ON THE DOI-NAGANUMA LIFTING ASSOCIATED
WITH IMAGINARY QUADRATIC FIELDS

TETSUYA ASAI

Introduction

Similarly to the real quadratic field case by Doi and Naganuma ([3], [9]) there is a lifting from an elliptic modular form to an automorphic form on $SL_2(C)$ with respect to an arithmetic discrete subgroup relative to an imaginary quadratic field. This fact is contained in his general theory of Jacquet ([6]) as a special case. In this paper, we try to reproduce this lifting in its concrete form by using the theta function method developed first by Niwa ([10]); also Kudla ([7]) has treated the real quadratic field case on the same line. The theta function method will naturally lead to a theory of lifting to an orthogonal group of general signature (cf. Oda [11]), and the present note will give a prototype of non-holomorphic case.

Let an imaginary quadratic number field be fixed once for all throughout this paper, and let $\mathfrak{o}$ denote the ring of integers of the field. For simplicity's sake we assume that the class number is one and the discriminant $-D$ is odd prime, less than -3. For a positive even integer ν we denote by $\mathscr{S}_{\nu+1}$ the space of cusp forms of weight $\nu + 1$ of Neben type χ with respect to $\Gamma_0(D)$, where χ denotes the Kronecker character $\left(\dfrac{-D}{*}\right)$. We shall show that each cusp form f in $\mathscr{S}_{\nu+1}$ can be lifted to a $C^{2\nu+1}$-valued automorphic form F on $SL_2(C)$ with respect to $SL_2(\mathfrak{o})$, belonging to an irreducible representation of $SU_2(C)$ of degree $2\nu + 1$, which is also an eigen-function of the Casimir operator with the eigen-value $\frac{1}{2}(\nu^2 - 1)$. We shall give the Fourier expansion of F explicitly. It may be remarkable that the lifted image F is cuspidal if and only if f is orthogonal to $\theta_{-D}^{(\nu)}$ in $\mathscr{S}_{\nu+1}$, where $\theta_{-D}^{(\nu)}(z) = \frac{1}{2} \sum_{r \in \mathfrak{o}} r^\nu \exp{(2\pi i r\bar{r}z)}$.

In our argument, a special polynomial of four variables, which is

---

nothing but a classical Laplace's spherical harmonics in essential, plays a fundamental role, so that we shall devote the first section to summarizing its properties. One of virtues of Niwa's method is in connecting a theta function with so-called Rankin's method, and we shall follow this, but in a little more direct fashion, namely we try to avoid using Eisenstein series there. The exceptional behaviour of $\theta_{-D}^{(\nu)}$ is related to the fact that $\theta_{-D}^{(\nu)}$ is the unique primitive form in $\mathscr{S}_{\nu+1}$ whose Fourier coefficients are all real. As an application of the Doi-Naganuma lifting we shall give a proof of this in the last section.

§ 1. The spherical harmonic polynomial

1.1. We shall denote by $\rho_n(g)$ the n-ply symmetric tensor product of $g \in GL_2(C)$, i.e.

$$^t((a, b)^t g)_n = \rho_n(g)\, ^t(a, b)_n$$

with indeterminants $a, b \in C$ and $(a, b)_n = (a^n, a^{n-1}b, \cdots, ab^{n-1}, b^n)$. Put $G = SL_2(C)$ and $K = SU_2(C)$. Each ρ_n gives an irreducible representation of K as is well known. Let us put

$$V = \{X \in M_2(C)\,;\, X = {}^t\bar{X}\}\,,$$

which is a vector space of dimension four over R, and so we regard $M_2(C)$ as $V_C = V \otimes_R C$. The group G acts on V in such a way that $X^g = {}^t\bar{g}Xg$ for $X \in V$ and $g \in G$. The action restricted to K gives a representation on V equivalent to $\rho_0 \oplus \rho_2$. Let us define two symmetric bilinear forms Q and R on V by

$$Q(X, Y) = -\mathrm{tr}\,(X\tilde{Y}),\ R(X, Y) = \mathrm{tr}\,(XY)\,;\qquad X, Y \in V\,,$$

where $\tilde{Y} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}{}^t Y \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. $Q(X)$ and $R(X)$ denote the associated quadratic forms $Q(X, X)$ and $R(X, X)$, respectively. The form Q is of signature $(3, 1)$ and G-invariant, i.e. $Q(X^g) = Q(X)$ for every $g \in G$, while the form R is positive definite and K-invariant, i.e. $R(X^\kappa) = R(X)$ for every $\kappa \in K$. We should note the form $R(X^g)$ is a minimal majorant of $Q(X)$ for each $g \in G$. For a non-negative integer ν, let $\mathscr{H}^\nu$ denote the C-linear space of polynomial functions on V spanned by $Q(X, A)^\nu$, where $A \in V_C$ such that $Q(X, A) = R(X, A)$ and $Q(A) = R(A) = 0$. An element of $\mathscr{H}^\nu$ is called a spherical harmonic polynomial with respect to Q and its majorant R. The dimension of $\mathscr{H}^\nu$ is $2\nu + 1$, in fact, we can get a natural basis

as follows.

Let us put $A = {}^t(a, b)(a, b)(_1 \; {}^{-1}) \in V_C$, then the form $Q(X, A)^\nu$ is a homogeneous polynomial of degree ν (resp. 2ν) with respect to X (resp. a, b). Hence we may define a homogeneous polynomial $\eta_{\nu, \alpha}(X)$ of degree ν as the coefficient of $a^{\nu - \alpha} b^{\nu + \alpha}$ in $Q(X, A)^\nu$ for each $\alpha, |\alpha| \leqq \nu$. We thus put:

$$(1) \qquad \begin{aligned} &\eta_{(\nu)}(X)^t(a, b)_{2\nu} = Q(X, A)^\nu \, , \\ &\eta_{(\nu)}(X) = (\eta_{\nu, -\nu}(X), \cdots, \eta_{\nu, \nu}(X)) \, . \end{aligned}$$

Obviously we have

LEMMA 1. *The polynomials* $\eta_{\nu, \alpha}(X), |\alpha| \leqq \nu$, *form a basis of* $\mathscr{H}^\nu$.

Since the forms Q and R are K-invariant, $\mathscr{H}^\nu$ is regarded as a representation space of K, which is irreducible. In particular, we have

LEMMA 2. $\eta_{(\nu)}(X^\kappa) = \eta_{(\nu)}(X)\rho_{2\nu}(\kappa)$ *for every* $\kappa \in K$.

Proof. We have $\eta_{(\nu)}(X^\kappa)^t(a, b)_{2\nu} = Q(X^\kappa, A)^\nu = Q(X, A^{\kappa^{-1}})^\nu$, which is equal to $\eta_{(\nu)}(X)^t((a, b)^t\kappa)_{2\nu} = \eta_{(\nu)}(X)\rho_{2\nu}(\kappa)^t(a, b)_{2\nu}$, since $A^{\kappa^{-1}} = {}^t((a, b)^t\kappa)((a, b)^t\kappa)$ $\cdot(_1 \; {}^{-1})$ for $\kappa \in K$.

We should note that Lemmas 1 and 2 characterize the polynomials $\eta_{\nu, \alpha}(X)$ up to a constant multiple, in fact, by a simple observation we can show that the polynomials $\eta_{\nu, \alpha}(X)$ are essentially the same as classical Laplace's spherical harmonics (cf. [4], Chap. XI).

