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TWO THEOREMS ON GENERALISED METRIC SPACES

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We prove that any compact space, and even any countably compact space having the weak topology with respect to a sequence of symmetrisable subspaces, is metrisable. This generalises results of Arhangel'skii and Nedev on metrisability of symmetrisable compact spaces. Also we define and study contraction functions on generalised metric spaces whose topology can be described in terms of a 'distance function' which is not quite a metric. In particular we present necessary and sufficient conditions for a space of countable pseudo-character to be submetrisable in terms of real-valued contraction functions on this space.

1. INTRODUCTION

This article consists of two sections. The purpose of the first section is to generalise the well-known result on metrisability of a symmetrisable compact space. In the second section we study submetrisable spaces and obtain necessary and sufficient conditions for a space of countable pseudo-character to be submetrisable in terms of real-valued *contraction functions on this space.*

We start with the necessary definitions and examples.

1.1 DEFINITION: A space X has the *weak topology with respect to a sequence of subspaces* $\{X_n : n \in \mathbb{N}\}$ (or the topology of X *is determined by this sequence of subspaces*) if a set U is open in X whenever $U \cap X_n$ is open in X_n for all $n \in \mathbb{N}$.

Consider the disjoint sum of countably many copies of a convergent sequence taken with its limit point and shrink all limit points to one point. Such a quotient space is known as a Frechet fan [**5**]. The Frechet fan is an example of a non-metrisable space having the weak topology with respect to a sequence of metrisable subspaces. On the other hand, the following corollary to a more general theorem of Filippov holds:

1.2 COROLLARY. [**6, 8**] *Every compact space having the weak topology with respect to a sequence of metrisable subspaces is metrisable.*

To see this, suppose that X and $\{X_n : n \in \mathbb{N}\}$ are as in Definition 1.1, X is compact and X_n is metrisable for each $n \in \mathbb{N}$. Then the canonical mapping $\oplus\{X_n : n \in \mathbb{N}\} \to X$ sending a point to 'itself' is a quotient s-map (in fact, it is a countable-to-one quotient map). Thus X is metrisable [**6**, Theorem 2.2].

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Now we turn to another metrisation theorem for compact spaces.

1.3 Definition: A function $d : X \times X \to \mathbb{R}^+$ is called a symmetric on X if, for each $x, y \in X$,

(a) $d(x,y) = d(y,x)$,
(b) $d(x,y) = 0$ if and only if $x = y$.

Since the triangle inequality is not present in the definition of a symmetric d, the usual ε-balls

$$B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$$

might not form a base for a topology on X. However, the following definition gives another way to relate the ε-balls to a topology.

1.4 Definition: A space X is said to be symmetrisable if there exists a symmetric d on X such that $U \subseteq X$ is open if and only if whenever $x \in U$ there is $\varepsilon > 0$ such that $B_d(x,\varepsilon) \subseteq U$.

Not every symmetrisable space is metrisable [1]. However, as Arhangelskii's theorem shows, the situation is different in the presence of compactness.

1.5 Theorem. [1] *A symmetrisable compact space is metrisable.*

The main result of the second section is the following generalisation of the previous theorem:

1.6 Theorem. *If X is a compact space having the weak topology with respect to a sequence of symmetrisable subspaces, then X is metrisable.*

What makes our result nontrivial is the fact that there are non-symmetrisable spaces with the weak topology with respect to a sequence of symmetrisable (even metrisable) subspaces for example, the Frechet fan [5].

The third section of this article deals with submetrisable spaces. A space X with the topology $\mathcal{T}$ is said to be *submetrisable* if there is some metric topology $\mathcal{T}^*$ on X such that $\mathcal{T}^* \subseteq \mathcal{T}$. If X is a submetrisable space then we simply say that X has a coarser metric topology. It follows easily that any submetrisable space X has a G_δ-*diagonal* (that is, the diagonal of X is a G_δ-set in X^2). This is because X^2 also has a coarser metric topology, hence the diagonal (as a closed subset) is a G_δ-set. It is readily seen that any space having a G_δ-diagonal has a *countable pseudo-character* (that is, each point of this space is a G_δ-set).

Martin [9] introduced and investigated spaces of real-valued *contraction* functions defined on symmetrisable spaces. In particular, he used spaces of functions to find conditions for an arbitrary symmetrisable space to be submetrisable [9, Theorem 3.1]. However, this result can not be viewed as a characterisation theorem for submetrisable spaces for it is restricted to the class of symmetrisable spaces only.

