TENSOR PRODUCTS OF BANACH ALGEBRAS

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1. Introduction. This paper is concerned with a generalization of some recent theorems of Hausner **(1)** and Johnson **(4; 5)**. Their result can be summarized as follows: *Let G be a locally compact abelian group, A a commutative Banach algebra, $B^1 = B^1(G, A)$ the (commutative Banach) algebra of A-valued, Bochner integrable functions on G, $\mathfrak{M}_1$ the maximal ideal space of A, $\mathfrak{M}_2$ the maximal ideal space of $L^1(G)$ (the [commutative Banach] algebra of complex-valued, Haar integrable functions on G), $\mathfrak{M}_3$ the maximal ideal space of B^1. Then $\mathfrak{M}_3$ and the Cartesian product $\mathfrak{M}_1 \times \mathfrak{M}_2$ are homeomorphic when the spaces $\mathfrak{M}_i$, $i = 1, 2, 3$, are given their weak* topologies. Furthermore, the association between $\mathfrak{M}_3$ and $\mathfrak{M}_1 \times \mathfrak{M}_2$ is such as to permit a description of any epimorphism $E_3: B^1 \to B^1/M_3$ in terms of related epimorphisms $E_1: A \to A/M_1$ and $E_2: L^1(G) \to L^1(G)/M_2$, where M_i is in $\mathfrak{M}_i$, $i = 1, 2, 3$.*

On the other hand, Hausner **(2)** (and the author, independently) showed that a similar result is valid for generalized continuous function algebras. One form of the theorem is the following: *Let X be a compact Hausdorff space, A a commutative Banach algebra, $D = C(X, A)$ the (commutative Banach) algebra of A-valued continuous functions on X, $\mathfrak{M}_1$ the maximal ideal space of A, $\mathfrak{M}_2$ the maximal ideal space of $C(X)$ (the [commutative Banach] algebra of complex-valued continuous functions on X), $\mathfrak{M}_3$ the maximal ideal space of D. Then $\mathfrak{M}_3$ and the Cartesian product $\mathfrak{M}_1 \times \mathfrak{M}_2$ are homeomorphic when the spaces $\mathfrak{M}_i$, $i = 1, 2, 3$, are given their weak* topologies. Furthermore, the association between $\mathfrak{M}_3$ and $\mathfrak{M}_1 \times \mathfrak{M}_2$ is such as to permit a description of any epimorphism $E_3: D \to D/M_3$ in terms of related epimorphisms $E_1: A \to A/M_1$ and $E_2: C(X) \to C(X)/M_2$, where M_i is in $\mathfrak{M}_i$, $i = 1, 2, 3$.*

The crucial point in the latter theorem is the proof that D is spanned by "simple" functions, that is, functions which are linear combinations, with coefficients in A, of complex-valued continuous functions on X. On the other hand, the very definition of B^1 shows that it is spanned by "simple" functions, that is, this time, functions which are linear combinations, with coefficients in A, of complex-valued, Haar integrable functions on G. Clearly, in each instance, the collection of "simple" functions is an algebra which is a tensor

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product of A and some complex function algebra, and the object of discussion is the completion of this tensor product with respect to an appropriate norm.

2. Tensor products. Let A_1 and A_2 be Banach algebras and let $A_3' = A_1 \otimes A_2$ be their algebraic tensor product **(0)**. As is well known, **(6)** there are generally many norms which can be given to A_3' in terms of the norms of A_1 and A_2. Our first result is about one of these norms.

THEOREM 1. *Let* $||\ldots||_i$ *be the norms in* A_i, $i = 1, 2$. *Then the "greatest cross norm"* **(6)** *defined in* A_3' *by*

$$\left|\left|\sum_{i=1}^{n} a_1^{(i)} \otimes a_2^{(i)}\right|\right|_3' = \inf \sum_{i=1}^{n} ||a_1^{(i)}||_1\, ||a_2^{(i)}||_2\,,$$

where the inf *is taken over the equivalence class which defines*

$$\sum_{i=1}^{n} a_1^{(i)} \otimes a_2^{(i)},$$

is a Banach algebra norm which satisfies the relationship

$$\mathfrak{P}' \qquad ||\, a_1 \otimes a_2\, ||_3' = ||\, a_1\, ||_1\, ||\, a_2\, ||_2.$$