1.2. We write a general element X of V as follows:

$$(2) \qquad X = \begin{pmatrix} m & r \\ \bar{r} & n \end{pmatrix}; \; m, n \in \mathbf{R}, r \in \mathbf{C} \, ,$$

and this parametrization will be kept throughout the paper. It is, then, easy to see that $\eta_{(0)}(X) = 1$ and $\eta_{(1)}(X) = (-\bar{r}, m - n, r)$, while we can compute more by use of a recursion formula

$$(3) \qquad \eta_{\nu, \alpha}(X) = (-\bar{r})\eta_{\nu-1, \alpha+1}(X) + (m - n)\eta_{\nu-1, \alpha}(X) + (r)\eta_{\nu-1, \alpha-1}(X) \, ,$$

where we understand $\eta_{0,0}(X) = 1$ and $\eta_{\nu, \alpha}(X) = 0$ if $|\alpha| > \nu$. The following is an explicit formula which we need later.

LEMMA 3. *For* $|\alpha| \leqq \nu$, *we have*

(4) $\eta_{\nu,\alpha}(X) = \nu! \sum_{\beta,\gamma} \frac{1}{(\alpha+\beta)!\cdot\gamma!} 2^{-\beta} r^{\alpha} L_{\beta}^{(\alpha)}(2r\bar{r}) H_{\gamma}(m-n)$,

where $L_{\beta}^{(\alpha)}$ and H_{γ} are Laguerre's and Hermite's polynomials respectively, and the sum is taken over all non-negative integers β, γ such that $2\beta + \gamma = \nu - \alpha$ and $\alpha + \beta \geqq 0$.

Proof. We put $q = m - n$ for abbreviation. By the definition (1) we have

$$\exp((-\bar{r}a^2 + qab + rb^2)t) = \sum_{\nu=0}^{\infty} \sum_{\alpha=-\nu}^{\nu} \eta_{\nu,\alpha}(X) a^{\nu-\alpha} b^{\nu+\alpha} \frac{t^{\nu}}{\nu!} \ .$$

It follows from a generating function formula of H_{γ} (e.g. [8], p. 253) that

$$\exp\left(qabt - \frac{1}{2}a^2b^2t^2\right) = \sum_{\gamma=0}^{\infty} H_{\gamma}(q) a^{\gamma} b^{\gamma} \frac{t^{\gamma}}{\gamma!} \ .$$

On the other hand, we have

$$\exp\left((-\bar{r}a^2 + rb^2)t + \frac{1}{2}a^2b^2t^2\right) = \sum_{\ell=0}^{\infty} \exp(-\bar{r}a^2t)\left(1 + \frac{1}{2}r^{-1}a^2t\right)^{\ell} \frac{(rb^2t)^{\ell}}{\ell!}$$

$$= \sum_{\ell=0}^{\infty} \sum_{\beta=0}^{\ell} 2^{-\beta} r^{\ell-2\beta} L_{\beta}^{(\ell-2\beta)}(2r\bar{r}) a^{2\beta} b^{2\ell-2\beta} \frac{t^{\ell}}{(\ell-\beta)!}$$

by use of a generating function formula of $L_{\beta}^{(\alpha)}$ (e.g. [8], p. 242). Equating these, we obtain the proof.

For convenience' sake let us define another quadratic form on V by

(5) $P(X) = Q(X) + R(X)$,

then it can be immediately seen that P is K-invariant, and $0 \leqq P(X) \leqq 2R(X)$ for $X \in V$.

LEMMA 4. $|\eta_{\nu,\alpha}(X)|^2 \leqq \binom{2\nu}{\nu-\alpha}\binom{2\nu}{\nu}^{-1} P(X)^{\nu}$ *for* $X \in V$.

Proof. It is sufficient to prove

(6) $\sum_{\alpha=-\nu}^{\nu} \binom{2\nu}{\nu-\alpha}^{-1} |\eta_{\nu,\alpha}(X)|^2 = \binom{2\nu}{\nu}^{-1} P(X)^{\nu}$.

Denote by J a diagonal constant matrix of degree $2\nu + 1$ satisfying $(a,b)_{2\nu} J^2 {}^t(\bar{a}, \bar{b})_{2\nu} = (a\bar{a} + b\bar{b})^{2\nu}$, then $J\rho_{2\nu}(\kappa)J^{-1}$ is a unitary matrix for every

$\kappa \in K$. Hence the left-hand side of (6), which is $\eta_{(\nu)}(X)J^{-2}\,{}^{t}\overline{\eta_{(\nu)}(X)}$, is K-invariant as well as the right-hand side. We may, therefore, assume $X = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$. Then both sides are equal to $\binom{2\nu}{\nu}^{-1}(m-n)^{2\nu}$. This completes the proof.

1.3. For the purpose to give another property of $\eta_{\nu,a}(X)$ we recall some notions on Lie derivatives. Let S be an arbitrary element of the Lie algebra of G, i.e. $S \in M_2(C)$ with $\mathrm{tr}(S) = 0$. A differential operator S' (or S'') on $C^{\infty}(G)$ is defined by $S'f(g) = \left[\dfrac{\partial}{\partial t}f(g \cdot \exp(tS))\right]_{t=0}$ $\left(\text{or by}\right.$ replacing $\dfrac{\partial}{\partial t}$ with $\dfrac{\partial}{\partial \bar{t}}$, accordingly.$\Big)$ On the other hand, a function $F \in C^{\infty}(V)$ induces a function $F_X(g) = F(X^g) \in C^{\infty}(G)$ for each $X \in V$. Hence S' (or S'') operates on $C^{\infty}(V)$ in such way that $(S'F)_X = S'F_X$ (or $(S''F)_X = S''F_X$). It can be easily seen that S' (or S'') on $C^{\infty}(V)$ is equal to $m'\dfrac{\partial}{\partial m} + n'\dfrac{\partial}{\partial n} + r'\dfrac{\partial}{\partial r} + r''\dfrac{\partial}{\partial \bar{r}}$, where XS (or ${}^{t}\bar{S}X) = \begin{pmatrix} m' & r' \\ r'' & n' \end{pmatrix}$ with the parametrization $X = \begin{pmatrix} m & r \\ \bar{r} & n \end{pmatrix} \in V$. The elements $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ form a basis of the Lie algebra, and the Casimir operators C' and C'' are given by $C' = A'B' + B'A' + \frac{1}{2}U'U'$ and $C'' = A''B'' + B''A'' + \frac{1}{2}U''U''$, while both the operators C' and C'' coincide with each other on $C^{\infty}(V)$. Further, by an easy computation we can obtain a formula on $C^{\infty}(V)$:

$$(7) \qquad C' = C'' = \tfrac{1}{2}L^2 + L - \tfrac{1}{2}Q(X)\varDelta_Q \,,$$

where $L = m\dfrac{\partial}{\partial m} + n\dfrac{\partial}{\partial n} + r\dfrac{\partial}{\partial r} + \bar{r}\dfrac{\partial}{\partial \bar{r}}$ and $\varDelta_Q = 2\dfrac{\partial^2}{\partial r \partial \bar{r}} - 2\dfrac{\partial^2}{\partial m \partial n}$ (cf. [13], p. 95).

LEMMA 5. *Put*

$$(8) \qquad \eta^{*}_{\nu,a}(X) = \eta_{\nu,a}(X)P(X)^{-(\nu+\frac{1}{2})} \,,$$

then we have $C'\eta^{*}_{\nu,a} = C''\eta^{*}_{\nu,a} = \frac{1}{2}(\nu^2 - 1)\eta^{*}_{\nu,a}$.

Proof. Since $\eta^{*}_{\nu,a}(X)$ is a homogeneous function of degree $-(\nu+1)$, $L\eta^{*}_{\nu,a} = -(\nu+1)\eta^{*}_{\nu,a}$. On the other hand, $\varDelta_Q\eta^{*}_{\nu,a} = 0$ as well as $\varDelta_Q\eta_{\nu,a} = 0$. These combined with the formula (7) complete the proof.