The purpose of the second part of this note is to exploit Martin's construction by defining the space of contraction functions $L_d(X)$ on a space of countable pseudo-character X and to prove the following result

1.7 Theorem. *The following conditions are equivalent for a space X:*

(i) *X is submetrisable.*

(ii) *$L_d(X)$ separates points of X for some separately continuous symmetric d.*

(iii) *$L_d(X)$ separates points of X for some continuous symmetric d.*

Since every submetrisable space has a countable pseudo-character, we therefore obtain conditions by which submetrisable spaces can be characterised. Note that such conditions in terms of sequences of covers for a space with a G_δ-diagonal are well-known and can be found in [**1**, **4**] and [**7**].

In order to get our main results, the concepts of a *ψ-metric* and a *ψ-metrisable* space are introduced. These concepts enable us to characterise spaces of countable pseudo-character in terms of a "distance function" which is not quite a metric.

Throughout the paper all spaces are assumed to be Hausdorff; 'metric space' means 'metrisable space' (a standard abuse of terminology). For a space X of countable pseudo-character we use the notation $\psi(X) \leqslant \omega$. The space of all real-valued functions on a space X with the Tychonoff product topology is denoted by $\mathbb{R}^X$. The symbols $\mathbb{R}^+$ and $\mathbb{N}$ denote the natural numbers and the set of positive real numbers. All other terminology and notations are contained in [**7**] and [**5**].

2. A Metrisation Theorem for Compact Spaces

The main result of this section is the following

2.1 Theorem. *If X is a compact space having the weak topology with respect to a sequence of symmetrisable subspaces, then X is metrisable.*

To prove Theorem 2.1 we use the technique of the intersection of topologies and some ideas developed in [**1**] and [**10**].

2.2 Definition: Let topologies $\mathfrak{T}$ and $\{\mathfrak{T}_s : s \in S\}$ are defined on a set X. Then $\mathfrak{T}$ is the intersection of topologies $\{\mathfrak{T}_s : s \in S\}$ if a set U is open in $(X, \mathfrak{T})$ if and only if U is open in $(X, \mathfrak{T}_s)$ for each $s \in S$.

The following lemma is straightforward.

2.3 Lemma. *If X is a compact space having the weak topology with respect to a sequence of symmetrisable subspaces, then the topology of X is the intersection of a sequence of symmetrisable topologies.*

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2.4 Lemma. *If the topology $\mathfrak{T}$ of a compact space X is the intersection of a sequence of symmetrisable topologies, then every closed subset of X is a G_δ-set.*

Proof: Let $\mathfrak{T}$ be the intersection of symmetrisable topologies $\{\mathfrak{T}_n : n \in \mathbb{N}\}$ on X and let d_k be a symmetric with respect to the topology $\mathfrak{T}_k$. Consider any closed $F \subseteq X$ and an open cover $\{V_s : s \in S\}$ of $X^* = X \setminus F$ with the relative topology induced by $\mathfrak{T}$. Clearly we can assume that the set S is well ordered and for each $s \in S$, we have $[V_s]_{\mathfrak{T}} \subseteq X^*$.

For each $n, k \in \mathbb{N}$ and $s \in S$, let us define

$$U_s = V_s \Big\backslash \bigcup \{V_t : t < s,\ t \in S\}$$

and

$$M^n_{s,k} = \{x \in X^* : x \in U_s,\ d_k(x, X \setminus V_s) > 1/n\}.$$

It is readily seen that the collection $\{M^n_{s,k} : s \in S\}$ is disjoint, $M^n_{s,k} \subseteq M^{n+1}_{s,k}$ and $\{M^n_{s,k} : s \in S,\ n \in \mathbb{N}\}$ forms a cover of X^* for every $n, k \in \mathbb{N}$ and $s \in S$.

