of semi-Markovian operators, i.e., positive operators T on L_1 such that $\sup_n \|T^n\| < \infty$.

On the other hand, the author of the present paper has extended the above result to the case of absolutely continuous normed Köthe spaces [**4**], which include the Orlicz spaces with delta two property and all the L_p spaces $(1 \leqq p < \infty)$ as special cases.

As we have mentioned in [**4**], the lattice structures play a very important role in this kind of theorem. In this paper we shall show that the theorem in [**4**] is still valid in general *KB* spaces—Dedekind complete normed lattices with the properties:

i) If $\langle x_n \rangle$ is a decreasing sequence with $\inf_n x_n = 0$, then $\lim_n \|x_n\| = 0$.
ii) If $\langle x_n \rangle$ is an increasing sequence with $\sup_n \|x_n\| < \infty$, then $\sup_n x_n$ exists.

First of all, we list some known results in Riesz space theory. All the propositions which are headed by parentheses ( ) can be found in [**7**], unless otherwise specified.

As usual the least upper bound (greatest lower bound) of any two elements x and y in a Riesz space is denoted as $x \vee y$ (respectively $x \wedge y$). We also write $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, $|x| = x \vee (-x)$. The supremum (respectively infimum) of a set A will be denoted as sup A (respectively inf A). Furthermore, an increasing (respectively, decreasing) sequence $\langle x_n \rangle$ with a supremum (respectively, infimum) x is denoted as $x_n \uparrow x$ (respectively, $x_n \downarrow x$). We shall also let **R** and **N** be the real number system and the set of natural numbers respectively.

By a *band* N in a Riesz space L, we mean a linear subspace such that (iii) and (iv) hold.

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iii) If $x \in L$, $y \in N$ and $|x| \leqq |y|$, then $x \in N$.
iv) If $A \subset N$ and sup A exists in L then sup $A \in N$.

For $A \subseteq L$, the set $A^{\perp} = \{x \in L: |x| \wedge |y| = 0 \text{ for all } y \in A\}$ is called the *orthogonal complement* of A.

For a Dedekind complete Riesz space L and a band N in L, we have:

(1) The projection $P_N : L \to N$ determined by the formula $P_N(x) = \sup \{y \in N: 0 \leqq y \leqq x\}$ for $x \geqq 0$ has the properties that $P_N(z) \in N$, $z - P_N(z) \in N^{\perp}$ for all $z \in L$ and $0 \leqq P_N(x) \leqq x$ for $x \geqq 0$.

(2) For an arbitrary subset A in L, $A^{\perp}$ is a band in L, and $A^{\perp\perp}$ is the smallest band containing A.

(3) Let x, y be two positive elements in L and $N = \{x\}^{\perp\perp}$. Then $P_N(y) = \sup_n (y \wedge nx)$; in particular, $y \in \{x\}^{\perp\perp}$ if and only if $y = \sup_n (y \wedge nx)$.

We denote by $L^{\sim}$ the class of regular linear functionals on L, i.e., the linear functionals which can be represented as a difference of two positive linear functionals. We denote by $L^{(i)}$ the class of integrals on L, i.e., the functionals f in $L^{\sim}$ which satisfy the condition

v) if $x_n \downarrow 0$ in L, then $\lim_n f(x_n) = 0$.

It is well known that both $L^{(i)}$ and $L^{\sim}$ are Dedekind complete Riesz spaces. The following proposition will play an important role in the proof of our main theorem. For the proof of this proposition we refer to [**4**].

(4) Let L be a Dedekind complete Riesz space. If $0 \leqq f \in L^{(i)}$, $0 \leqq h \in L^{\sim}$ and $f \wedge h = 0$, then for any $y \geqq 0$ in L and any positive real number $\epsilon > 0$, there exists x in L with $0 \leqq x \leqq y$ and $h(x) = 0$, $f(y - x) < \epsilon$.

Throughout this paper we assume that M is an arbitrary KB space. In addition to the propositions stated above, it has also the following properties:

(5) $M^{(i)} = M^{\sim} = M'$, where M' is the class of (norm) bounded functionals on M.