Proof. In **(6)** the validity of the last equality is shown. We prove here the fact that if p, q are in A_3', then $||pq||_3' \leqslant ||p||_3'||q||_3'$. To this end, let $r > 0$ be given. Then there is a choice of $a_1^{(i)}$, $a_2^{(i)}$, $b_1^{(j)}$, $b_2^{(j)}$ for which

$$\sum_{i=1}^{n} a_1^{(i)} \otimes a_2^{(i)} \qquad \text{and} \qquad \sum_{j=1}^{m} b_1^{(j)} \otimes b_2^{(j)}$$

define the respective equivalence classes of p and q and for which

$$||p||_3'||q||_3' \geqslant \left(\sum_{i=1}^{n} ||a_1^{(i)}||_1||a_2^{(i)}||_2\right)\left(\sum_{j=1}^{m} ||b_1^{(j)}||_1||b_2^{(j)}||_2\right) - r.$$

The last expression is

$$\sum_{i,j} ||a_1^{(i)}||_1||b_1^{(j)}||_1||a_2^{(i)}||_2||b_2^{(j)}||_2 - r$$

which majorizes

$$\sum_{i,j} ||a_1^{(i)}b_1^{(j)}||_1||a_2^{(i)}b_2^{(j)}||_2 - r.$$

Obviously, the last expression majorizes $||pq||_3' - r$. Since r is an arbitrary positive number, we see $||p||_3'||q||_3' \geqslant ||pq||_3'$. This completes the proof.

THEOREM 2. *Let* A_3 *be the completion of* A_3' *endowed with the "greatest cross norm"* $||\ldots||_3'$. *Let* A_i *be commutative and let* $\mathfrak{M}_i$ *be their respective maximal ideal spaces with their respective weak* topologies,* $i = 1, 2$. *Then* A_3 *is a commutative Banach algebra. Its norm* $||\ldots||_3$ *satisfies the analogue*

$$\mathfrak{P} \quad ||a_1 \otimes a_2||_3 = ||a_1||_3||a_2||_3$$

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of $\mathfrak{P}'$. If $||\ldots||_3''$ is any tensor product norm relative to which A_3' is a normed algebra, with no dense reg. max. ideal (for example, if $||\cdots||_3''$ is the greatest cross norm), and if A_3 is the completion of A_3' relative to $||\cdots||_3''$, then A_3 is a commutative Banach algebra and its maximal ideal space $\mathfrak{M}_3$ in its weak topology is homeomorphic with the Cartesian product $\mathfrak{M}_1 \times \mathfrak{M}_2$. Let t be the homeomorphism the existence of which is asserted: $t: \mathfrak{M}_3 \to \mathfrak{M}_1 \times \mathfrak{M}_2$, and let $t(M_3) = (M_1, M_2)$. Then the epimorphisms*

$$E_1: A_1 \to A_1/M_1, \quad E_2: A_2 \to A_2/M_2, \quad E_3: A_3 \to A_3/M_3,$$

are uniquely determined by the respective maximal ideals M_i, $i = 1, 2, 3$; E_1 and E_2 together determine E_3 and conversely.

Proof. The commutativity of A_3 and the validity of $\mathfrak{P}$ are clear consequences of the hypotheses.

Since A_i/M_i, $i = 1, 2, 3$, is the complex numbers, and since each epimorphism E_i, $i = 1, 2, 3$ commutes with multiplication by complex numbers ($cE_i(a) = E_i(ca)$, c complex, a in A_i) and since the complex numbers admit no non-trivial automorphism which commutes with multiplication by complex numbers, the uniqueness of the E_i follows.

We now proceed to set up a $1-1$ correspondence between $\mathfrak{M}_3$ and $\mathfrak{M}_1 \times \mathfrak{M}_2$. After this has been accomplished, the correspondence will be shown to be a homeomorphism. With a view to greater ultimate generality, we shall, however, show how to establish the kind of correspondence we need between $\mathfrak{M}_1 \times \mathfrak{M}_2$ and a part of $\mathfrak{M}_3$ under conditions far less restrictive than those imposed in the hypothesis of Theorem 2. This correspondence will serve when the hypothesis of Theorem 2 is in force and will in fact prove to be the homeomorphism which is sought. What follows then is an interlude, justified and required by economy.

