ESSENTIAL NORMAL AND CONJUGATE EXTENSIONS OF INVERSE SEMIGROUPS

by FRANCIS PASTIJN

(Received 10 October, 1980)

1. Introduction. In the following we use the notation and terminology of [**6**] and [**7**].

If S is an inverse semigroup, then E_S denotes the semilattice of idempotents of S. If a is any element of the inverse semigroup, then a^{-1} denotes the inverse of a in S. An inverse subsemigroup S of an inverse semigroup S' is self-conjugate in S' if for all $x \in S'$, $x^{-1}Sx \subseteq S$; if this is the case, S' is called a conjugate extension of S. An inverse subsemigroup S of S' is said to be a full inverse subsemigroup of S' if $E_S = E_{S'}$. If S is a full self-conjugate inverse subsemigroup of the inverse semigroup S', then S is called a normal inverse subsemigroup of S', or, S' is called a normal extension of S.

When considering congruences on inverse semigroups one is naturally led to consider normal inverse subsemigroups. Indeed, if ρ is any congruence on an inverse semigroup S, then the kernel of ρ, that is the union of all ρ-classes which contain idempotents, forms a normal subsemigroup of S [**10**]. Normal extensions of inverse semigroups were investigated in [**1**] and [**8**]. The conjugate extensions of inverse semigroups form a generalization of normal extensions. They were studied in [**9**].

Let S be an inverse semigroup, and let $\Omega(S)$ be the translational hull of S. Recall that $\Omega(S)$ is an inverse semigroup as well ([**7**, V.4.6]), which may be considered to be the greatest dense ideal extension of S ([**7**, III.5.9]). The set $\Psi(S)$ consisting of isomorphisms among subsemigroups of S of the form $\lambda S \rho$ with $(\lambda, \rho) \in E_{\Omega(S)}$, forms an inverse subsemigroup of the symmetric inverse semigroup on the set S; $\Psi(S)$ is called the conjugate hull of S ([**9**, Section 3]). The set $\Phi(S)$ of isomorphisms among subsemigroups of S of the form $\lambda_e S \rho_e = eSe$ with $e \in E_S$ forms an inverse subsemigroup of $\Psi(S)$, called the normal hull of S ([**8**, Section 4]). For any $s \in S$, let θ^s be the isomorphism

$$\theta^s : sSs^{-1} \to s^{-1}Ss,\ x \to s^{-1}xs.$$

The mapping $\Theta : S \to \Phi(S)$, $s \to \theta^s$ is a homomorphism of S onto a normal subsemigroup $\Theta(S)$ of $\Phi(S)$; one can show that $\Psi(S)$ is a conjugate extension of $\Theta(S)$ [**8**], [**9**]. The kernel of the congruence $\Theta\Theta^{-1}$ on S is denoted by $M(S)$, and $M(S)$ is called the metacenter of S. Obviously $\Theta\Theta^{-1}$ is idempotent-separating, and so $M(S)$ is a semilattice of groups.

LEMMA 1.1. *Let S be an inverse semigroup. An element c of S belongs to $M(S)$ if and only if $cc^{-1} = c^{-1}c = e$ for some $e \in E_S$ and $c(exe) = (exe)c$ for all $x \in S$.*

Proof. Let c be any element of $M(S)$. Then there exists an idempotent e of S such that $\Theta(c) = \Theta(e)$. Hence $\operatorname{dom} \theta^c = \operatorname{dom} \theta^e$ and $\operatorname{im} \theta^c = \operatorname{im} \theta^e$, from which we have $c^{-1}Sc = eSe = cSc^{-1}$. Therefore $cc^{-1} = c^{-1}c = e$. Every element of $\operatorname{dom} \theta^c = \operatorname{dom} \theta^e = eSe$ has the form *exe*, $x \in S$. Further,

$$exe = (exe)\theta^e = (exe)\theta^c = c^{-1}(exe)c.$$

We conclude that $c(exe) = (exe)c$ for all $x \in S$.

Conversely, if $cc^{-1} = c^{-1}c = e$ for some $e \in E_S$, and $c(exe) = (exe)c$ for all $x \in S$, then obviously $\Theta(c) = \Theta(e)$, and $c \in M(S)$.

