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ON A STRUCTURE DEFINED BY A TENSOR FIELD F OF TYPE (1, 1) SATISFYING $F^2=0$

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Professor Eliopoulous studied almost tangent structure on manifolds M_{2n} in [1], [2], [3]. An almost tangent structure F is a field of class C^∞ of linear operations on M_{2n} such that at each point x in M_{2n}, F_x maps the complexified tangent space T_x^c into itself and that F_x is of rank n everywhere and satisfies $F^2=0$. The present author in [4] studied a (1, 1) tensor F on a riemannian manifold M_{2n} which satisfies $F^2=0$ and is such that the rank of F is n everywhere. In this paper we study a differentiable manifold M_n with a (1, 1) tensor field F so that $F^2=0$ and that the rank of F is a constant r everywhere. A positive definite riemannian structure always exists on M_n. Such a riemannian structure is an $0(n)$-structure, thus the structural group of the tangent bundle TM_n is reduced to $0(n)$. We shall prove the following:

THEOREM. *A necessary and sufficient condition for M_n to admit a* (1, 1) *tensor field F with constant rank $r>0$ such that $F^2=0$ is that the structural group of tangent bundle of M_n be reduced to the group* $0(r)\underline{\underline{\times}}0(r)\times 0(n-2r)$, *where* $0(r)\times 0(r)$ *denotes the group of diagonal product of* $0(r)$ [5], *that is*

$$0(r)\underline{\underline{\times}}0(r):\begin{pmatrix}A & 0\\ 0 & A\end{pmatrix},\qquad A\in 0(r).$$

1. Let T_x be the tangent space of M_n at x; $F_xT_x=B_x$ and B_x' be the orthogonal distribution to B_x with respect to a chosen riemannian metric. Thus $T_x=B_x\oplus B_x'$. Since $F^2=0$ we have $F_xB_x=0$ and $B_x=F_xT_x=F_x(B_x\oplus B_x')=F_x(B_x')$. This shows that $\dim B_x'\geq \dim B_x$ and hence $r=\text{rank of } F\leq n/2$. There is a subdistribution D_x of B_x', $\dim D_x=n-2r$, such that $F_xD_x=0$. Let C_x be the subdistribution in B_x orthogonal to D_x. Then $B_x'=C_x\oplus D_x$ and $FC_x=B_x$. In a local coordinate neighbourhood at the point x one can write the operator F and distributions B, C and D by

$$\begin{aligned}
&F_i^j \quad (i,j=1,2,\ldots,n)\\
&B_a^j \quad (a,b=1,2,\ldots,r)\\
&C_{\bar a}^j \quad (\bar a=a+r)\\
&D_\alpha^j \quad (\alpha,\beta=2r+1,\ldots,n).
\end{aligned}$$

One can further assume that $F_j^iC_{\bar a}^j=B_a^i$.

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The matrix $(B_a^j, C_{\bar{a}}^j, D_\alpha^j)$ has an inverse. Let

$$(B_a^j, C_{\bar{a}}^j, D_\alpha^j)^{-1} = \begin{pmatrix} B^a \\ C_i^{\bar{a}} \\ D_i^\alpha \end{pmatrix}$$

and

$$B_{ji} = B_j^a B_i^a, \qquad C_{ji} = C_j^{\bar{a}} C_i^{\bar{a}}, \qquad D_{ji} = D_j^\alpha D_i^\alpha;$$
$$A_{ji} = B_{ji} + C_{ji} + D_{ji}.$$

The following statements are justified by calculations, almost identical to those in [4].

$(B_a^j, C_{\bar{a}}^j, D_\alpha^j)$ are orthogonal with respect to the metric A_{ji}. For any vector fields X, Y on M_n we define $\bar{A}$, $\bar{B}$, $\bar{g}$ as follows

$$\bar{B}(X, Y) = B_{ji} X^j Y^i, \qquad \bar{A}(X, Y) = A_{ji} X^j Y^i,$$
$$\bar{g}(X, Y) = \tfrac{1}{2}\{\bar{A}(X, Y) + \bar{A}(FX, FY) + \bar{B}(X, Y)\}.$$

Then B, C, D are orthogonal with respect to $\bar{g}$ and

$$\bar{g}(X, Y) = \bar{g}(FX, FY) \quad \text{for any } X, Y \in C.$$

Thus we have proved

LEMMA. *If in M_n there is a* $(1, 1)$ *tensor field F of constant rank $r>0$ which satisfies $F^2=0$ then $2r \le n$ and there exist complementary distributions B, C, and D of dimensions r, r and $n-2r$ and a positive definite riemannian metric $\bar{g}$ with respect to which B, C, and D are mutually orthogonal and such that* (i) $\bar{g}(X, Y) = \bar{g}(FX, FY)$ *for any $X, Y \in C$,* (ii) *F maps an orthonormal basis of C onto an orthonormal basis of B.*

2. **Proof of the theorem.** With respect to the orthonormal basis B_a, $C_{\bar{a}}$, $2^{-1/2}D^\alpha$ in the above lemma, the tensors $\bar{g}$ and F have the following components:

$$\bar{g} = \begin{pmatrix} E_r & 0 & 0 \\ 0 & E_r & 0 \\ 0 & 0 & E_{n-2r} \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 & 0 \\ E_r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.1}$$

Where E_r, E_{n-2r} denote the $r \times r$ and $(n-2r) \times (n-2r)$ unit matrices. We call such a frame $(B_a, C_{\bar{a}}, 2^{-1/2} D_\alpha)$ an adapted frame of the F structure. Now take another adapted frame $\{\bar{B}_a, \bar{C}_{\bar{a}}, 2^{-1/2}\bar{D}_\alpha\}$ to which the metric tensor $\bar{g}$ and the tensor F have the same components as (2, 1). Put

$$\bar{e}_i = \gamma_i^j e_j, \qquad \gamma_i^j \in 0(2n) \qquad \left(e_a = B_a,\ e_{\bar{a}} = C_{\bar{a}},\ e_\alpha = \frac{1}{\sqrt{2}} D_\alpha\right).$$

Then we can easily find that γ has the form

$$\gamma = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \qquad \alpha \in 0(r) \qquad \beta \in 0(n-2r).$$

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Thus the group of the tangent bundle of M_n can be reduced to $0(r)\underline{\underline{\times}}0(r)\times 0(n-2r)$.

Conversely if the group of the tangent bundle of the manifold can be reduced to $0(r)\underline{\underline{\times}}0(r)\times 0(n-2r)$ then we can define a positive definite riemannian metric $\bar{g}$ and a tensor field F of type (1, 1) with constant rank r having (2, 1) as components with respect to the adapted frames. Then we have that $F^2=0$ and that the rank of F is r. This completes the proof of the theorem.

References

1. H. A. Eliopoulos, *Structures presque tangentes sur les variétés différentiables*, C. R. Acad. Sc. Paris, **255** (1962), 1563–1565.

2. ——, *Euclidean structures compatible with almost tangent structures*, Acad. Roy. Belg. Bull. Cl. Sci. (5) **50** (1964), 1174–1182.

3. ——, *On the general theory of differentiable manifolds with almost tangent structure*, Canad. Math. Bull. **8** (1965), 721–748.

4. C. S. Houh, *On a Riemannian manifold M_{2n} with an almost tangent structure*, Canad. Math. Bull. **12** (1969), 759–769.

5. H. Wakakuwa and S. Hashimoto, *Remark on almost tangent structure*. Tensor, **20** (1969), 270–272.

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