The function $\eta^*_{\nu,a}(X)$ is also called a spherical harmonics of degree $-(\nu + 1)$, which coincides with $\eta_{\nu,a}(X)$ on the surface: $P(X) = 1$.

§2. Poincaré series and theta series

2.1. Let $\mathfrak{o}$ be the ring of integers of an imaginary quadratic field of odd prime discriminant $-D$. It is, as in the introduction, assumed that the class number is one and the units are 1 and -1 only. Let us define a lattice $\mathfrak{L}$ of V over $\mathbf{Z}$ by $\mathfrak{L} = V \cap M_2(\mathfrak{o})$, and denote by $\mathfrak{L}(\ell)$ the subset of $\mathfrak{L}$ consisting of all elements of determinant $-\ell$ for each $\ell \in \mathbf{Z}$. Let $\varLambda$ denote the discrete subgroup $SL_2(\mathfrak{o})$ of G. Each subset $\mathfrak{L}(\ell)$ is $\varLambda$-invariant. For each positive integer ℓ we define a $\mathbf{C}^{2\nu+1}$-valued function on G by

$$(9) \qquad h_{(\nu),\ell}(g) = \sum_{X \in \mathfrak{L}(\ell)} \eta^*_{(\nu)}(X^g) , \qquad (g \in G) ,$$

where $\eta^*_{(\nu)}(X) = (\eta^*_{\nu,-\nu}(X), \cdots, \eta^*_{\nu,\nu}(X))$ and $\eta^*_{\nu,a}(X)$ is given by (8). This series is, as we shall see below, absolutely convergent for $\nu > 1$, hence the function $h_{(\nu),\ell}$ can be regarded as a type of Poincaré series, considering that the set $\mathfrak{L}(\ell)$ is a union of some $\varLambda$-orbits. This combined with Lemmas 2 and 5 leads to

LEMMA 6. *For each positive integer ℓ and $\nu > 1$, it holds*
(i) $h_{(\nu),\ell}(\gamma g \kappa) = h_{(\nu),\ell}(g)\rho_{2\nu}(\kappa)$ *for $\gamma \in \varLambda$ and $\kappa \in K$.*
(ii) $C'h_{(\nu),\ell} = C''h_{(\nu),\ell} = \frac{1}{2}(\nu^2 - 1)h_{(\nu),\ell}.$

For the purpose to prove that the series (9) is convergent absolutely and uniformly on any compact subset of G, it is sufficient to combine Lemma 4 with next two lemmas.

LEMMA 7. *If ℓ is positive, a series $\sum_{X \in \mathfrak{L}(\ell)} P(X)^{-s}$ is absolutely convergent for $\mathrm{Re}\, s > 1$.*

Proof. By noting $P(X) = 4r\bar{r} + (m - n)^2$ for $X = \begin{pmatrix} m & r \\ \bar{r} & n \end{pmatrix} \in V$, divide the summation on $\mathfrak{L}(\ell)$ into the following three parts: one with $r = 0$, one with $r\bar{r} = \ell$ and the others. The first, then, is a finite sum, and it can be easily seen that the second and the third are majorized by some constant multiples of $\sum_{n \in \mathbf{Z}} (4\ell + n^2)^{-\mathrm{Re}\, s}$ and $\sum_{r \in \mathfrak{o} - \{0\}} (r\bar{r})^{-(\mathrm{Re}\, s - \varepsilon)}$ respectively, where ε is chosen so that $\mathrm{Re}\, s - 1 > \varepsilon > 0$. These imply the lemma.

LEMMA 8. *For arbitrary $g \in G$ there exists a positive constant c such that $P(X^g) \geqq cP(X)$ for every $X \in V$ with non-positive determinant.*

Proof. Since $P(X)$ is K-invariant, we can assume that g is diagonal: $g = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$. Take c to be the minimum of $|\beta|^4$ and $|\beta|^{-4}$, then we have $P(X^g) = 4r\bar{r} + (m|\beta|^2 - n|\beta|^{-2})^2 \geqq P(X) + (m^2 + n^2)(c - 1)$, which is not less than $cP(X)$ because $P(X) \geqq m^2 + n^2$ and $c \leqq 1$.

Remark. $h_{(\nu),\ell}$ vanishes for every odd ν. Also $h_{(\nu),0}$ can be defined by (9) in whose summation $X = 0$ is excluded, and it is an Eisenstein series satisfying the properties in Lemma 6.

2.2. For a function $f(z)$ on the upper half complex plane $\mathfrak{H}$ and a real matrix $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of positive determinant (i.e. $\sigma \in GL_2^+(R)$), we write

$$(f \,|\, [\sigma]_k)(z) = (\det(\sigma))^{k/2}(cz + d)^{-k}f(\sigma z)$$

for $k \in Z$, where $\sigma z = (az + b)(cz + d)^{-1}$. We define a differential operator δ_λ for $\lambda \in R$ following Shimura ([12]) by

$$\delta_\lambda = \frac{1}{2\pi i}\left(\frac{\lambda}{2iy} + \frac{\partial}{\partial z}\right) = \frac{1}{2\pi i}y^{-\lambda}\frac{\partial}{\partial z}y^\lambda ,$$

where $z = x + iy$, and we also put

$$\delta_\lambda^\ell = \delta_{\lambda + 2(\ell-1)} \cdots \delta_{\lambda+2}\delta_\lambda \qquad \text{for } 0 \leqq \ell \in Z .$$

The "raising" operator δ_λ^ℓ acts on functions on $\mathfrak{H}$ and satisfies that $\delta_\lambda^\ell(f \,|\, [\sigma]_\lambda) = (\delta_\lambda^\ell f)\,|\,[\sigma]_{\lambda+2\ell}$ for every $\sigma \in GL_2^+(R)$. We need the following

LEMMA 9. (i) $\delta_\lambda^\ell \exp(2\pi i a z) = \ell!\,(-4\pi y)^{-\ell}L_\ell^{(\lambda-1)}(4\pi a y)\exp(2\pi i a z)$, ($\lambda$, $a \in R$).

(ii) *Put* $t_r(z\,;\,m, n) = \sqrt{y}(8\pi y)^{-r/2}H_r(\sqrt{2\pi y}(m - n))\exp(\pi i(x(-2mn) + iy(m^2 + n^2)))$, *then* $\delta_r^\ell t_r = t_{r+2\ell}$, ($0 \leqq \gamma \in Z$, $m, n \in R$). *Here $L_\ell^{(\lambda)}$ and H_r are Laguerre's and Hermite's polynomials respectively.*

Proof. We can easily prove (i) or (ii) by induction on ℓ, using a formula (cf. [8], p. 241, p. 252):

$$x\frac{d}{dx}L_\ell^{(\alpha)}(x) = (\ell + 1)L_{\ell+1}^{(\alpha)}(x) - (\ell + \alpha + 1 - x)L_\ell^{(\alpha)}(x)$$

or

$$x\frac{d}{dx}H_\ell(x) = (x^2 - (\ell + 1))H_\ell(x) - H_{\ell+2}(x) ,$$

accordingly.