It is easy to see that $M(S)$ is a semilattice of abelian groups. The metacenter $M(S)$ is said to be idempotent if $M(S) = E_S$. This is the case if and only if Θ is a monomorphism, and then $\Psi(S)$ [$\Phi(S)$] may be considered to be a conjugate [normal] extension of S.

A conjugate [normal] extension S' of S is said to be essential if the only congruence on S' whose restriction to S is the identity relation on S must be the identity relation on S'. We are now in the position to restate some of the most important results of [**8**], [**9**].

THEOREM 1.2. *Let S be an inverse semigroup with idempotent metacenter. Then $\Psi(S)$ [$\Phi(S)$] is the greatest essential conjugate [normal] extension of $\Theta(S)(\cong S)$. Every inverse subsemigroup of $\Psi(S)$ [$\Phi(S)$] containing $\Theta(S)$ is an essential conjugate [normal] extension of $\Theta(S)(\cong S)$. Every essential conjugate [normal] extension of S is isomorphic to an inverse subsemigroup of $\Psi(S)$ [$\Phi(S)$] containing $\Theta(S)$.*

The purpose of this paper is to show that, if there exists a greatest essential conjugate [normal] extension of the inverse semigroup S, then the metacenter of S must be idempotent. We thereby not only establish a kind of converse for the above mentioned results, but also we create an analogue for the same type of result concerning dense ideal extensions of a semigroup S, where the translational hull $\Omega(S)$ takes the place of $\Psi(S)$ [$\Phi(S)$] under the hypothesis that S be weakly reductive [**3**, Section 1], [**11**].

If S is a group, then the metacenter $M(S)$ is actually the center of S. Our result will generalize Gluskin's Theorem 3.4 of [**2**] which states that a group S has a greatest essential (group-) extension if and only if the center of S is trivial. If this is the case then the automorphism group $\mathrm{Aut}(S) = \Phi(S)$ of S is the greatest essential group-extension of the group of inner automorphisms $\mathrm{In}(S) = \Theta(S)(\cong S)$ of S. Our Section 2 contains a generalization of Theorem 3.2 of [**2**] and Section 3 generalizes and corrects the wrong proof of Theorem 3.4 of [**2**].

2. A construction of an essential normal extension. Let S be an inverse semigroup, and let us suppose that the metacenter of S is not idempotent . Let c be an element in the metacenter $M(S)$ of S such that $c^2 \neq c$. In this section we shall construct an essential normal extension B of S, such that c is in the metacenter of B.

The element c belongs to a maximal subgroup of S. We have remarked above that $c^k exe = exec^k$ for every integer k, $e = cc^{-1}$, and all $x \in S$. Note that for all $m, n \in \mathbb{Z}$ and all $x \in S$ we have $c^m xc^n = c^{m+n}xe$. The cyclic group $\langle c\rangle$ which is generated by c is contained in the maximal subgroup H_e of S. Let $\langle u\rangle$ and $\langle v\rangle$ be cyclic groups which are generated by u and v respectively, and let us suppose that $u \to c[v \to c]$ extends to a isomorphism of $\langle u\rangle$ onto $\langle c\rangle[\langle v\rangle$ onto $\langle c\rangle]$.

Let us consider the set $SeS\times\langle u\rangle\times\langle v\rangle$. Let $e_u[e_v]$ denote the identity of $\langle u\rangle[\langle v\rangle]$. For any $xey\in SeS$, let $\overline{xey}=(xey, e_u, e_v)$. Obviously the mapping

$$SeS\rightarrow SeS\times\langle u\rangle\times\langle v\rangle,\ xey\rightarrow\overline{xey}$$

is a one-to-one mapping of SeS onto the subset $SeS\times\{e_u\}\times\{e_v\}$ of $SeS\times\langle u\rangle\times\langle v\rangle$. There exists a set $\bar{S}=\{\bar{x}\mid x\in S\}$ such that the mappings $S\rightarrow\bar{S}$, $x\rightarrow\bar{x}$ is a bijection, and

$$\bar{S}\cap(SeS\times\langle u\rangle\times\langle v\rangle)=SeS\times\{e_u\}\times\{e_v\}=\{\overline{xey}\mid x, y\in S\}.$$

We shall put $S'=\bar{S}\cup(SeS\times\langle u\rangle\times\langle v\rangle)$. Before we introduce a multiplication on S', we prove the following lemma.

