Canad. Math. Bull. Vol. **19** (4), 1976

SOME ARITHMETIC PROPERTIES OF A SPECIAL SEQUENCE OF INTEGERS

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1. Leeming [3] has defined a sequence of polynomials $\{Q_{4n}(x)\}$ and a sequence of integers $\{Q_{4n}\}$ by means of

$$\frac{\cosh xt+\cos xy}{\cosh t+\cos t}=\sum_{n=0}^{\infty} Q_{4n}(x)\frac{t^{4n}}{(4n)!} \tag{1.1}$$

and

$$Q_{4n}=Q_{4n}(0). \tag{1.2}$$

Thus

$$\frac{2}{\cosh t+\cos t}=\sum_{n=0}^{\infty} Q_{4n}\frac{t^{2n}}{(4n)!} \tag{1.3}$$

and equivalently

$$\sum_{k=0}^{n}\binom{4n}{4k}Q_{4k}=0 \qquad (n>0). \tag{1.4}$$

Moreover

$$(-4)^n Q_{4n}=\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}E_{2k}E_{4n-2k}, \tag{1.5}$$

where the E_{2k} are the Euler numbers defined by [5, Ch. 2]

$$\frac{1}{\cosh t}=\sum_{n=0}^{\infty} E_{2n}\frac{t^{2n}}{(2n)!}. \tag{1.6}$$

Leeming shows also that the Q_{4n} are all odd and that

$$(-1)^n Q_{4n}>0 \qquad (n=0,1,2,\ldots). \tag{1.7}$$

It is well known (see for example [4, Ch. 14]) that the Euler numbers satisfy Kummer's congruence:

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}E_{2n+s(p-1)}\equiv 0 \pmod{p^r} \qquad (2n\geq r), \tag{1.8}$$

Received by the editors June 10, 1976.
Supported in part by NSF grant GP-37924X.

https://doi.org/10.4153/CMB-1976-063-4 Published online by Cambridge University Press

where p is an arbitrary odd prime. For $p=2$, Frobenius [2, p. 477] stated the following result: If $a \geq 0$, $b>0$, $r>0$, then the power of 2 dividing the number

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} E_{2a+2sb}$$

is the same as the power of 2 dividing the number $(2b)^r r!$. For proof see [1]. Frobenius [2, p. 477] proved that

$$E_{2n} \equiv 1-2n+8\binom{n}{2} \pmod{16} \tag{1.9}$$

as well as more precise results.

In the present note we shall prove first the following result corresponding to (1.8). Let p be an odd prime and let $m \geq 0$. Then

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} Q_{4n+sm} \equiv 0 \pmod{p^r} \qquad (4n \geq r), \tag{1.10}$$

where m is divisible by $p-1$ or p^2-1 according as $p \equiv 1$ or $3 \pmod 4$. In particular

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} Q_{4n+s(p-1)} \equiv 0 \pmod{p^r} \qquad (4n \geq r,\ p \equiv 1\ (mod\ 4)), \tag{1.11}$$

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} Q_{4n+s(p^2-1)} \equiv 0 \pmod{p^r} \qquad (4n \geq r,\ p \equiv 3\ (mod\ 4)). \tag{1.12}$$

Also, corresponding to (1.9), we show that

$$Q_{4n} \equiv 1-2n+8\binom{n}{2} \qquad \pmod{16}; \tag{1.13}$$

in particular

$$Q_{4n} \equiv (-1)^n \pmod 4, \qquad Q_{4n} \equiv 1-2n \pmod 8. \tag{1.14}$$

Note that, by (1.9) and (1.13),

$$Q_{4n} \equiv E_{2n} \pmod{16}. \tag{1.15}$$

Thus it would be of interest to know whether $Q_{4n}-E_{2n}$ is divisible by some high power of 2.

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2. **Proof of (1.10).** We have

$$\frac{2}{\cosh t+\cos t}=\frac{4}{e^t+e^{-t}+e^{it}+e^{-it}}$$

$$=\frac{4}{4-(1-e^t)-(1-e^{-t})-(1-e^{it})-(1-e^{-it})}$$

$$(2.1)\qquad =\sum_{a,b,c,d=0}^{\infty} 4^{-a-b-c-d}(a, b, c, d)(1-e^t)^a(1-e^{-t})^b \cdot (1-e^{it})^c(1-e^{-it})^d,$$

where

$$(a, b, c, d)=\frac{(a+b+c+d)!}{a!\, b!\, c!\, d!}.$$

Expanding (2.1) we get

$$\sum_{a,b,c,d=0}^{\infty} 4^{-a-b-c-d}(a, b, c, d) \sum_{a',b',c',d'} (-1)^{a'+b'+c'+d'}\binom{a}{a'}\binom{b}{b'}\binom{c}{c'}\binom{d}{d'} \cdot e^{(a'-b'+(c'-d')i)t}.$$