Let M'' be the space of bounded linear functionals on M'. It is well known that both M' and M'' are Banach lattices. If we define for any $x \in M$ an $\hat{x} \in M''$ by $\hat{x}(f) = f(x)$, $f \in M'$, and let $\hat{M} = \{\hat{x}: x \in M\}$, then we have:

(6) The mapping $x \mapsto \hat{x}$ is a norm-preserving one-to-one linear transformation from M to M'' such that $x \leqq y$ if and only if $\hat{x} \leqq \hat{y}$;
$\inf_{x \in A} \hat{x} = \widehat{\inf A}$ if inf A exists in M; and
$\sup_{x \in A} \hat{x} = \widehat{\sup A}$ if sup A exists in M.
Furthermore, all the elements in $\hat{M}$ are integrals on M'.

(7) $\hat{M}$ is a band in M''.

For our convenience, we introducing the following:

Definition. An element x in a KB space is said to be *absolutely continuous with respect to* y ($x \ll y$) if $x \in \{y\}^{\perp\perp}$. Two elements x and y are said to be *equivalent* ($x \sim y$) if $x \ll y$ and $y \ll x$.

PROPOSITION 1. *$x \ll y$ in M if and only if $f(|y|) = 0$ implies that $f(|x|) = 0$ for any $0 \leqq f \in M'$.*

LEMMA. *If u, v are two positive elements in M such that $u \wedge v = 0$, then there is a functional $0 \leqq g \in M'$ such that $g(u) > 0$ and $g(v) = 0$.*

Proof. By the Hahn-Banach theorem there is an $f \in M'$ such that $0 < ||u|| = f(u) = f^+(u) - f^-(u)$, so $f^+(u) > 0$. From (5) we know that $f^+ \in M'$. Let N be the band generated by u. If we define $g = f^+ \circ P_N$, then

$$\begin{aligned} g(u) &= f^+(P_N(u)) = f^+(u) > 0 \\ g(v) &= f^+(P_N(v)) = f^+(\sup_n (v \wedge nu)) \quad \text{by (3)} \\ &= f^+(0) = 0. \end{aligned}$$

Proof of Proposition 1. Since $x \ll y \Leftrightarrow x \in \{y\}^{\perp\perp} \Leftrightarrow |x| \in \{|y|\}^{\perp\perp} \Leftrightarrow |x| \ll |y|$, we can assume that both x and y are positive.

If $x \ll y$, then by (3) $x = \sup_n (x \wedge ny)$. For $0 \leqq f \in M'$, $f(y) = 0$ implies that $f(x \wedge ny) = nf(1/nx \wedge y) = 0$ for all $n \in \mathbf{N}$. Since f is also a member of $M^{(i)}$, it follows that $f(x) = 0$.

Conversely, suppose $x \notin \{y\}^{\perp\perp}$. Then by definition there is $0 \leqq z \in \{y\}^{\perp}$ with $z \wedge x > 0$. Let $u = z \wedge x$; then we have $u \wedge y = 0$. By the lemma there is $0 \leqq g \in M'$ such that $g(u) > 0$ and $g(y) = 0$. Hence $g(x) \geqq g(u) > 0$ and $g(y) = 0$. This completes the proof.

Definition. A *semi-Markovian operator* T is a positive linear operator on a KB space into itself such that $\sup_n ||T^n|| < \infty$.

The following proposition is an abstraction of Lemma 1 in [**5**]; it also appeared in [**4**] where we assume that the underlying space is a normed Köthe space.

PROPOSITION 2. *Let $x \ll u$ be two positive elements in M, $T: M \to M$ be a semi-Markovian operator. Then for any $0 \leqq f \in M'$, $\inf_n f(T^n x) > 0$ implies $\inf_n f(T^n u) > 0$.*

Proof. Let $a = \inf_n f(T^n x)$, $b = \sup_n ||T^n||$. Since $x \ll u$, it follows from (3) that $x = \sup_k (x \wedge ku)$. Hence $||x - x \wedge ku|| \downarrow 0$ according to i). We choose a positive integer k_1 such that $||x - x \wedge k_1 u|| < a/2b||f||$.