2.3. Now let us define a theta series with respect to the indefinite quadratic form Q on V. Since the minimal majorant $R(X^g)$ is parametrized by $g \in G$, our function has double variables $z = x + iy \in \mathfrak{H}$ and $g \in G$. Namely we put

(10)
$$\theta_{\nu,a}(z, g) = \sqrt{y} \sum_{X \in \mathfrak{L}} \eta_{\nu,a}(X^g) \exp(\pi i(xQ(X) + iyR(X^g))) ,$$
$$\theta_{(\nu)}(z, g) = (\theta_{\nu,-\nu}(z, g), \cdots, \theta_{\nu,\nu}(z, g)) ,$$

where $\mathfrak{L} = V \cap M_2(\mathfrak{o})$ as before. Obviously the series is absolutely convergent. We may, for non-triviality, assume that ν is even, since $\theta_{(\nu)}(z, g)$ vanishes for odd ν. For abbreviation Γ will stand for the subgroup $\Gamma_0(D)$ of $SL_2(\mathbf{Z})$ consisting of all $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $D \mid c$, and χ denotes the character of Γ defined by Kronecker's symbol, i.e. $\chi(\sigma) = \left(\dfrac{-D}{d}\right)$ for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

LEMMA 10. *For a non-negative even integer ν, it holds that*
(i) $\theta_{\nu,a}(z, g) = \chi(\sigma)\theta_{\nu,a}(z, g)|[\sigma]_{\nu+1}$ *for every* $\sigma \in \Gamma$.
(ii) $\theta_{(\nu)}(z, \gamma g\kappa) = \theta_{(\nu)}(z, g)\rho_{2\nu}(\kappa)$ *for every* $\gamma \in \Lambda$ *and* $\kappa \in K$.
(ii.a) $\theta_{(\nu)}(z, g^0) = \theta_{(\nu)}(z, g)\rho_{2\nu}(^1 {}_{-1})$, *where* $g^0 = (^1 {}_{-1})g(^1 {}_{-1})$.
(ii.b) $\theta_{(\nu)}(z, \bar{g}) = \theta_{(\nu)}(z, g)\rho_{2\nu}(_1 {}^{-1})$.

Proof. Due to Lemma 1.2 and Proposition 1.6 of Shintani [13], the theta transformation formula depends only on the form Q and the lattice $\mathfrak{L}$, hence it is sufficient to prove (i) in the case that $\alpha = \nu$ and $g = 1$ (the identity). Since $\eta_{\nu,\nu}(X) = r^\nu$ for $X = \begin{pmatrix} m & r \\ \bar{r} & n \end{pmatrix} \in \mathfrak{L}$, we can immediately see that $\theta_{\nu,\nu}(z, 1) = 2\theta_{-D}^{(\nu)}(z)\theta_0(z)$, where $\theta_{-D}^{(\nu)}(z) = \frac{1}{2}\sum_{r \in \mathfrak{o}} r^\nu \cdot \exp(2\pi i r\bar{r}z)$ and $\theta_0(z) = \sqrt{y}\sum_{m,n \in \mathbf{Z}} \exp(\pi i(x(-2mn) + iy(m^2 + n^2)))$. As is well known, $\theta_{-D}^{(\nu)} = \chi(\sigma)\theta_{-D}^{(\nu)}|[\sigma]_{\nu+1}$ for $\sigma \in \Gamma$, and $\theta_0 = \theta_0|[\sigma]_0$ for $\sigma \in SL_2(\mathbf{Z})$ since $\theta_0(z) = \sum_{c,d \in \mathbf{Z}} \exp(-\pi y^{-1}|cz + d|^2)$ as derived by Poisson's summation formula. We have thus proved (i). For the proof of (ii) we have only to consider Λ-invariance of $\mathfrak{L}$, K-invariance of $R(X^g)$ and Lemma 2. We can derive (ii.a) in the same way as (ii), though $(^1 {}_{-1})$ belongs

to neither Λ nor K. Finally from the definition (1) of $\eta_{\nu,\alpha}$ we have $\eta_{\nu,\alpha}(\overline{X}) = (-1)^{\alpha}\eta_{\nu,-\alpha}(X)$, which imply (ii.b).

2.4. For a diagonal $g \in G$ the theta function $\theta_{\nu,\alpha}(z,g)$ splits into a convenient form. To describe this, we define two more theta series by

$$(11) \qquad \theta^{(\alpha)}_{-D,\beta}(z) = \tfrac{1}{2}\sum_{r \in \mathfrak{o}} r^{\alpha}y^{-\beta}L^{(\alpha)}_{\beta}(4\pi r\bar{r}y)\exp(2\pi i r\bar{r}z)$$

for $0 \leqq \alpha, \beta \in \mathbf{Z}$, and

$$(12) \qquad \theta_{r}(z,v) = v^{r+1}y^{-r}\sum_{c,d \in \mathbf{Z}}(cz+d)^{r}\exp(-\pi v^2 y^{-1}|cz+d|^2)$$

for $0 \leqq \gamma \in \mathbf{Z}$, where $z = x + iy \in \mathfrak{H}$ and $0 < v \in \mathbf{R}$. We abbreviate $\theta^{(\alpha)}_{-D,0}$ to $\theta^{(\alpha)}_{-D}$ simply. We should notice that $\theta^{(\alpha)}_{-D,\beta}$ (or θ_{r}) vanishes for odd α (or odd γ). Due to Lemma 9 we have $\theta^{(\alpha)}_{-D,\beta} = (-4\pi)^{\beta}(\beta!)^{-1}\delta^{\beta}_{\alpha+1}\theta^{(\alpha)}_{-D}$ and $\theta_{r} = (-4)^{r/2}\delta^{r/2}_{0}\theta_{0}$ for even γ, so that $\theta^{(\alpha)}_{-D,\beta} = \chi(\sigma)\theta^{(\alpha)}_{-D,\beta}|[\sigma]_{\alpha+2\beta+1}$ for $\sigma \in \Gamma$ and $\theta_{r} = \theta_{r}|[\sigma]_{r}$ for $\sigma \in SL_{2}(\mathbf{Z})$. There is another expression for θ_{r}:

$$(13) \qquad \theta_{r}(z,v) = (-4)^{r/2}\sum_{m,n \in \mathbf{Z}}t_{r}(z\,;\,mv,nv^{-1})\,,$$

where $t_{r}(z\,;\,m,n)$ is the same as defined in Lemma 9. We can obtain (13) directly by Poisson's summation formula, or by applying the raising operator $\delta^{r/2}_{0}$ to the simpler case of $\gamma = 0$.

Now let us put $g(v) = \dfrac{1}{\sqrt{v}}\left(\begin{smallmatrix} v & \\ & 1 \end{smallmatrix}\right)$ for $0 < v \in \mathbf{R}$. Then we have $\theta_{\nu,\alpha}(z,g(v)) = (-1)^{\alpha}\theta_{\nu,-\alpha}(z,g(v))$ which, especially, vanishes for odd α.

LEMMA 11. *For a non-negative even integer* α,

$$\theta_{\nu,\alpha}(z,g(v)) = 2(\nu!)\sum_{\beta,r}\frac{1}{(\alpha+\beta)!\cdot\gamma!}i^{r}(4\pi)^{-\beta}\theta^{(\alpha)}_{-D,\beta}(z)\theta_{r}(z,v)\,,$$

where the sum is taken over $\beta \geqq 0$, $\gamma \geqq 0$ *with* $2\beta + \gamma = \nu - \alpha$.

Proof. Observing that

$$xQ(X) + iyR(X^{g}) = (2r\bar{r}z) + (x(-2mn) + iy(m^2v^2 + n^2v^{-2}))$$

for $g = g(v)$ and $X = \left(\begin{smallmatrix} m & r \\ \bar{r} & n \end{smallmatrix}\right) \in \mathfrak{L}$, and considering the explicit formula for $\eta_{\nu,\alpha}(X)$ in Lemma 3, we can easily obtain the lemma.