Hence, by (1.3),

$$Q_n = \sum_{\substack{a,b,c,d\\ a',b',c',d'}} (-1)^{a'+b'+c'+d'}(a, b, c, d)\binom{a}{a'}\binom{b}{b'}\binom{c}{c'}\binom{d}{d'}$$

$$(2.2)\qquad \cdot (a'-b'+(c'-d')i)^n,$$

where $Q_n=0$ if n is not divisible by 4. It is clear, by finite differences, that we may assume in (2.2) that $a+b+c+d\le n$.

It follows from (2.2) that, for arbitrary $m>0$,

$$\sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s}Q_{n+sm} = \sum_{\substack{a,b,c,d\\ a',b',c',d}} (-1)^{a'+b'+c'+d'}(a, b, c, d)\binom{a}{a'}\binom{b}{b'}\binom{c}{c'}\binom{d}{d'}$$

$$\cdot (a'-b'+(c'-d')i)^n\{(a'-b'+(c'-d-)i)^m-1\}^r.$$

Now if $a+bi$ is an arbitrary Gaussian integer and p is a prime, $p\equiv 1 \pmod 4$, then

$$(a+bi)^p\equiv a+bi \pmod p;$$

however if $p\equiv 3 \pmod 4$, then

$$(a+bi)^{p^2}\equiv a+bi \pmod p.$$

Therefore we have

$$(2.3)\qquad \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s}Q_{n+sm}\equiv 0 \bmod p^r) \qquad (n\ge r),$$

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where $p-1 \mid m$ if $p \equiv 1 \pmod 4$ while $p^2-1 \mid m$ if $p \equiv 3$. This result evidently includes (1.10).

3. **Proof of (1.13).** We have

$$1-2k+8\binom{k}{2}=1-6k+4k^2=1-5k+k(4k-1).$$

Put

$$S_n=\sum_{k=0}^{n}\binom{4n}{4k}(1-5k+k(4k-1)). \tag{3.1}$$

Then

$$S_n=S_n'-5S_n''+S''', \tag{3.2}$$

where

$$S_n'=\sum_{k=0}^{n}\binom{4n}{4k},$$

$$S_n''=\sum_{k=0}^{n}k\binom{4n}{4k}=n\sum_{k=1}^{n}\binom{4n-1}{4k-1}=n\sum_{k=0}^{n-1}\binom{4n-1}{4k},$$

$$S_n'''=\sum_{k=0}^{n}k(4k-1)\binom{4n}{4k}=n(4n-1)\sum_{k=1}^{n}\binom{4n-2}{4k-2}=n(4n-1)\sum_{k=0}^{n-1}\binom{4n-2}{4k}.$$

Since, for $m>0$,

$$\sum_{4k\le m}\binom{m}{4k}=\tfrac{1}{4}\{(1+1)^m+(1-1)^m+(1+i)^m+(1-i)^m\}$$
$$=\tfrac{1}{4}\{2^m+(1+i)^m+(1-i)^m\},$$

we get

$$S_n'=\tfrac{1}{4}\{2^{4n}+(-1)^n2^{2n+1}\},$$
$$S_n''=\tfrac{1}{4}n\{2^{4n-1}+(-1)^n2^{2n}\},$$
$$S'''=\tfrac{1}{4}n(4n-1)\cdot 2^{4n-2}.$$

Hence, by (3.2),

$$S_n=2^{4n-2}-5n\cdot 2^{4n-3}+n(4n-1)\cdot 2^{4n-4}+(-1)^n2^{2n-1}-5n(-1)^n2^{2n-2}. \tag{3.3}$$

It follows from (3.3) that

$$S_n=0 \qquad (n=1,2,3) \tag{3.4}$$

and

$$S_n\equiv 0 \pmod{16} \qquad (n\ge 4), \tag{3.5}$$

so that

$$S_n \equiv 0 \pmod{16} \qquad (n = 1, 2, 3, \ldots). \tag{3.6}$$

Therefore by (3.1) and (1.4), we have

$$Q_{4n} \equiv 1 - 2n + 8\binom{n}{2} \qquad \pmod{16}. \tag{3.7}$$

According to the numerical data in [3], the Q_{4n} have the following residues (mod 16):

n	0	1	2	3	4	5	6
Q_{4n}	1	−1	5	3	9	7	−3

These results are evidently in agreement with (3.7).

References

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https://doi.org/10.4153/CMB-1976-063-4 Published online by Cambridge University Press