During this interlude we shall not assume that A_1 and A_2 are commutative. $\mathfrak{M}_1$ and $\mathfrak{M}_2$ will denote their respective spaces of (two-sided) regular maximal ideals. For each pair (M_1, M_2) in $\mathfrak{M}_1 \times \mathfrak{M}_2$, let E_1 and E_2 be *some* epimorphisms $E_i: A_i \to A_i/M_i$, $i = 1, 2$. For p in A_3', define E_3' by the formula

$$E_3'(p) = \sum_{i=1}^{n} E_1(a_1^{(i)}) \otimes E_2(a_2^{(i)}),$$

a member of the tensor product $(A_1/M_1) \otimes (A_2/M_2)$, where

$$\sum_{i=1}^{n} a_1^{(i)} \otimes a_2^{(i)}$$

is some representation of p. Clearly $E_3'(p)$ does not depend on the representation of p and is an epimorphism of A_3'. $E_3': A_3' \to (A_1/M_1) \otimes (A_2/M_2)$. Let A_3' have the norm $||\cdots||''$ and let $E_3'(A_3')$ have the quotient space norm (which is independent of the choice of E_1 and E_2). The quotient space norm is admissible as a true norm since A_3' has no dense reg. max. ideal. Then, relative to these topologies, E_3' is a bounded (hence uniformly continuous) transformation of A_3' and has a unique extension E_3, an epimorphism of A_3 onto the completion of $(A_1/M_1) \otimes (A_2/M_2)$ relative to its (quotient space) norm.

Let $M_3 = E_3^{-1}(0)$. M_3 is an ideal in A_3. If u_i are identities modulo M_i in A_i, $(i = 1, 2)$, then $E_3(u_1 \otimes u_2)$ is an identity in $E_3(A_3)$, whence M_3 is regular. Consequently, there is a regular maximal ideal N_3 which contains M_3. We shall show that N_3 and M_3 are the same.

For this purpose, we define two mappings G_i of A_i into $E_3(A_3)$, $i = 1, 2$, as follows:

$$G_i(a_i) = E_3(a_i u), \qquad i = 1, 2,$$

where a_i is in A_i and u is an identity modulo M_3. Clearly $G_i(a_i)$ is independent of the choice of u. Let F_3 be *some* epimorphism, $F_3: A_3 \to A_3/N_3$, and let H_i, $i = 1, 2$, be engendered by F_3 as G_i are engendered by E_3. We will show that $M_i = H_i^{-1}(0) = G_i^{-1}(0) = E_i^{-1}(0)$, $(i = 1, 2)$.

If a_1 is in M_1, then $E_1(a_1) = 0$, whence $E_3(a_1 u) \equiv E_3(a_1 u_1 \otimes u_2) = E_1(a_1 u_1) \otimes E_2(u_2) = 0$ (where $u = u_1 \otimes u_2$). Thus $G_1(a_1) = 0$ and hence M_1 is contained in $G_1^{-1}(0)$. Since $G_1^{-1}(0)$ is a proper ideal and M_1 is a maximal ideal we see that $M_1 = G_1^{-1}(0)$.

Since $a_1 u_1 \otimes u_2$ is a member of M_3 which is a subset of N_3, it follows that $F_3(a_1 u_1 \otimes u_2) = 0 = H_1(a_1)$. We see that M_1 is contained in $H_1^{-1}(0)$ which is a proper ideal of A_1. Since M_1 is maximal, it follows that $M_1 = H_1^{-1}(0)$. Of course, by definition, $M_1 = E_1^{-1}(0)$. Analogously, we can show $M_2 = H_2^{-1}(0) = G_2^{-1}(0) = E_2^{-1}(0)$.

In order to continue we shall require the following lemmas.

Lemma 1. *Let A be a Banach algebra, I a closed ideal of A and let E and E'' be two epimorphisms, $E: A \to A/I$, $E'': A \to A/I$. Then, relative to the quotient space (norm) topology of A/I, there is an isometric automorphism α of A/I, α commutes with complex multiplication and $E = \alpha E''$.*

Proof. For b in A/I, let $E''(a) = b$ and let $\alpha(b) = E(a)$. If $E''(a') = b$, then $a' - a$ is in I, whence $E(a') = E(a)$ and thus $\alpha(b)$ is uniquely defined. If $E(a'') = b$, then $\alpha E''(a'') = E(a'') = b$, whence α is an automorphism, which clearly commutes with complex multiplication. Finally, if $||\cdots||_A$ and $||\cdots||$ are the respective norms of A and A/I, we see

$$||\alpha(b)|| = ||E(a)|| = \inf \{||a + i||_A | i \text{ in } I\}.$$

On the other hand,

$$||b|| = ||E''(a)|| = \inf \{||a + i||_A | i \text{ in } I\},$$

whence $||\alpha(b)|| = ||b||$ and α is an isometry.