§3.   The Doi-Naganuma lifting

3.1.   We hereafter make the assumption that ν is even and positive. $\mathscr{S}_{\nu+1}$ will, as in the introduction, denote the space of all holomorphic cusp forms of weight $\nu + 1$ of Neben type χ with respect to Γ, so that f in $\mathscr{S}_{\nu+1}$ satisfies $f = \chi(\sigma)f\,|\,[\sigma]_{\nu+1}$ for $\sigma \in \Gamma$, where $\Gamma = \Gamma_0(D)$ and $\chi(\sigma) = \left(\dfrac{-D}{d}\right)$ for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Let f be a cusp form in $\mathscr{S}_{\nu+1}$, and let us consider the following integral of Petersson's inner product type in view of (i) in Lemma 10:

$$(14) \qquad F_\alpha(g) = 2^{\nu-2} i^{-(\nu+1)} D^{\nu/2} \int_{\Gamma\backslash\mathfrak{H}} \theta_{\nu,\alpha}(z,g) \overline{f_T(z)} y^{\nu-1} dx dy \,,$$

and define a $C^{2\nu+1}$-valued function $F(g)$ on G by

$$(15) \qquad F(g) = (F_{-\nu}(g), \cdots, F_\nu(g)) \,,$$

where f_T in $\mathscr{S}_{\nu+1}$ is defined by $f_T(z) = \overline{(f\,|\,[W]_{\nu+1})(-\bar{z})}$ with $W = (_{-D} \ ^1)$. In an obvious manner we can see that $\theta_{\nu,\alpha}(z,g) = O\,(\exp(-\varepsilon y))$ when $y \to \infty$ for some positive ε and similar estimations hold at any other cusps, hence the integral (14) is absolutely convergent. The correspondence of f with F defines a linear map, which we shall denote by I, from $\mathscr{S}_{\nu+1}$ to a space of some functions on G. This is the definition of the Doi-Naganuma lifting in our case.

THEOREM 1.   *The lifted image $F = I(f)$ satisfies the followings*:
(i) $F(\gamma g \kappa) = F(g)\rho_{2\nu}(\kappa)$ *for* $\gamma \in \Lambda$ *and* $\kappa \in K$.
(i.a) $F(g^0) = F(g)\rho_{2\nu}(^1 \ _{-1})$, *where* $g^0 = (^1 \ _{-1})g(^1 \ _{-1})$.
(i.b) $F(\bar{g}) = F(g)\rho_{2\nu}(_1 \ ^{-1})$.
(ii) $C'F = C''F = \frac{1}{2}(\nu^2 - 1)F$, *where* C', C'' *are the Casimir operators on* G.

Proof.   (i), (i.a) and (i.b) are immediate consequences of (ii), (ii.a) and (ii.b) in Lemma 10, respectively. To prove (ii), we should recall that the space $\mathscr{S}_{\nu+1}$ is generated by Poincaré series for $1 \leqq \ell \in \mathbf{Z}$:

$$p_\ell(z) = \sum_{\sigma \in \Gamma_\infty \backslash \Gamma} \chi(\sigma) \exp(2\pi i \ell z)\,|\,[\sigma]_{\nu+1} \,,$$

where σ runs over Γ modulo the stabilizer Γ_∞ of ∞ in Γ. We can easily obtain

$$\int_{\Gamma\backslash\mathfrak{H}} \theta_{(\nu)}(z,g)\overline{p_\ell(z)}y^{\nu-1}dxdy = \pi^{-(\nu+\frac{1}{2})}\Gamma(\nu+\tfrac{1}{2})h_{(\nu),\ell}(g) \ ,$$

where $h_{(\nu),\ell}(g)$ is the same as defined by (9). The computation is valid for $\nu > 1$. This combined with Lemma 6 implies (ii).

3.2. To give the Fourier expansion of the lifted image $F = I(f)$ we need some notations as follows. We should first recall that Λ has the unique non-equivalent cusp, say ∞. Let us put $g(u,v) = \dfrac{1}{\sqrt{v}}\begin{pmatrix} v & u \\ & 1 \end{pmatrix}$ for $u \in C$ and $0 < v \in R$, so that every element of G has unique expression as $g(u,v)\kappa$ with some $g(u,v)$ and $\kappa \in K$. We may abbreviate $g(0,v)$ to $g(v)$ as before. We put $S(u) = u + \bar{u}$ for $u \in C$. $K_\alpha(v)$ denotes the modified Bessel function of order α (e.g. [8], Chap. III). For each $\alpha \in Z$ a grössen character ξ^α is defined by $\xi^\alpha(r) = r^\alpha |r|^{-\alpha}$ for $r \in \mathfrak{o} - \{0\}$. Finally we put $\omega = \dfrac{1}{\sqrt{-D}}$, so that (ω) is the complementary ideal of $\mathfrak{o}$.

THEOREM 2. *Let* $f \in \mathscr{S}_{\nu+1}$ *and* $F = (F_{-\nu}, \cdots, F_\nu)$ *be the lifted image* $I(f)$. *Suppose we have*

$$f(z) = \sum_{n=1}^\infty a(n) \exp(2\pi i n z) \ , \qquad f_T(z) = \sum_{n=1}^\infty \overline{b(n)} \exp(2\pi i n z) \ ,$$

and put

$$C(0) = 2^{\nu-1}D^{\nu/2}\int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{\theta^{(\nu)}_{-D}(z)}y^{\nu-1}dxdy \ ,$$
$$C(r) = C_1(r) + C_2(r) \qquad \text{for } r \in \mathfrak{o} - \{0\};$$
$$C_1(r) = \sum_{n|(r)} n^\nu a(n^{-2}r\bar{r}) \ , \qquad C_2(r) = (-i)D^{\nu/2}\sum_{n|(r\omega)} n^\nu b(n^{-2}D^{-1}r\bar{r}) \ .$$

Then

(16) $$F_\alpha(g(u,v)) = \sum_{r\in\mathfrak{o}} C(r)\phi_\alpha(v,r) \exp(2\pi i S(\bar{r}\omega u))$$

for $|\alpha| \leqq \nu$, *where*

$$\phi_\alpha(v,0) = \delta_{\nu,|\alpha|}v \ , \quad \phi_\alpha(v,r) = \binom{2\nu}{\nu - \alpha}\xi^\alpha(r)K_\alpha(4\pi |r\omega| v)v^{\nu+1} \qquad \text{for } r \neq 0 \ .$$

We shall prove Theorem 2 in 3.5 after some preliminary lemmas. On the other hand, Theorem 2 says that the image $F = I(f)$ is not always cuspidal even though f is a cusp form. To make clear we state

this as follows, while the proof is obvious:

THEOREM 3. *Let $\mathscr{S}^1_{\nu+1}$ be the orthogonal complement of $\theta^{(\nu)}_D$ with respect to the Petersson metric in $\mathscr{S}_{\nu+1}$, then the lifted image $F = I(f)$ of f in $\mathscr{S}^1_{\nu+1}$ satisfies the followings:*
(iii)   *$F(g)^t\overline{F(g)}$ is bounded on G.*

(iv) $\displaystyle\int_{C/\mathfrak{o}} F(g(u,1)\cdot g)dud\overline{u} = 0$ *for every $g \in G$.*

3.3.   The next and following two lemmas are preparations to prove Theorem 2.

LEMMA 12.   *Under the same conditions and notations as in Theorem 2,*

(17) $$F_\alpha(g(v)) = \sum_{r \in \mathfrak{o}} C(r)\phi_\alpha(v,r) \ .$$

Proof.   We first notice that $F_\alpha(g(v))$ vanishes for odd α as well as $\theta_{\nu,\alpha}(z, g(v))$, and $F_\alpha(g(v)) = F_{-\alpha}(g(v))$ for even α. Hence we may assume that α is even and non-negative. We prove (17) by a direct computation of the integral (14) for $g = g(v)$. Owing to Lemma 11, it is reduced to a computation of an integral