Suppose that for some $m \in \mathbb{N}$ the set of non-empty elements of $\{M^m_{s,1} : s \in S\}$ is uncountable. Choose any point $x_s \in M^m_{s,1}$ for every $s \in S$. For every $\varepsilon > 0$, let

$$O_{\varepsilon,1}F = \{x \in X : d_1(x, F) \leqslant \varepsilon\}.$$

It is clear that $F = \bigcap \{O_{1/n,1}F : n \in \mathbb{N}\}$ and some $S_1 \subseteq S$ can be chosen in such a way that the set $\{x_s : s \in S_1\}$ is uncountable and $d_1(\{x_s : s \in S_1\}, F) > \varepsilon$. We shall show that $\{x_s : s \in S_1\}$ is closed and discrete in $(X, \mathcal{T}_1)$. Suppose there is a set $\{x_t : t \in T_1\}$ which is not closed in $(X, \mathcal{T}_1)$ and

$$\{x_t : t \in T_1\} \subseteq \{x_s : s \in S_1\}.$$

Then there exists $x \in X$ such that $d_1(x, \{x_t : t \in T_1\}) = 0$. Since $d_1(\{x_t : t \in T_1\}, F) > \varepsilon$, it easily follows that $x \in X^*$. First assume

$$x \in \bigcup \{M^m_{s,1} : s \in S_1\} \subseteq X^*$$

and let

$$s' = \min\{s \in S_1 : x \in M^m_{s,1}\}.$$

Then $d_1(x_t, x) > 1/m$ for any $t \in T_1$, $t \neq s'$ and so $d_1(x, \{x_t : t \in T_1\}) > 0$. Therefore the only option is

$$x \notin \bigcup \{M^m_{s,1} : s \in S_1\} \subseteq X^*.$$

Thus $d_1(x_s, x) > 1/m$ for each $s \in S_1$ and $d_1(x, \{x_t : t \in T_1\}) > 0$. This contradiction shows that for each $T_1 \subseteq S_1$, the set $\{x_t : t \in T_1\}$ is closed in $(X, \mathcal{T}_1)$. Thus the set $\{x_s : s \in S_1\}$ is closed and discrete in $(X, \mathcal{T}_1)$.

Further, consider $\{M^n_{s,2} : s \in S_1,\ n \in \mathbb{N}\}$ which is a cover of X. For each $s \in S_1$, the point x_s is contained in some set $M^{n(s)}_{s,2}$. Since $x_s \in U_s$ for every $s \in S$ and the collection of sets $\{U_s : s \in S\}$ is disjoint, the map $x_s \to M^{n(s)}_{s,2}$ is one to one. As $\{x_s : s \in S_1\}$ is uncountable, there are a subset $\{x_s : s \in S_2\} \subseteq \{x_s : s \in S_1\}$ and $l \in \mathbb{N}$ such that $x_s \in M^l_{s,2}$ for all $s \in S_2$. For each $\varepsilon > 0$, let

$$O_{\varepsilon,2}F = \{x \in X : d_2(x, F) \leqslant \varepsilon\}.$$

Since $F = \bigcap\{O_{1/n,2}F : n \in \mathbb{N}\}$, there is $\delta > 0$ such that $d_2(\{x_s : s \in S_2\}, F) > \delta$. Now similar arguments show that the set $\{x_s : s \in S_2\}$ is discrete and closed in $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$.

We inductively define sequences

$$\{S_i \subseteq S : S_{i+1} \subseteq S_i,\ i \in \mathbb{N}\}$$

such that

(1) $\{x_s : s \in S_i\}$ is uncountable for all $i \in \mathbb{N}$;

(2) $\{x_s : s \in S_i\}$ is discrete and closed in $(X, \mathcal{T}_n)$ for all $n \leqslant i$ and $i \in \mathbb{N}$.

Finally, for each $i \in \mathbb{N}$, pick a point

$$a_i \in \{x_s : s \in S_i\} \subseteq X^*.$$

If the set $\{a_i : i \in \mathbb{N}\}$ is not closed in $(X, \mathcal{T})$, then there is $k \in \mathbb{N}$ such that $\{a_i : i \in \mathbb{N}\}$ is not closed in $(X, \mathcal{T}_k)$, which is impossible due to property (2) above. So $\{a_i : i \in \mathbb{N}\}$ is discrete and closed in $(X, \mathcal{T})$. But $(X, \mathcal{T})$ is a compact space, a contradiction.

Hence, for each $m \in \mathbb{N}$, the set of non-empty elements of $\{M^m_{s,1} : s \in S\}$ is countable. So $\{M^n_{s,k} : s \in S, n \in \mathbb{N}\}$ is a countable cover of X^* for all $k \in \mathbb{N}$ and any open cover of X^* has a countable subcover. It follows that F is G_δ-set. $\square$

The following lemma is of some independent interest.