Lemma 2. *Let A_i be Banach algebras, I_i closed ideals in A_i, E_i, E_i'' epimorphisms, $E_i: A_i \to A_i/I_i$, $E_i'': A_i \to A_i/I_i$, α_i the isometric automorphisms*

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(*Lemma* 1) *for which* $E_i = \alpha_i E_i''$, $i = 1, 2$, *and let* α *be the tensor product* $\alpha_1 \otimes \alpha_2$. *If* $A_1 \otimes A_2$ *is given some tensor product norm with respect to which* $A_1 \otimes A_2$ *becomes a normed algebra, then* α *is an isometric automorphism of* $(A_1/I_1) \otimes (A_2/I_2)$ *relative to the quotient space norm described earlier. If* E' *and* $(E')''$ *are the respective epimorphisms engendered by* E_1, E_2 *and* E_1'', E_2'', *then* $E' = \alpha(E')''$.

Proof. The fact that α is an automorphism is clear, as is the relationship $E' = \alpha(E')''$. If b is in $(A_1/I_1) \otimes (A_2/I_2)$, then a representative of b is an expression of the form

$$\sum_{i=1}^{n} E_1''(a_1^{(i)}) \otimes E_2''(a_2^{(i)}).$$

A representative of $\alpha(b)$ is the expression

$$\sum_{i=1}^{n} E_1(a_1^{(i)}) \otimes E_2(a_2^{(i)}).$$

The argument given in Lemma 1 may be repeated *mutatis mutandis* to show that $\alpha(b)$ and b have the same norm.

LEMMA 3. *Let* A *be a normed algebra and let* α *be an isometric automorphism of* A *which commutes with complex multiplication. If the completion* $\bar{A}$ *of* A *is simple, so is the completion* $\overline{\alpha A}$ *of* αA.

Proof. Since α is an isometry, it may be extended in a unique fashion to an isometric automorphism $\bar{\alpha}$ of $\bar{A}$ which commutes with complex multiplication. Clearly $\bar{\alpha}(\bar{A}) = \overline{\alpha A}$, whence the simplicity of $\bar{A}$ implies the simplicity of $\overline{\alpha A}$.

From the preceding paragraphs and lemmas we can conclude that there exist isometric automorphisms α_i, β_i of A_i/M_i which commute with complex multiplication and which satisfy the relations $H_i = \alpha_i G_i = \beta_i E_i$, $(i = 1, 2)$. If β is the tensor product $\beta_1 \otimes \beta_2$, then β is an isometry of (A_1/M_1) and the following relationship is valid:

$$F_3(A_3') = \beta((A_1/M_1) \otimes (A_2/M_2)) = \beta E_3(A_3').$$

Since the completion of $F_3(A_3')$ is $F_3(A_3)$ which is simple, and since β^{-1} is an isometry, we see (Lemma 3) that the completion of $\beta^{-1}F_3(A_3')$ is simple and hence that the completion of $E_3(A_3')$ is simple. Hence $M_3 = E_3^{-1}(0)$ is a regular maximal ideal, and thus $M_3 = N_3$.

We have shown how to associate with each pair (M_1, M_2) *a* maximal ideal M_3. The method of association demands that we show that M_3 is uniquely determined in this manner by the pair (M_1, M_2), regardless of which epimorphisms E_1, E_2, etc., are used in the construction.

To this end, suppose that E_1'', E_2'' are chosen in place of E_1, E_2, at the beginning of our construction. Then there are automorphisms γ_i of A_i/M_i such that $E_i'' = \gamma_i E_i$, $(i = 1, 2)$. Let E_3'' be engendered by E_1'', E_2'' as E_3

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is engendered by E_1, E_2, and let E_3'' engender G_1'', G_2'' as E_3 engenders G_1, G_2. Then there are automorphisms π_i of A_i/M_i which satisfy $G_i'' = \pi_i E_i$, $i = 1, 2$. If we set π equal to the tensor product $\pi_1 \otimes \pi_2$, we see that $E_3'' = \pi E_3$ and hence that $(E_3'')^{-1}(0) = E_3^{-1}(0)$. Thus M_3 is uniquely determined, even though the epimorphisms involved in its determination are not unique.