(18) $$\int_{\Gamma\backslash\mathfrak{H}} \theta^{(\alpha)}_{-D,\beta}(z)\theta_r(z,v)\overline{f_T(z)}y^{\nu-1}dxdy$$

with $2\beta + \gamma = \nu - \alpha, \beta \geqq 0, \gamma \geqq 0$. By the definition (12) of θ_r, it is plain to see that

$$\theta_r(z,v) = \delta_{r,0}v + \theta^1_r(z,v) + D^{r/2}\theta^1_r(Dz,v)|[W]_r \ ,$$

where $\theta^1_r(z,v) = 2\sum_{n=1}^\infty n^r \sum_{\sigma \in \Gamma_\infty\backslash\Gamma} k(z,n)|[\sigma]_r$, $k(z,n) = v^{r+1}y^{-r}\exp(-\pi v^2 n^2 y^{-1})$ and $W = (_{-D} \ ^1)$. Hence the integral (18) decomposes into three parts. For the first, because of Lemma 6 of [12] we have, when $\gamma = 0$,

$$\int_{\Gamma\backslash\mathfrak{H}} \theta^{(\alpha)}_{-D,\beta}(z)\overline{f_T(z)}y^{\nu-1}dxdy = 0 \qquad \text{unless} \ \ \beta = 0 \ (\alpha = \nu) \ .$$

By usual method the second part is computed as follows:

$$\int_{\Gamma\backslash\mathfrak{H}} \theta^{(\alpha)}_{-D,\beta}(z)\overline{f_T(z)}\theta^1_r(z,v)y^{\nu-1}dxdy$$

$$= 2\sum_{n=1}^\infty n^r \int_{\Gamma\backslash\mathfrak{H}} \theta^{(\alpha)}_{-D,\beta}(z)\overline{f_T(z)}k(z,n)y^{\nu-1}dxdy$$

$$= 2v^{\nu+1} \sum_{n=1}^{\infty} n^{\tau} \int_0^{\infty} dy\, y^{\nu-\tau-1} \exp\left(-\pi v^2 n^2 y^{-1}\right) \int_0^1 \theta_{-D,\beta}^{(\alpha)}(z)\overline{f_T(z)} dx \ .$$

Here we need a formula (cf. [5], p. 175 (33), though there is a misprint.):

$$\int_0^{\infty} \exp\left(-at - \frac{b}{t}\right) L_\beta^{(\alpha)}(at) t^{\alpha+\beta-1} dt = (-1)^\beta \frac{2}{\beta!} a^{-\alpha/2} b^{\alpha/2+\beta} K_\alpha(2\sqrt{ab})$$

for $a, b > 0$. We thus obtain

$$\int_{\Gamma \backslash \mathfrak{H}} \theta_{-D,\beta}^{(\alpha)}(z)\overline{f_T(z)}\theta_r^1(z, v) y^{\nu-1} dx dy$$

$$= (-\pi)^\beta (\beta!)^{-1} 2^{1-\alpha} i^{\alpha+1} D^{-\nu/2} \sum_{r \in \mathfrak{o}-\{0\}} \xi^\alpha(r) C_2(r) K_\alpha(4\pi |r\omega| v) v^{\nu+1} \ .$$

The same computation holds for the third part. Thus by using a relation $\nu! \sum_{2\beta+\tau=\nu-\alpha} 2^\tau (\beta!\, \tau!\, (\alpha + \beta)!)^{-1} = \binom{2\nu}{\nu - \alpha}$, we can complete the proof of (17).

3.4. **LEMMA 13.** *Suppose that $F = (F_{-\nu}, \cdots, F_\nu)$ is a function on G, satisfying the properties* (i), (i.a), (i.b) *and* (ii) *in Theorem 1, and an additional condition that $F_\alpha(g(v)) = \delta_{\nu,|\alpha|} B(0) v + O\left(\exp\left(-\varepsilon v\right)\right)$ for $v \to \infty$ with some constants $\varepsilon > 0$ and $B(0)$. Then, F_α has a Fourier expansion as follows*:

$$(19) \qquad F_\alpha(g(u, v)) = \sum_{r \in \mathfrak{o}} B(r)\phi_\alpha(v, r) \exp\left(2\pi i S(\bar r \omega u)\right) \ ,$$

where $\phi_\alpha(v, r)$ is the same as in Theorem 2 and the coefficient $B(r)$ $(= B(-r) = B(\bar r))$ does not depend on α.

Proof. This lemma is due to Weil [14], Chap. VIII. In fact, put formally $F_\alpha(g(u, v)) = \sum_{r \in \mathfrak{o}} \psi_\alpha(v, r) \exp\left(2\pi i S(\bar r \omega u)\right)$, then each term satisfies the Beltrami operator's equation (E_0) in [14], p. 72 for $r = 0$ or (E) in p. 74 for $r \neq 0$. So we have $\psi_\alpha(v, 0) = \delta_{\nu,\alpha} B(0) v$. For $r \neq 0$ it first follows that we can put $\psi_\nu(v, r) = B(r)\phi_\nu(v, r)$, and then we obtain $\psi_\alpha(v, r) = B(r)\phi_\alpha(v, r)$ recursively by using a formula $x K_\alpha'(x) + \alpha K_\alpha(x) = -x K_{\alpha-1}(x)$ (cf. [8], p. 67) and by noting a special role of the factor $\binom{2\nu}{\nu - \alpha}\xi^\alpha(r)$. It follows from (i.a) and (i.b) that $B(r) = B(-r) = B(\bar r)$.

LEMMA 14. *For a non-negative integer ℓ, it holds that*

(20) $\left[\dfrac{\partial^\ell}{\partial \bar{u}^\ell}\theta_{\nu,\nu}(z, g(u, v))\right]_{u=0} = 2(-2\pi i)^\ell v^{-\ell} y^\ell \theta_{-D}^{(\nu+\ell)}(z)\theta_\ell(-z, v)$.

Proof. We first note that $\eta_{\nu,\nu}(X^g) = (mu + r)^\nu$ does not depend on $\bar{u}$ for $X = \begin{pmatrix} m & r \\ \bar{r} & n \end{pmatrix} \in \mathfrak{L}$ and $g = g(u, v)$. Put $Y = \pi y R(X^g)$ which is a quadratic polynomial of $\bar{u}$, $Y_1 = \dfrac{\partial}{\partial \bar{u}}Y$ and $Y_2 = \dfrac{\partial^2}{\partial \bar{u}^2}Y$, so that it holds

$(-1)^\ell \exp(Y)\dfrac{\partial^\ell}{\partial \bar{u}^\ell}\exp(-Y) = Y_2^{\ell/2}H_\ell(Y_1 Y_2^{-1/2})$ with Hermite's polynomial H_ℓ. Hence

$$\left[\dfrac{\partial^\ell}{\partial \bar{u}^\ell}\exp(-Y)\right]_{u=0} = (-rv^{-1})^\ell(2\pi y)^{\ell/2}H_\ell(\sqrt{2\pi y}(mv + nv^{-1}))$$
$$\exp(-\pi y(2r\bar{r} + m^2 v^2 + n^2 v^{-2})) ,$$

which leads to the proof of (20).