Since $x = k_1 u + (x - k_1 u) \leqq k_1 u + (x - x \wedge k_1 u)$, it follows that $a \leqq f(T^n x) \leqq k_1 f(T^n u) + f(T^n(x - x \wedge k_1 u)) \leqq k_1 f(T^n u) + ||f||b||x - x \wedge k_1 u|| \leqq k_1 f(T^n u) + a/2$ for any $n \in \mathbf{N}$.

Therefore $0 < a/2k_1 \leqq \inf_n f(T^n u)$, completing the proof.

We shall utilize the concept of Banach limits in the proof of our main

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theorem. A Banach limit LIM is a linear functional defined on the space of all bounded real sequences with the following properties:

vi) $\mathrm{LIM}\ (a_n) = \mathrm{LIM}\ (a_{n+1})$
vii) $\underline{\lim}\ a_n \leqq \mathrm{LIM}\ (a_n) \leqq \overline{\lim}\ a_n$

The existence of a Banach limit can be deduced from the Hahn-Banach theorem as shown in [**2**].

THEOREM. *Let T be a semi-Markovian operator on a KB space M. If there exists $x \geqq 0$ in M such that $f(x) > 0$ implies* $\inf_n f(T^n x) > 0$ *for any* $0 \leqq f \in M'$, *then there is a* $y \geqq 0$ *in M with* $x \ll y$ *and* $Ty = y$.

Conversely, if $Ty = y > 0$ *in M and x is an arbitrary positive element with* $x \sim y$, *then* $f(x) > 0$ *implies* $\inf_n f(T^n x) > 0$ *for any* $0 \leqq f \in M'$.

Proof. Assume that $\inf_n f(T^n x) > 0$ for any $0 \leqq f \in M'$ with $f(x) > 0$. We define $\lambda : M' \to \mathbf{R}$ by $\lambda(f) = \mathrm{LIM}\, f(T^n x) = \mathrm{LIM}\ (T^{*n} f)(x)$, where LIM is a Banach limit, and $T^* : M' \to M'$ is the conjugate of T.

Since

$$|\lambda(f)| = |\mathrm{LIM}\ f(T^n x)| \leqq \mathrm{LIM}\ |f(T^n x)| \leqq ||f||\ \sup_n ||T^n x|| \leqq ||f||\ ||x||\ \sup_n ||T^n||,$$

it follows that $\lambda \in M''$. It is also obvious that $\lambda \geqq 0$.

On the other hand, from (7) we know that $\hat{M}$ is a band of M'', so by (1) there exists a positive element $u \in M$ such that $\hat{u} = \sup \{\hat{z} \in \hat{M} : 0 \leqq \hat{z} \leqq \lambda\}$ and $v = \lambda - \hat{u} \in \hat{M}^{\perp}$. (In fact $\hat{u} = P_{\hat{M}}(\lambda)$).

We claim that $\widehat{Tu} \leqq \lambda$. Since $\widehat{Tu}(f) = f(Tu) = (T^* f)(u) = \hat{u}(T^* f) \leqq \lambda(T^* f) = \mathrm{LIM}\ (T^{*n+1} f)(x) = \mathrm{LIM}\ (T^{*n} f)(x) = \lambda(f)$ for any $0 \leqq f \in M'$. Therefore $\widehat{Tu} \leqq \hat{u}$. It follows from (6) that $Tu \leqq u$.

Let $y = \inf_n T^n u$. Since $(T^n u - y) \downarrow 0$, we have $||T^n u - y|| \downarrow 0$ and $||Ty - y|| \leqq ||Ty - T^n u|| + ||T^n u - y|| \leqq ||T||\ ||y - T^{n-1} u|| + ||T^n u - y|| \to 0$. So $Ty = y$.