LEMMA 2.1. *If $xey=x_1ey_1$ holds in S, then $xc^ky=x_1c^ky_1$* for all $c^k\in\langle c\rangle$.

Proof. We have

$$xc^ky=xex^{-1}xec^ky=xc^kex^{-1}xey=xc^kex^{-1}x_1ey_1=xex^{-1}x_1c^ky_1$$

and analogously,

$$x_1c^ky_1=xc^kyy_1^{-1}ey_1.$$

Therefore

$$x_1c^ky_1=(xc^ky)(y_1^{-1}ey_1)=(xex^{-1})(x_1c^ky_1)(y_1^{-1}ey_1)$$
$$=(xex^{-1})x_1c^k(ey_1)(ey_1)^{-1}(ey_1)=(xex^{-1})(x_1c^ky_1)=xc^ky.$$

On $SeS\times\langle u\rangle\times\langle v\rangle$ we define a multiplication by putting

$$(x_1ey_1, u^{m_1}, v^{n_1})(x_2ey_2, u^{m_2}, v^{n_2})=(x_1c^{m_2n_1}y_1x_2ey_2, u^{m_1+m_2}, v^{n_1+n_2}).$$

It follows from Lemma 2.1 and from the fact that $\langle u\rangle\cong\langle c\rangle\cong\langle v\rangle$, that the above given multiplication is well-defined. One may check that for all $x_1ey_1, x_2ey_2\in SeS$, and all integers m, n

$$(x_1ey_1, u^m, v^n)\overline{x_2ey_2}=(x_1ey_1x_2ey_2, u^m, v^n)=\overline{x_1ey_1}(x_2ey_2, u^m, v^n),$$

and in particular,

$$\overline{x_1ey_1}\ \overline{x_2ey_2}=\overline{x_1ey_1x_2ey_2}.$$

Therefore we may extend the above given multiplication to the whole of S' in the following way: for all $\bar{z}_1,\bar{z}_2\in\bar{S}$ and all $(xey, u^m, v^n)\in SeS\times\langle u\rangle\times\langle v\rangle$, we put

$$\bar{z}_1\bar{z}_2=\overline{z_1z_2},$$
$$\bar{z}_1(xey, u^m, v^n)=(z_1xey, u^m, v^n),$$
$$(xey, u^m, v^n)\bar{z}_2=(xeyz_2, u^m, v^n).$$

It immediately follows that the mapping $S\rightarrow S'$, $x\rightarrow\bar{x}$ embeds S isomorphically into S'. Further, $SeS\times\langle u\rangle\times\langle v\rangle$ is an ideal of S'.

LEMMA 2.2. *S' is an inverse semigroup, and $\bar{S}(\cong S)$ is a full inverse semigroup of S'.*

Proof. It is routine to check associativity. We shall only check the "hardest" case here. Let

$$\begin{aligned} A &= [(x_1ey_1, u^{m_1}, v^{n_1})(x_2ey_2, u^{m_2}, v^{n_2})](x_3ey_3, u^{m_3}, v^{n_3}) \\ &= (x_1c^{m_2n_1}y_1x_2ey_2, u^{m_1+m_2}, v^{n_1+n_2})(x_3ey_3, u^{m_3}, v^{n_3}) \\ &= (x_1c^{m_2n_1+m_3(n_1+n_2)}y_1x_2ey_2x_3ey_3, u^{m_1+m_2+m_3}, v^{n_1+n_2+n_3}), \end{aligned}$$