3.5. *Proof of Theorem* 2. Since our function $F = I(f)$ satisfies the assumption of Lemma 13, we may write $F_\alpha(g(u, v))$ in the form of (19). A simple observation of this and (17) follows that $B(0) = C(0)$. Next we must prove that $B(r) = C(r)$ for any $r \in \mathfrak{o} - \{0\}$. Observing that $\xi^\alpha(\mathfrak{a}), B(\mathfrak{a})$ and $C(\mathfrak{a})$ are well defined for each ideal $\mathfrak{a} = (r)$ if α is even, it is sufficient for our purpose to show

(21) $\sum\limits_{\mathfrak{a}} \xi^\alpha(\mathfrak{a})B(\mathfrak{a})N\mathfrak{a}^{-s} = \sum\limits_{\mathfrak{a}} \xi^\alpha(\mathfrak{a})C(\mathfrak{a})N\mathfrak{a}^{-s}$

for all even $\alpha \in Z$ and $s \in C$ with sufficiently large real part, where $\mathfrak{a}$ runs over all non-zero integral ideals of $\mathfrak{o}$. For $|\alpha| \le \nu$ we can get (21) by the Mellin transform

$$\int_0^\infty (F_\alpha(g(v)) - \delta_{\nu,|\alpha|}C(0)v)v^{2s-(\nu+2)}dv ,$$

which, in fact, is equal to

$$\frac{1}{2}\begin{pmatrix} 2\nu \\ \nu - \alpha \end{pmatrix}D^s(2\pi)^{-2s}\Gamma\left(s + \frac{\alpha}{2}\right)\Gamma\left(s - \frac{\alpha}{2}\right)\sum \xi^\alpha(\mathfrak{a})C(\mathfrak{a})N\mathfrak{a}^{-s}$$

if we use (17), or the same in which $C(\mathfrak{a})$ is replaced by $B(\mathfrak{a})$ if we use (19). For $|\alpha| > \nu$ we may assume $\alpha = \nu + \ell$ with positive even ℓ because $\xi^{-\alpha}(\mathfrak{a}) = \xi^\alpha(\bar{\mathfrak{a}}), B(\mathfrak{a}) = B(\bar{\mathfrak{a}})$ and $C(\mathfrak{a}) = C(\bar{\mathfrak{a}})$. On one hand, it follows from (19) that

$$\left[\frac{\partial^\ell}{\partial\overline{u}^\ell}F_\nu(g(u,v))\right]_{u=0} = (-2\pi)^\ell D^{-\ell/2}\sum_{r\in\mathfrak{o}-\{0\}}r^\ell B(r)\phi_\nu(v,r) \ .$$

On the other hand, this is equal to

$$\int_{\Gamma\backslash\mathfrak{H}}\left[\frac{\partial^\ell}{\partial\overline{u}^\ell}\theta_{\nu,\nu}(z,g(u,v))\right]_{u=0}\cdot\overline{f_T(z)}y^{\nu-1}dxdy \ ,$$

which becomes $(-2\pi)^\ell D^{-\ell/2}\sum_{r\in\mathfrak{o}-\{0\}}r^\ell C(r)\phi_\nu(v,r)$ through a similar computation to Lemma 12 by using Lemma 14. We can therefore complete the proof of (21) for $\alpha=\nu+\ell$ by Mellin transform again.

3.6. We give here some supplementary remarks on Dirichlet series and their Euler products. Let $F=I(f)$ be the lifted image of a cusp form f in $\mathscr{S}_{\nu+1}$, so that F has the Fourier expansion as in Theorem 2. Let us put

$$\Phi_\alpha(s) = D^s(2\pi)^{-2s}\Gamma\Big(s+\frac{\alpha}{2}\Big)\Gamma\Big(s-\frac{\alpha}{2}\Big)\sum_\mathfrak{a}\xi^\alpha(\mathfrak{a})C(\mathfrak{a})N\mathfrak{a}^{-s}$$

for even $\alpha\in Z$ and $s\in C$ with sufficiently large real part. Because ${}^t g^{-1} = \Big(\begin{smallmatrix}&-1\\1&\end{smallmatrix}\Big)g\Big(\begin{smallmatrix}&1\\-1&\end{smallmatrix}\Big)$ for $g\in G$, $F({}^t g^{-1}) = F(g)\rho_{2\nu}\Big(\begin{smallmatrix}&1\\-1&\end{smallmatrix}\Big) = F(\bar{g})$ and so $F(g(v)) = F(g(v^{-1}))$. Hence a variant expression of the Mellin transform

$$\int_1^\infty (F_\alpha(g(v)) - \delta_{\nu,|\alpha|}C(0)v)(v^{2s-(\nu+1)} + v^{(\nu+1)-2s})\frac{dv}{v}$$

$$- \delta_{\nu,|\alpha|}C(0)\Big(\frac{1}{2s-\nu} - \frac{1}{\nu+2-2s}\Big)$$

gives the meromorphic continuation of $\Phi_\alpha(s)$ and the functional equation $\Phi_\alpha(s) = \Phi_\alpha(\nu+1-s)$ for $|\alpha|\leqq\nu$. In particular, $\Phi_\alpha(s)$ is entire for $|\alpha| < \nu$, while $\Phi_{\pm\nu}(s)$ is entire if and only if $f\in\mathscr{S}_{\nu+1}^1$. For $\alpha=\nu+\ell$ with positive even ℓ(It should be noticed $\Phi_\alpha = \Phi_{-\alpha}$ in our case.), by Rankin's method in the convolution of f and $\theta_{-D}^{(\alpha)}$ it is also possible to get the meromorphic continuation of $\Phi_\alpha(s)$ and the functional equation $\Phi_\alpha(s) = \Phi_\alpha(\nu+1-s)$, while we can say no more about the holomorphy except the fact $\Big(s-\frac{\alpha}{2}\Big)\Big(s-\frac{\alpha}{2}+1\Big)\cdots\Big(s-\frac{\alpha}{2}+(\ell-1)\Big)\Phi_\alpha(s)$ is entire.

When f in $\mathscr{S}_{\nu+1}$ is a normalized primitive form (i.e. a common eigen-function of all Hecke operators with $a(1)=1$), it is well known

that $f_T = -iD^{-\nu/2}a(D)f$ and so $b(n) = iD^{-\nu/2}\overline{a(Dn)}$. Hence we can easily show

$$\sum_{\mathfrak{a}} \xi^\alpha(\mathfrak{a})C(\mathfrak{a})N\mathfrak{a}^{-s} = \prod_{\mathfrak{p}} (1 - \xi^\alpha(\mathfrak{p})C(\mathfrak{p})N\mathfrak{p}^{-s} + \xi^{2\alpha}(\mathfrak{p})N\mathfrak{p}^{\nu-2s})^{-1}$$

for every even integer α. It should be also remarked that $C(\mathfrak{p}) = a(p)$, $a(p^2) + p^\nu$ or $a(p) + \overline{a(p)}$ according as $\left(\dfrac{-D}{p}\right) = 1, -1$ or 0 for each prime $\mathfrak{p}\,|\,p$. In particular, it holds

$$\sum_{\mathfrak{a}} C(\mathfrak{a})N\mathfrak{a}^{-s} = \sum_{n=1}^{\infty} a(n)n^{-s} \cdot \sum_{n=1}^{\infty} \overline{a(n)}n^{-s} \ .$$

§4. A characterization of $\theta_{-D}^{(\nu)}(z)$

4.1. As an application of the Doi-Naganuma lifting we give a proof of the following

THEOREM 4. *Let f be a normalized primitive form in $\mathscr{S}_{\nu+1}$, and assume that all the eigen-values for Hecke operators are real, then $f = \theta_{-D}^{(\nu)}$.*

Remark. Our proof will be based on the fact that the lifted image $I(f)$ is cuspidal if and only if f is orthogonal to $\theta_{-D}^{(\nu)}$, i.e. $f \in \mathscr{S}_{\nu+1}^1$. In contrast with this, the lifted image of a holomorphic cusp form is always a Hilbert modular cusp form in the real quadratic field case. By using this fact we may derive an analogous result as follows: There are no such primitive forms of Neben type $\left(\Gamma_0(\varDelta), \left(\dfrac{\varDelta}{*}\right)\right)$ as all the eigen-values for Hecke operators are real, where $\varDelta$ is a discriminant of a real quadratic field. In fact, we have already treated the case that the class number is one and $\varDelta$ is odd (cf. [2], especially Cor. to Prop. 5).