Next we show that $x \ll u$. By Proposition 1 it is the same as showing that $f(u) = 0 \Rightarrow f(x) = 0$ for $0 \leqq f \in M'$. Suppose to the contrary that there were $0 \leqq f \in M'$ with $f(u) = 0$ and $f(x) > 0$. We let $\epsilon = f(x)/2$. Since $\hat{x} \wedge v = 0$, by (4) there is $g \in M'$ such that $0 \leqq g \leqq f$ and $v(g) = 0$, $\hat{x}(f - g) < \epsilon$. So $g(x) > f(x) - \epsilon = f(x)/2 > 0$. By the assumption we then have $\inf_n g(T^n x) > 0$. Therefore

(*) $\lambda(g) = \mathrm{LIM}\ g(T^n x) \geqq \inf_n g(T^n x) > 0$

On the other hand, since $0 \leqq g \leqq f$ and $f(u) = 0$, so $g(u) = 0$. It follows that $\lambda(g) = g(u) + v(g) = 0$, a contradiction to (*). This proves that $x \ll u$.

To show that $x \ll y$, we let f be an arbitrary positive element in M' with $f(x) > 0$. From the assumption we have $\inf_n f(T^n x) > 0$. So by Proposition 2 we have $\inf_n f(T^n u) > 0$. Moreover, since $T^n u \downarrow y$ and $f \in M' = M^{(i)}$, we have $f(T^n u) \downarrow f(y)$. So $f(y) > 0$. Therefore, by Proposition 1 it must be that $x \ll y$.

Conversely, let $y > 0$ be a fixed point for T and $0 \leqq x \in M$ with $x \sim y$. For any $0 \leqq f \in M'$ such that $f(x) > 0$, we have $f(y) > 0$. So $\inf_n f(T^n y) = f(y) > 0$. By Proposition 2 we have $\inf_n f(T^n x) > 0$.

COROLLARY 1. *Let M be a KB space with a unit e* (i.e. $e \geqq 0$ *and* $\{e\}^{\perp\perp} = M$), *T a semi-Markovian operator on M. Then a necessary and sufficient condition for the existence of a positive fixed point y with $\{y\}^{\perp\perp} = M$ is that* $\inf_n f(T^n e) > 0$ *for any $f > 0$ in M'.*

Proof. The sufficiency is trivial. From the theorem above we have a positive fixed point y with $e \ll y$. Since $\{e\}^{\perp\perp} = M$, it follows that $\{y\}^{\perp\perp} = M$.

Conversely, let $Ty = y > 0$ and $\{y\}^{\perp\perp} = M$. Then $y \sim e$, so by the theorem we have $\inf_n f(T^n e) > 0$ for any $0 \leqq f \in M'$ with $f(e) > 0$. Since $\{e\}^{\perp\perp} = M$, from Proposition 1 we know that $0 < f \in M'$ implies $f(e) > 0$. This completes the proof.

Given a positive element x in M that satisfies the condition in the theorem, it is natural to ask whether the fixed point y obtained in this way is equivalent to x. In general we can only confirm that x is absolutely continuous with respect to y. However, if the operator satisfies the additional condition that $T(z) \ll x$ for all $z \ll x$, then the fixed point y is equivalent to x.

COROLLARY 2. *Let x be a positive element in a KB space M, T a semi-Markovian operator such that $Tz \ll x$ for any $z \ll x$. Then a necessary and sufficient condition for the existence of a positive fixed point y equivalent to x is that* $\inf_n f(T^n x) > 0$ *for any $0 \leqq f \in M'$ with $f(x) > 0$.*

Proof. The condition $Tz \ll x$ for any $z \ll x$ implies that $\{x\}^{\perp\perp}$ is invariant under T. Since $\{x\}^{\perp\perp}$ is a KB subspace of M and the restriction of T on $\{x\}^{\perp\perp}$ is also semi-Markovian, the result follows from Corollary 1.