and

$$\begin{aligned} B &= (x_1ey_1, u^{m_1}, v^{n_1})[(x_2ey_2, u^{m_2}, v^{n_2})(x_3ey_3, u^{m_3}, v^{n_3})] \\ &= (x_1ey_1, u^{m_1}, v^{n_1})(x_2c^{m_3n_2}y_2x_3ey_3, u^{m_2+m_3}, v^{n_2+n_3}) \\ &= (x_1c^{(m_2+m_3)n_1}y_1x_2c^{m_3n_2}y_2x_3ey_3, u^{m_1+m_2+m_3}, v^{n_1+n_2+n_3}) \\ &= (x_1c^{m_2n_1+m_3n_1+m_3n_2}y_1x_2ey_2x_3ey_3, u^{m_1+m_2+m_3}, v^{n_1+n_2+n_3}). \end{aligned}$$

Therefore $A = B$.

The idempotents of S' must all be of the form $\bar{f}$, $f \in E_S$, and thus $E_{S'} = E_{\bar{S}}$. One may verify that for every $(xey, u^m, v^n) \in SeS \times \langle u \rangle \times \langle v \rangle$, $(y^{-1}c^{mn}x^{-1}, u^{-m}, v^{-n})$ is an inverse of (xey, u^m, v^n). Therefore S' is regular. Since $E_{S'}$ is a semilattice, it follows that S' is an inverse semigroup. Since $E_{S'} = E_{\bar{S}}$, we have that $\bar{S}$ is a full inverse subsemigroup of S'.

Let us put $\bar{u} = (e, u, e_v)$ and $\bar{v} = (e, e_u, v)$. Then $\bar{u}^{-1} = (e, u^{-1}, e_v)$ and $\bar{v}^{-1} = (e, e_u, v^{-1})$. Since $\bar{u}\bar{u}^{-1} = \bar{u}^{-1}\bar{u} = \bar{e} = \bar{v}\bar{v}^{-1} = \bar{v}^{-1}\bar{v}$, it follows that $\bar{e}$, $\bar{c}$, $\bar{u}$ and $\bar{v}$ all belong to the same maximal subgroup of S'.

LEMMA 2.3. *S' is generated (as an inverse semigroup) by the elements of $\bar{S} \cup \{\bar{u}, \bar{v}\}$. For every $\bar{x} \in \bar{S}$ we have*

$$\bar{u}^{-1}\bar{x}\bar{u} = \bar{v}^{-1}\bar{x}\bar{v} = \bar{u}\bar{x}\bar{u}^{-1} = \bar{v}\bar{x}\bar{v}^{-1} = \bar{e}\bar{x}\bar{e} = \overline{exe}.$$

$\bar{S}$ ($\cong S$) is a normal subsemigroup of S', and $\bar{c}$ is in the metacenter of S'.

Proof. Let (xey, u^m, v^n) be any element of $SeS \times \langle u \rangle \times \langle v \rangle$. Then

$$(xey, u^m, v^n) = \overline{xe}\bar{u}^m\bar{v}^n\overline{ey};$$

hence S' is generated (as an inverse semigroup) by the elements of $\bar{S} \cup \{\bar{u}, \bar{v}\}$. Let $\bar{x} \in \bar{S}$. Then

$$\begin{aligned} \bar{u}^{-1}\bar{x}\bar{u} &= (e, u^{-1}, e_v)\bar{x}(e, u, e_v) \\ &= (exe, e_u, e_v) = \overline{exe} = \bar{e}\bar{x}\bar{e}. \end{aligned}$$

The other equalities stated above can be proved in a similar fashion. It follows that $\bar{S}$ is a self-conjugate inverse subsemigroup of S'. From Lemma 2.2 it now follows that $\bar{S}$ is a normal inverse subsemigroup of S'.

The element $\bar{c}$ belongs to the maximal subgroup of S' which has identity $\bar{e}$. Every element of $\bar{e}S'\bar{e}$ is of the form (exe, u^m, v^n) for some $exe \in eSe$, $u^m \in \langle u \rangle$ and $v^n \in \langle v \rangle$. Since

$$\begin{aligned}\bar{c}(exe, u^m, v^n) &= (cexe, u^m, v^n)\\ &= (exec, u^m v^n)\\ &= (exe, u^m, v^n)\bar{c},\end{aligned}$$

it follows from Lemma 1.1 that $\bar{c}$ is in the metacenter of S'.