4.2. We quote a lemma on Rankin's convolution. For the proof and some other details we can refer to [2], p. 91 and [1], Th. 3, the latter of which, however, contains an obvious mistake in its statement.

Let $f_j(z) = \sum_{n=1}^{\infty} a_j(n) \exp(2\pi inz)$ $(j = 1$ or $2)$ be a normalized primitive form in $\mathscr{S}_{\nu+1}$, so that the corresponding Dirichlet series $\phi_j(s) = \sum_{n=1}^{\infty} a_j(n)n^{-s}$ has the Euler product as follows:

$$\phi_j(s) = \prod_p \phi_{j,p}(s) \, ;$$

$$\phi_{j,p}(s)^{-1} = \begin{cases} (1 - \xi_j V)(1 - \eta_j V) & \text{if } p \neq D, \\ (1 - a_j(p)V) & \text{if } p = D, \end{cases}$$

where $V = p^{-s}$, and ξ_j, η_j are two roots of the equation $x^2 - a_j(p)x + \left(\dfrac{-D}{p}\right)p^\nu = 0$ for each rational prime p. The convolution of these is defined by

$$\psi(s; f_1, \bar{f}_2) = \prod_p \psi_p(s);$$

$$\psi_p(s)^{-1} = \begin{cases} (1 - \xi_1\bar{\xi}_2 V)(1 - \xi_1\bar{\eta}_2 V)(1 - \eta_1\bar{\xi}_2 V)(1 - \eta_1\bar{\eta}_2 V) & \text{if } p \neq D, \\ (1 - a_1(p)\overline{a_2(p)}V)(1 - \overline{a_1(p)}a_2(p)V) & \text{if } p = D. \end{cases}$$

LEMMA 15. $\psi(s; f_1, \bar{f}_2)$ *can be meromorphically continued to the whole complex s-plane and satisfies a functional equation. It is entire if $f_1 \neq f_2$ and it has a simple pole at $s = \nu + 1$ if $f_1 = f_2$.*

4.3. *Proof of Theorem 4.* Suppose that $f(z) = \sum_{n=1}^{\infty} a(n) \exp(2\pi i n z)$ is the Fourier expansion and put $f_\rho(z) = \sum_{n=1}^{\infty} \overline{a(n)} \exp(2\pi i n z)$. We have to prove that $f \neq f_\rho$ if f is a normalized primitive form in $\mathscr{S}^1_{\nu+1}$. Owing to Lemma 15, it is sufficient to prove that $\psi(s; f, \bar{f}_\rho)$ is entire. Let $F = I(f)$ denote the lifted image whose Fourier expansion is, we may assume, given by (16). We consider a Dirichlet series associated with F defined by

$$H(s) = \zeta_{-D}(2s - 2\nu) \sum_{\mathfrak{a}} C(\mathfrak{a})^2 N\mathfrak{a}^{-s},$$

where ζ_{-D} is the Dedekind zeta function of the imaginary quadratic field of discriminant $-D$. By comparing their Euler products, we can obtain

$$H(s) = \psi(s; f, \bar{f})\psi(s; f, \bar{f}_\rho).$$

This is the same as Proposition 6 in [2], though only the real quadratic field case is treated there, hence we omit the detail here. Therefore we have only to prove:

LEMMA 16. *If $f \in \mathscr{S}^1_{\nu+1}$, then $H(s)$ can be meromorphically continued to the whole complex s-plane and is holomorphic except a simple pole at $s = \nu + 1$.*

Proof. Observing that $F(g)J^{-2}\,{}^t\overline{F(g)}$ is left Λ- and right K-invariant where J denote the same matrix as in the proof of Lemma 4, let us

consider an integral

$$\Omega(s) = \int_{\Lambda \backslash G/K} F(g) J^{-2}\, {}^t\overline{F(g)} E^*(g, s - \nu) d\dot{g} \ ,$$

where the invariant volume $d\dot{g}$ is given by $v^{-3}dud\bar{u}dv$ for $g = g(u, v)\kappa$ ($\kappa \in K$). In the above we put $E^*(g, s) = D^s(2\pi)^{-2s}\Gamma(2s)\zeta_{-D}(2s)E(g, s)$ with $E(g, s) = \sum_{\gamma \in \Lambda_\infty \backslash \Lambda} v(\gamma g)^s$ (Re $s > 1$), where $v(g) = v$ for $g = g(u, v)\kappa$ and Λ_∞ is the subgroup consisting of all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Lambda$ with $c = 0$. As is well known, $E^*(g, s)$ can be meromorphically continued to the whole complex s-plane and is holomorphic except two simple poles at $s = 0, 1$. We first notice that $F(g)$ decreases rapidly when $v(g) \to \infty$ because $f \in \mathscr{S}^1_{\nu+1}$. By so-called Rankin's method we obtain, on one hand, that the integral $\Omega(s)$ is absolutely convergent for all $s \in C$ except two simple poles at $s = \nu, \nu + 1$, and on the other hand, that $\Omega(s)$ is a constant multiple of $D^{2s}(2\pi)^{-4s}\Gamma(s)^2\Gamma(s - \nu)^2 H(s)$ when Re s is sufficiently large. In these computations we need a formula

$$\int_0^\infty K_\alpha(v)^2 v^{2s-1} dv = 2^{2s-3}\Gamma(s + \alpha)\Gamma(s - \alpha)\Gamma(s)^2\Gamma(2s)^{-1}$$

for Re $s > 2|\alpha|$ (e.g. [5], p. 334 (45)), and an elementary identity

$$\sum_{\alpha=-\nu}^{\nu} \binom{2\nu}{\nu - \alpha}\Gamma(s + \alpha)\Gamma(s - \alpha)\Gamma(2s)^{-1} = \Gamma(s - \nu)^2\Gamma(2s - 2\nu)^{-1} \ .$$

Thus we complete the proof of Lemma 16, and so, that of Theorem 4.

REFERENCES

[1] T. Asai, On the Fourier coefficients of automorphic forms at various cusps and some applications to Rankin's convolution, J. Math. Soc. Japan **28** (1976), 48–61.

[2] ——, On certain Dirichlet series associated with Hilbert modular forms and Rankin's method, Math. Ann. **226** (1977), 81–94.

[3] K. Doi and H. Naganuma, On the functional equation of certain Dirichlet series, Invent. Math. **9** (1969), 1–14.

[4] A. Erdélyi et al., Higher transcendental functions, Vol. 2, McGraw-Hill, 1953.

[5] ——, Tables of integral transforms, Vol. 1, McGraw-Hill, 1954.

[6] H. Jacquet, Automorphic forms on $GL(2)$, II, Lec. Notes in Math. **278**, Springer, 1972.

[7] S. Kudla, Theta-functions and Hilbert modular forms, to appear.

[8] W. Magnus et al., Formulas and theorems for the special functions of mathematical physics, 3rd ed., Springer, 1966.

[9] H. Naganuma, On the coincidence of two Dirichlet series associated with cusp forms of Hecke's "Neben"-type and Hilbert modular forms over a real quadratic

field, J. Math. Soc. Japan **25** (1973), 547–555.

[10] S. Niwa, Modular forms of half integral weight and the integral of certain theta-functions, Nagoya Math. J. **56** (1974), 147–161.

[11] T. Oda, On modular forms associated with indefinite quadratic forms of signature $(2, n\text{-}2)$, to appear.

[12] G. Shimura, The special values of zeta functions associated with cusp forms, Comm. Pure appl. Math. **29** (1976), 783–804.

[13] T. Shintani, On construction of holomorphic cusp forms of half integral weight, Nagoya Math. J. **58** (1975), 83–126.

[14] A. Weil, Dirichlet series and automorphic forms, Lec. Notes in Math. **189**, Springer, 1971.

Department of Mathematics
Faculty of General Education
Ehime University