2.5 Lemma. *If X is a compact space having the topology of intersection of a sequence of symmetrisable topologies, then the product topology of the space $X \times X$ is also the intersection of a sequence of symmetrisable topologies*

Proof: Let $\{\mathfrak{T}_n : n \in \mathbb{N}\}$ be a sequence of symmetrisable topologies on $(X, \mathfrak{T})$ and $\mathfrak{T} = \bigcap\{\mathfrak{T}_n : n \in \mathbb{N}\}$. Suppose that, for each $n \in \mathbb{N}$, the topology $\mathfrak{T}_n$ is generated by a symmetric d_n. For each $m, k \in \mathbb{N}$, we define a function d_{mk} on $X \times X$ by

$$d_{mk}((x_1, x_1'), (x_2, x_2')) = d_m(x_1, x_2) + d_k(x_1', x_2'),$$

where $(x_1, x_1'), (x_2, x_2') \in X \times X$.

For every $m, k \in \mathbb{N}$, let $\mathcal{T}_{mk}$ be the following topology on $X \times X$: a set F is closed in $(X \times X, \mathcal{T}_{mk})$ if and only if $d_{mk}(x, F) = 0$ for every $x \in (X \times X) \setminus F$. It is easy to show that, for all $m, k \in \mathbb{N}$, the space $(X \times X, \mathcal{T}_{mk})$ is symmetrisable with respect to the symmetric d_{mk}.

To show that the topology $\mathcal{T} \times \mathcal{T}$ is the intersection of topologies $\{\mathcal{T}_{mk} : m, k \in \mathbb{N}\}$, we first note that $(X \times X, \mathcal{T} \times \mathcal{T})$ is sequential [**8**, Example 10.2]. Next, let $A \subseteq X \times X$ and $[A]_{\mathcal{T} \times \mathcal{T}} \neq A$. As $(X \times X, \mathcal{T} \times \mathcal{T})$ is sequential, there is a point $(x_0, x_0') \in [A]_{\mathcal{T} \times \mathcal{T}} \neq A$ and a sequence $\{(x_i, x_i') \in A : i \in \mathbb{N}\}$ converging to (x_0, x_0'). Therefore, for each $m, k \in \mathbb{N}$, the sequences $\{x_i : i \in \mathbb{N}\}$ and $\{x_i' : i \in \mathbb{N}\}$ converge to the points x_0 and x_0' in the topologies $\mathcal{T}_m$ and $\mathcal{T}_k$ respectively. Finally, by the definition of the function d_{mk}, it follows that the sequence $\{(x_i, x_i') : i \in \mathbb{N}\}$ converges to (x_0, x_0') in the topology $\mathcal{T}_{mk}$. $\square$

Proof of the Theorem: According to the Lemma 2.4 and Lemma 2.5, the compact space $X \times X$ has a G_δ-diagonal (as the diagonal is a closed subset of $X \times X$) and thus is metrisable [**12**]. $\square$

As a matter of fact, a much more general result holds. Recall that a space X is said to be countably compact if any countable cover of open sets of X contains a finite subcover.

2.6 Corollary. *If X is a countably compact space having the weak topology with respect to a sequence of symmetrisable subspaces, then X is metrisable.*

Proof: Let $F = \emptyset$ in the proof of the Lemma 2.4. Then for any open cover $\{V_s : s \in S\}$ it is possible to find a countable collection $\{M_{s,1}^n : s \in S, n \in \mathbb{N}\}$ such that

(1) $\{M_{s,1}^n : s \in S, n \in \mathbb{N}\}$ is a cover of X;
(2) $M_{s,1}^n \subseteq V_s$ for all $s \in S$.

Hence any open cover of X contains a countable subcover and so X is compact. An application of the theorem gives the result. $\square$

2.7 Remark. As we have mentioned already, the canonical mapping $\oplus\{X_n : n \in \mathbb{N}\} \to X$ is countable-to-one and quotient. It is thus tempting to suggest that every compact space which is a quotient countable-to-one image of a symmetrisable space is metrisable. However, this is false in general due to a result of Reed [**11**]. He showed that any first countable space is an open countable-to-one image of a Moore (and so symmetrisable) space. Therefore, every non-metrisable first countable compact space is a quotient countable-to-one image of a symmetrisable space and our result can't be easily generalised in this direction unless some other specific properties of the canonical mapping are used.