LEMMA 2.4. *The semigroup $\bar{S}$ intersects every $\mathcal{H}$-class of S'. The maximal subgroup $H_{\bar{e}}$ of S' which contains $\bar{e}$ consists of the elements (x, u^m, v^n), where $x\mathcal{H}e$ in S $u^m \in \langle u \rangle$ and $v^n \in \langle v \rangle$.*

Proof. Let $(xey, u^m, v^n) \in SeS \times \langle u \rangle \times \langle v \rangle$. Then

$$(xey, u^m, v^n)(xey, u^m, v^n)^{-1} = \overline{xey}\,\overline{xey}^{-1} = \overline{xey(xey)^{-1}},$$
$$(xey, u^m, v^n)^{-1}(xey, u^m, v^n) = \overline{xey}^{-1}\overline{xey} = \overline{(xey)^{-1}xey},$$

and so $(xey, u^m, v^n)\,\mathcal{H}\,\overline{xey}$ in S'. Hence $\bar{S}$ intersects all $\mathcal{H}$-classes of S'. Since $SeS \times \langle u \rangle \times \langle v \rangle$ is an ideal of S', it follows that the $\mathcal{H}$-class of $\bar{e}$ in S' consists of the elements (z, u^m, v^n), where $z\mathcal{H}e$ is in S, $u^m \in \langle u \rangle$ and $v^n \in \langle v \rangle$.

LEMMA 2.5. *If ρ is any congruence on S' which separates the elements of $\bar{S}$, then ρ separates the elements of $\bar{S} \cup H_{\bar{e}}$.*

Proof. Since the restriction of ρ to $\bar{S}$ is the identity on $\bar{S}$, and since $E_{S'} = E_{\bar{S}}$, we have that ρ is idempotent-separating. Hence $\rho \subseteq \mathcal{H}$. This implies that every element of $H_{\bar{e}}$ can only be ρ-related to an element of $H_{\bar{e}}$. Therefore it suffices to show that the restriction of ρ to $H_{\bar{e}}$ is the identity on $H_{\bar{e}}$, or, that the normal subgroup $\bar{e}\rho$ of $H_{\bar{e}}$ which consists of all elements of S' which are ρ-related to $\bar{e}$ is trivial. Let us suppose that (x, u^m, v^n) belongs to $\bar{e}\rho$. Then $x\mathcal{H}e$ in S. An easy calculation shows that

$$\overline{c^{-m}}(x, u^m, v^n) = \bar{v}^{-1}(x, u^m, v^n)\bar{v} \in \bar{e}\rho$$

and

$$\overline{c^n}(x, u^m, v^n) = \bar{u}^{-1}(x, u^m, v^n)\bar{u} \in \bar{e}\rho,$$

and thus $\overline{c^{-m}}, \overline{c^n} \in \bar{e}\rho$. Since ρ separates the elements of $\bar{S}$, we have $\overline{c^{-m}} = \overline{c^n} = \bar{e}$, and thus $u^m = e_u$, $v^n = e_v$. It follows that $(x, u^m, v^n) = \bar{x} \in \bar{S}$. Again, since ρ separates the elements of $\bar{S}$, we have $\bar{e} = (x, u^m, v^n)$. We conclude that $\bar{e}\rho = \{\bar{e}\}$, and consequently, ρ separates the elements of $\bar{S} \cup H_e$.

We want to remark that part of the proof of Lemma 2.5 is based on ideas from Theorem 3.2 of [**2**].

We summarize our findings in the following theorem.

THEOREM 2.6. *Let S be an inverse semigroup, and let c be an element of the metacenter of S such that $c^2 \neq c$. Then there exists an idempotent e of S such that c belongs to the* *maximal subgroup of S with identity e. There exists an essential normal extension B of S such that the following are satisfied:*

(i) *B is generated as an inverse semigroup by the elements of* $S\cup\{a,b\}$, *where* $a,b\in B\backslash S$ *and* $a\neq b$.