Let (X, Σ, m) be an arbitrary σ-finite measure space, it is easy to see that for any $1 \leqq p < \infty$, the space $L_p(X, \Sigma, m)$ is a KB space. We shall use f, g, h to denote the measurable functions on (X, Σ, m) and let $S(f)$ to denote the set $\{x \in X: f(x) \neq 0\}$ for a measurable function f on (X, Σ, m). Furthermore, for $A, B \in \Sigma$, the notation $A \subset B$ means that almost all elements of A are in B. In this case the theorem can be restated as follows:

COROLLARY 3. *Let T be a semi-Markovian operator on $L_p(X, \Sigma, m)$* $(1 \leqq p < \infty)$. *If there exists $0 \leqq f \in L_p(X, \Sigma, m)$ such that $m(A \cap S(f)) > 0$ implies* $\inf_n \int_A T^n f \, dm > 0$, *then there exists $0 \leqq h \in L_p(X, \Sigma, m)$ with $S(f) \subset S(h)$ such that $Th = h$.*

Conversely, if $Th = h \geqq 0$ and $f \geqq 0$ is a function in $L_p(X, \Sigma, m)$ such that $S(f) = S(h)$, then $\inf_n \int_A T^n f \, dm > 0$ *for any $A \in \Sigma$ with $m(A \cap S(f)) > 0$.*

Proof. We note that the absolute continuity of f with respect to h is equivalent to the condition that $S(f) \subset S(h)$. On the other hand, since the dual space of $L_p(X, \Sigma, m)$ is precisely $L_q(X, \Sigma, m)$ where $q = p/(p-1)$ if $p \neq 1$,

$q = \infty$ if $p = 1$. The condition that $\int gf\,dm > 0$ implies $\inf_n \int gT^n f\,dm > 0$ for any $0 \leqq g \in L_q(X, \Sigma, m)$ is equivalent to the condition that $m(A \cap S(f)) > 0$ implies $\inf_n \int_A T^n f\,dm > 0$ for any $A \in \Sigma$. This the corollary follows from the theorem above.

When $p = 1$, $||T|| \leqq 1$ and $S(f) = X$, this proposition is equivalent to the theorem of Dean and Sucheston [**1**] and independently, Neveu [**5**].

We can also apply this theorem to a more general kind of KB spaces—the absolutely continuous normed Köthe spaces. This kind of function space includes as special cases the Orlicz spaces with delta two property [**4**]. The application of our main theorem to such spaces will give us exactly the results in [**4**].

It is natural to ask whether our main theorem can be improved to include the following statement:

viii) Let T be a semi-Markovian operator on a KB space M, x be a positive element in M. If there exists a positive fixed point y such that $x \ll y$, then $\inf_n f(T^n x) > 0$ for any $0 \leqq f \in M'$ such that $f(x) > 0$.

We note that viii) is different from the second part of our theorem, as the condition $x \sim y$ is reduced to $x \ll y$.

Unfortunately, statement viii) is not true. The rest of this paper is devoted to a counter example of viii).

Let T: $L_1(X, \Sigma, m) \rightarrow L_1(X, \Sigma, m)$ be the positive contraction induced by a nonsingular measurable transformation τ on (X, Σ, m) such that $Tf = dm_f/dm$, where m_f is the measure defined as $m_f(E) = \int_{\tau^{-1}(E)} f\,dm$ for any $E \in \Sigma$. It is well known that T has a non-zero positive invariant function in $L_1(X, \Sigma, m)$ if and only if τ has a nontrivial finite invariant measure on (X, Σ) which is absolutely continuous with respect to m.

For a fixed $A \in \Sigma$ we defined a measure m_A such that $m_A(E) = m(A \cap E)$ for any $E \in \Sigma$. The following statement can then be deduced from viii).

ix) Let τ be a nonsingular measurable transformation on (X, Σ, m) and A a measurable set with finite positive measure. If there exists a finite invariant measure μ such that $m_A \ll \mu \ll m$, then $\inf_n m_A(\tau^{-n}(E)) > 0$ for any $E \in \Sigma$ with $m(E \cap A) > 0$.

Now we let X be the closed interval $[-1, 1]$, m be the Lebesgue measure in $[-1, 1]$, τ be defined as $\tau(x) = -x$ for $x \in [-1, 1]$. Clearly τ is invariant under m itself. If we let $A = [0, 1]$, then $m_A(\tau^{-1}(A)) = m([0, 1] \cap [-1, 0]) = 0$, so $\inf_n m_A(\tau^{-n}(A)) = 0$, a contradiction to statement ix).

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