3. Contraction Functions on Submetrisable Spaces

Recall that if $d : X \times X \to \mathbb{R}^+$ is a function, then $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ is the ball of radius ε centred at x, with respect to the function d.

3.1 Definition: A function $d : X \times X \to \mathbb{R}^+$ is called a *ψ-metric* on the set X if, for each $x, y \in X$, $d(x,y) = 0$ if and only if $x = y$.

A topological space X is said to be *ψ-metrisable* if there is a *ψ-metric* d on X such that, for each $x \in X$ and $\varepsilon > 0$, the ball $B_d(x,\varepsilon)$ is open.

3.2 Lemma. *The following conditions are equivalent for any space X:*

(i) $\psi(X) \leqslant \omega$.

(ii) *X is ψ-metrisable.*

(iii) *There is a symmetric function (definition 1.3) $d : X \times X \to \mathbb{R}^+$ such that $x \in \operatorname{Int} B_d(x,\varepsilon)$ for all $\varepsilon > 0$ and $x \in X$.*

Proof: (iii)$\Rightarrow$ (i). Clearly $\{x\} = \bigcap_n U_n(x)$, where $U_n(x) = \operatorname{Int} B_d(x, 1/n)$.

(i)$\Rightarrow$ (ii). Suppose that $\psi(X) = \omega$. Then for each $x \in X$ we can choose a sequence $(U_n(x))$ of open sets in X such that $U_{n+1}(x) \subseteq U_n(x)$ for all $n \in \omega$ and $\{x\} = \bigcap_n U_n(x)$. Define a function $d : X \times X \to \mathbb{R}^+$ by

$$d(x,y) = 1/m \text{ if } m = \min\{n : y \notin U_n(x)\} \quad (\text{or } m = \min\{n : x \notin U_n(y)\}).$$

It is easy to show that the function d is as required.

(ii)$\Rightarrow$ (iii). Let $d : X \times X \to \mathbb{R}^+$ be a function as in part (ii) of the lemma. Define:

$$d_{sym}(x,y) = \min\{d(x,y), d(y,x)\}.$$

Evidently, d_{sym} is a symmetric on X. Moreover, for each $x \in X$ and $\varepsilon > 0$, it follows that

$$x \in B_d(x,\varepsilon) \subseteq B_{d_{sym}}(x,\varepsilon).$$

Since $B_d(x,\varepsilon)$ is open in X, it follows that $x \in \operatorname{Int} B_{d_{sym}}(x,\varepsilon)$. $\square$

3.3 Definition: Let $d : X \times X \to \mathbb{R}^+$ be a function. Then the *space of contraction real-valued functions on X with respect to d* is the following subspace $L_d(X)$ of $\mathbb{R}^X$:

$$L_d(X) = \{f \in \mathbb{R}^X : |f(x) - f(y)| \leqslant d(x,y) \text{ for all } x, y \in X\}.$$

For any ψ-function $d : X \times X \to \mathbb{R}^+$, one can easily show that

(a) $L_d(X)$ is a subspace of the space $C_p(X)$ [3] of all real-valued continuous functions on X;

(b) $L_d(X)$ is closed in $\mathbb{R}^X$;

(c) $L_d(X)$ is a perfect and nowhere dense subset of $\mathbb{R}^X$.

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Even when X is metrisable by a metric d not much is known about topological properties of the space $L_d(X)$. Since the main properties of the spaces $C_p(X)$ have been already investigated, a comparative study of the spaces $L_d(X)$ and $C_p(X)$ would be the first step in this direction.

In order to state the main results of this section recall that a subspace $E \subseteq \mathbb{R}^X$ separates points of X if, for each $x, y \in X$, there is an $f \in E$ such that $f(x) \neq f(y)$. As usual, if d and d^* are any functions from $X \times X$ into $\mathbb{R}^+$, then we write $d \leqslant d^*$ whenever $d(x,y) \leqslant d^*(x,y)$ for all $x, y \in X$.

3.4 Theorem. *The following conditions are equivalent for a space X:*

(i) *X is submetrisable,*

(ii) *X is ψ-metrisable (by a ψ-metric d) and $L_d(X)$ separates points of X.*

(iii) *$L_d(X)$ separates points of X for some separately continuous ψ-metric d.*

(iv) *$L_d(X)$ separates points of X for some continuous ψ-metric d.*

Proof: (i)$\Rightarrow$ (iv). Let $\rho : X \times X \to \mathbb{R}^+$ be the metric for a coarser metric topology on X. As ρ is continuous we need only prove that the space $L_\rho(X)$ separates points of X. Consider any points $a, b \in X$ such that $a \neq b$. Define $f(x) = \rho(x, a)$. Then

$$|f(x) - f(y)| = |\rho(x,a) - \rho(y,a)| \leqslant \rho(x,y),$$

that is, $f \in L_\rho(X)$. Further, $f(a) = \rho(a,a) = 0$ and $f(b) = \rho(b,a) \neq 0$.