The interlude is over and we continue the proof by using the complete hypothesis of our theorem.

We proceed to establish a correspondence between elements of $\mathfrak{M}_3$ and elements of $\mathfrak{M}_1 \times \mathfrak{M}_2$. If M_3 is in $\mathfrak{M}_3$ and if E_3 is the (unique) epimorphism, $E_3: A_3 \to A_3/M_3$, we can define G_i, $i = 1, 2$, as we did above in the more general context. This time we define M_i to be $G_i^{-1}(0)$, $(i = 1, 2)$. We will show that M_1 and M_2 are maximal ideals which engender, in the manner described above, a maximal ideal which is precisely M_3. The circle will thereby be closed.

The commutativity of A_1 and A_2 implies that A_3' (and hence A_3) is commutative. Let u be an identity modulo M_3. Then if p (in A_3'), represented by

$$\sum_{i=1}^{n} a_1^{(i)} \otimes a_2^{(i)},$$

is so near to u that $E_3(p) \neq 0$, we see that some term in the representation of $E_3(p)$ is not zero. Hence for some i_0,

$$G_1(a_1^{(i_0)}),\ G_2(a_2^{(i_0)})$$

are both not zero. It follows, since A_3/M_3 is the complex number system, that G_i are non-trivial epimorphisms, $G_i: A_i \to C$ (the complex number system), whence M_i are maximal ideals, $(i = 1, 2)$.

Clearly, if M_3'' is the maximal ideal engendered by the M_i in the manner described earlier, then M_3'' contains M_3, and, since M_3 is maximal, M_3'' and M_3 are the same.

Thus we have established a $1-1$ correspondence t between $\mathfrak{M}_3$ and $\mathfrak{M}_1 \times \mathfrak{M}_2$.

The homeomorphism between $\mathfrak{M}_3$ and $\mathfrak{M}_1 \times \mathfrak{M}_2$ can be established as follows. If a is in a commutative Banach algebra A, M is a maximal ideal of A, then $a^+(M)$ denotes the complex number into which a is mapped when A is reduced modulo M. If M_{03} is in $\mathfrak{M}_3$, if $t(M_{03}) = (M_{01}, M_{02})$ and if $N(M_{01}, M_{02})$ is a neighbourhood of (M_{01}, M_{02}) in $\mathfrak{M}_1 \times \mathfrak{M}_2$ we may assume $N(M_{01}, M_{02})$ is of the form $N(M_{01}) \times N(M_{02})$ where $N(M_{0i})$ are neighbourhoods of M_{0i} in M_i, $(i = 1, 2)$. But

$$N(M_{0i}) = \{M_i\ |a_{ji}^+(M_i) - a_{ji}^+(M_{0i})| < r_i, j = 1, 2, \ldots, J_i, r_i > 0\}.$$

Consider

$$\begin{aligned} N(M_{03}) = \\ \{M_3 |&(a_{j1} \otimes u_2)^+(M_3) - (a_{j1} \otimes u_2)^+(M_{03})| < r_1, j = 1, 2, \ldots, J_1, \\ &|(u_1 \otimes a_{j2})^+(M_3) - (u_1 \otimes a_{j2})^+(M_{03})| < r_2, j = 1, 2, \ldots, J_2\}, \end{aligned}$$

where u_i are identities modulo M_i, $i = 1, 2$, and $t(M_3) = (M_1, M_2)$. Since $(u_1 \otimes a_2)^+(M_3) = a_2^+(M_2)$, we see $t(N(M_{03}))$ is contained in $N(M_{01}, M_{02})$.