(ii) *The elements a, b and c belong to the maximal subgroup of B which has identity e.*

(iii) *For all* $s\in S$, *we have* $a^{-1}sa=asa^{-1}=ese=b^{-1}sb=bsb^{-1}$.

(iv) *The element c is in the metacenter of B.*

Proof. Let $\bar{S}$ and S' be as before. We consider the $\cap$-semilattice of all congruences on S' which separate the elements of $\bar{S}$. By Zorn's lemma, there exists a maximal element ρ of this semilattice. Obviously S'/ρ is an essential normal extension of $\bar{S}\rho^{\natural}$ by Lemma 2.3. We identify the inverse semigroup S with $\bar{S}\rho^{\natural}$, by the isomorphism $S\to\bar{S}\rho^{\natural}$, $x\to\bar{x}\rho^{\natural}$, and we put $S'/\rho=B$ and $a=\bar{u}\rho^{\natural}$, $b=\bar{v}\rho^{\natural}$. Then (i) follows from Lemma 2.3 and Lemma 2.5, (ii) follows from the remark preceding Lemma 2.3, (iii) follows from Lemma 2.3, and (iv) follows from Lemma 2.3 and Lemma 1.1.

3. On the existence of a greatest essential extension. Our first two results here state that an inverse semigroup A which has a metacenter which is not idempotent cannot have a greatest essential conjugate [normal] extension. We thereby solve a conjecture of Section 5 of **[8]** and Conjecture 5.4 of **[9]**.

THEOREM 3.1. *Let G be an essential conjugate extension of the inverse semigroup A, and let us suppose that the metacenter of A is not idempotent. Then there exists an essential conjugate extension of A which contains G properly.*

Proof. Let c be an element of the metacenter of A which is not an idempotent. Let $\mathscr{X}$ be the poset of all inverse subsemigroups of G which are essential normal extensions of A, and which have c in their metacenter. Clearly $A\in\mathscr{X}$. By Zorn's lemma, there exists a maximal element S of $\mathscr{X}$. Let e, a, b and B be as in Theorem 2.6, where a, $b\notin G$. By Theorem 2.6(iv) c is in the metacenter of B. For all $s\in S$, and all $x\in A$, we have

$$a^{-1}xa=axa^{-1}=b^{-1}xb=bxb^{-1}=exe\in A$$

by Theorem 2.6(iii), and $s^{-1}xs\in A$, and since B is generated by the elements of $S\cup\{a,b\}$, we may conclude that B is a normal extension of A.

Let ρ be a congruence on B whose restriction to A is the identity on A. Let τ be the restriction of ρ to S. Since τ is a congruence on S which separates the elements of A, and since S is an essential normal extension of A, we have that τ is the identity on S. Since B is an essential normal extension of S, it follows that ρ is the identity on B. We conclude that B is an essential normal extension of A.

Let us consider the inverse semigroup amalgam $[\{G,B\};S;\{\iota,\iota'\}]$ where $G\cap B=S$, and where ι and ι' are inclusion mappings. Since the class of inverse semigroups satisfies the strong amalgamation property **[4]**, **[6]**, there exists an inverse semigroup P which contains G and B as inverse subsemigroups, such that $G\cap B=S$ in P, and such that P is generated as an inverse semigroup by the elements of $G\cup B$. Hence, P is generated as an inverse semigroup by the elements of $G \cup \{a, b\}$. One can easily check that P is a conjugate extension of A, since both G and B are conjugate extensions of A.

Let $\mathcal{N}$ be the $\cap$-semilattice of all congruences on P which induce the identity on G, and let κ be a maximal element of $\mathcal{N}$; such a maximal element κ of $\mathcal{N}$ exists by Zorn's lemma. The restriction of κ to B is a congruence on B which separates the elements of A. Since B is an essential normal extension of A, it follows that the restriction of κ to B is the identity on B. Let $P/\kappa = D$. We have thus shown that the restrictions of $\kappa^\natural$ to G and B are monomorphisms of G and B respectively into D.