(iv)$\Rightarrow$ (iii). Trivial.

(iii)$\Rightarrow$ (ii). Follows from the Definition 3.1.

(ii)$\Rightarrow$ (i). Let X be ψ-metrisable by a ψ-metric d and suppose that for every $x, y \in X$ there is $f \in L_d(X)$ such that $f(x) \neq f(y)$. Define $\rho : X \times X \to \mathbb{R}^+$ as follows:

$$\rho(x,y) = \sup\{|g(x) - g(y)| : g \in L_d(X)\}.$$

It is readily seen that ρ is a metric on X. It remains to show that ρ is continuous (that is ρ generates a coarser metric topology). It follows from the definition of the space $L_d(X)$ that $\rho \leqslant d$. Thus for each $\varepsilon > 0$ and $x \in X$, we have

$$x \in B_d(x,\varepsilon) \subseteq B_\rho(x,\varepsilon).$$

Since $B_d(x,\varepsilon)$ is open, it follows that $x \in \operatorname{Int} B_\rho(x,\varepsilon)$ for each $\varepsilon > 0$ and $x \in X$. Finally, as ρ is a metric, it is possible, for every $y \in B_\rho(x,\varepsilon)$, to find some $\delta > 0$ such that

$$y \in \operatorname{Int} B_\rho(y,\delta) \subseteq B_\rho(x,\varepsilon).$$

This shows that every ε-ball $B_\rho(x,\varepsilon)$ is open and so X is submetrisable. □

3.5 Remark. It has been shown in Lemma 3.2 that $\psi(X) \leqslant \omega$ if and only if X is ψ-metrisable. Therefore part (ii) of the theorem gives a necessary and sufficient condition for a space of countable pseudo-character to be submetrisable.

Using Lemma 3.2 it is not difficult to modify the proof of the previous theorem to get the following

3.6 Theorem. *The following conditions are equivalent for a space X:*

(i) *X is submetrisable.*

(ii) *$L_d(X)$ separates points of X for some separately continuous symmetric d.*

(iii) *$L_d(X)$ separates points of X for some continuous symmetric d.*

It is known that a submetrisable pseudocompact space is metrisable [2]. Unfortunately, there appears to be no reference to the proof of this result available and so a sketch of the proof is given here.

Let a space $(X, \mathfrak{T})$ be pseudocompact and submetrisable that is, there exists a coarser metric topology $\mathfrak{T}^*$ on X. It follows that the space $(X, \mathfrak{T}^*)$ is a metrisable compact [5, Theorems 3.10.21 and 5.1.20]. If we prove that the space $(X, \mathfrak{T})$ is countably compact, then the rest will follow from [7]. Suppose that $(X, \mathfrak{T})$ is not countably compact. Then there is an infinite sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ which has no limit points in X. Since the space $(X, \mathfrak{T}^*)$ is metric and compact, one can choose a subsequence $\{x_{n(k)}\}_{k=1}^{\infty}$ converging in the topology $\mathfrak{T}^*$. Now, for each $x_{n(k)}$, it is possible to pick an open (in $\mathfrak{T}^*$ and so in $\mathfrak{T}$) set $U(x_{n(k)})$ such that the collection $\{U(x_{n(k)})\}_{k=1}^{\infty}$ is discrete in the subspace $X \setminus \{x\}$ with the topology induced from $\mathfrak{T}^*$. Finally, for each $x_{n(k)}$, there obviously exists a subset $V(x_{n(k)})$ of $U(x_{n(k)})$ such that the collection $\{V(x_{n(k)})\}_{k=1}^{\infty}$ is discrete in $(X, \mathfrak{T})$. This contradicts [5, Theorem 3.10.22].

3.7 Corollary. *A pseudocompact space X is metrisable if and only if there is a function $d : X \times X \to \mathbb{R}^+$ such that (a) d is separately continuous; (b) $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$; (c) $L_d(X)$ separates points of X.*

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