We want to show that D is an essential cojugate extension of $A\kappa^\natural$ ($\cong A$). Since P is a conjugate extension of A, it is clear that D is a conjugate extension of $A\kappa^\natural$. Let ν be a congruence on D whose restriction to $A\kappa^\natural$ is the identity on $A\kappa^\natural$. Let ν^* be the congruence on P which is defined by

$$x\nu^* y \Leftrightarrow (x\kappa^\natural)\,\nu\,(y\kappa^\natural).$$

Obviously $\kappa \subseteq \nu^*$, and ν^* separates the elements of A. Since G is an essential conjugate extension of A, it follows that the restriction of ν^* to G is the identity of G. Hence, $\nu^* \in \mathcal{N}$. By the maximality of κ in $\mathcal{N}$ we have $\nu^* = \kappa$ and so ν is the identity on D. We have proved that D is an essential conjugate extension of $A\kappa^\natural$ ($\cong A$) which contains $G\kappa^\natural$ ($\cong G$). In order to complete our proof, it suffices to show that D contains $G\kappa^\natural$ properly.

Let us suppose that $D = G\kappa^\natural$. It should be obvious that $B\kappa^\natural$ is an essential normal extension of $A\kappa^\natural$ which has $c\kappa^\natural$ in its metacenter; furthermore, $B\kappa^\natural$ must be an inverse subsemigroup of $G\kappa^\natural$ which contains $S\kappa^\natural$. By the maximality of S in $\mathscr{X}$ we have $S\kappa^\natural = B\kappa^\natural$; yet, this is impossible since S is properly contained in B, and since $\kappa^\natural$ is injective on B. Thus the assumption $D = G\kappa^\natural$ is false, and we conclude that $G\kappa^\natural$ is properly contained in D.

THEOREM 3.2. *Let G be an essential normal extension of the inverse semigroup A, and let us suppose that the metacenter of A is not idempotent. Then there exists an essential normal extension of A which contains G properly.*

Proof. The proof proceeds along the same lines as the proof of the foregoing theorem. The main difference consists in the fact that P may be considered to be an inverse semigroup such that $E_A = E_G = E_S = E_P$ [**5**]; consequently, P will be a normal extension of A, and $P/\kappa = D$ is an essential normal extension of $A\kappa^\natural$, which contains $G\kappa^\natural$ properly.

We now may conclude with the main result of our paper. The following is a combination of Theorem 1.2, Theorem 3.1 and Theorem 3.2.

THEOREM 3.3. *An inverse semigroup S has a greatest essential conjugate [normal] extension if and only if the metacenter of S is idempotent.*

ACKNOWLEDGEMENT. I thank M. Petrich for suggesting the problems treated in this paper.

REFERENCES

1. D. Allouch, *Sur les extensions de demi-groupes strictement réguliers*, Doctoral dissertation, Université de Montpellier (1979).

2. L. M. Gluskin, On dense embeddings, *Trudy Moskov. Mat. Obšč.* **29** (1973), 119–131 (Russian).

3. L. M. Gluskin, and B. M. Schein, Ideal extensions of irreductive semigroups, *Semigroup Forum* **9** (1974), 216–240.

4. T. E. Hall, Free products with amalgamation of inverse semigroups, *J. Algebra* **34** (1975), 375–385.

5. T. E. Hall, Amalgamation and inverse and regular semigroups,*Trans. Amer. Math. Soc.* **246** (1978), 395–406.

6. J. M. Howie, *An introduction to semigroup theory* (Academic Press, 1976).

7. M. Petrich, *Introduction to semigroups* (Merrill, Columbus, 1973).

8. M. Petrich, Extensions normales de demi-groupes inverses (submitted).

9. M. Petrich, The conjugate hull of an inverse semigroup, *Glasgow Math. J.* **21** (1980), 103–124.

10. H. E. Scheiblich, Kernels of inverse semigroup homomorphisms, *J. Austral. Math. Soc.* **18** (1974), 289–292.

11. L. N. Ševrin, Densely embedded ideals in semigroups, *Mat. Sb.* **79** (1969), 425–432 (Russian).

Dienst Hogere Meetkunde
Rijksuniversiteit te Gent
Krijgslaan 281
B-9000 Gent, Belguim