On the other hand, let

$$N(M_{03}) = \{M_3 | a_j(M_3) - a_j(M_{03})| < r, j = 1, 2, \ldots J\}.$$

Choose

$$P_j = \sum_{i=1}^{n_j} a_{i1}^{(j)} \otimes a_{i2}^{(j)} \quad \text{in} \quad A_3'$$

so that $||a_j - P_j||_3'' < r/3$, $j = 1, 2, \ldots, J$. Let

$$N(M_{01}) = \{M_1 \, |a_{i1}^{(j)+}(M_1) - a_{i1}^{(j)+}(M_{01})| < r/(6Jn(2R_1 + r)),$$
$$i = 1, 2, \ldots, n_j, j = 1, 2, \ldots, J\},$$

where

$$n = \sum_{j=1}^{J} n_j, R_1 = \sup_{i,j}\{||a_{i1}^{(j)}||_1\}.$$

Similarly let

$$N(M_{02}) = \{M_2| \, |a_{i2}^{(j)+}(M_2) - a_{i2}^{(j)+}(M_{02})| < r/6Jn(2R_2 + r)),$$
$$i = 1, 2, \ldots, n_j, j = 1, 2, \ldots, J\}.$$

Then if $M_3 = (M_1, M_2)$ is in $N(M_{01}) \times N(M_{02})$, we see

$$|a_j^+(M_3) - a_j^+(M_0)|$$
$$\leqslant |(a_j - P_j)^+(M_3)| + |(a_j - P_j)^+(M_{03})| + |P_j^+(M) - P_j^+(M_{03})|$$
$$< 2r/3 + \left| \sum_{i=1}^{n_j} a_{i1}^{(j)+}(M_1)a_{i2}^{(j)+}(M_2) - a_{i1}^{(j)+}(M_{01})a_{i2}^{(j)+}(M_{02}) \right| < r,$$

and hence $t^{-1}(N(M_{01}) \times N(M_{02})) \subset N(M_{03})$, and t is a homeomorphism. The proof of Theorem 2 is complete.

The following remarks are in order at this point.*

1. A little reflection shows that $B^1(G, A)$ is the completion of the tensor product of $L^1(G)$ and A relative to the norm:

$$\left|\left| \sum_{i=1}^{n} \lambda_i(x)a_i \right|\right|' = \int_G \left|\left| \sum_{i=1}^{n} \lambda_i(x)a_i \right|\right|_A dx.$$

2. The result of Hausner and the author shows that $C(X, A)$ is the completion of the tensor product of $C(X)$ and A relative to the norm:

$$\left|\left| \sum_{i=1}^{n} \lambda_i(x)a_i \right|\right|' = \sup\left\{ \left|\left| \sum_{i=1}^{n} \lambda_i(x)a_i \right|\right|_A \middle| x \in X \right\}.$$

*At the time of the writing of this paper, the author was unaware of the results of Willcox **(8)** and of the appearance of Hausner's paper **(3)**. Clearly the spirit expressed in the second paragraph, p. 876 of **(8)** has motivated much of our study.

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3. The tensorial approach explains and unifies a collection of phenomena and symmetries hitherto observed without comprehension.

For example, Johnson **(5)** shows that $B^1(G, L^1(H))$ and $L^1(G \times H)$ are isomorphic if G and H are locally compact abelian groups. From our standpoint $B^1(G, L^1(H))$ is the completion of the tensor product $L^1(G) \otimes L^1(H)$. Relative to this format, Johnson's theorem is essentially the statement that $L^1(G) \otimes L^1(H)$ (completed) and $L^1(G \times H)$ are isomorphic. The symmetry and truth of this statement are clarified by the tensorial viewpoint.

4. When either of A_1 or A_2 is non-commutative, the most important topologies for the associated spaces of two-sided regular maximal ideals are the kernel-hull topologies **(7)**. In general, under these circumstances, A will be non-commutative and even if the "natural" 1–1 map $t: \mathfrak{M}_3 \to \mathfrak{M}_1 \times \mathfrak{M}_2$ can be constructed, the question of the bi-continuity of t seems to be open.

5. The property $\mathfrak{P}$ of $|| \cdots ||_3$ is irrelevant to the existence of the homeomorphism t. The impact of Theorem 1 and the associated part of Theorem 2 is the existence of norms for A_3' and A_3 relative to which they become normed or Banach algebras.

6. When A_1 and A_2 are not assumed to be commutative, the following results obtain:

(i) *If A_1 and A_2 have "approximate identities," then the* 1–1 *mapping t: $\mathfrak{M}_3 \to \mathfrak{M}_1 \times \mathfrak{M}_2$ can be constructed.*

(ii) *If A_1 and A_2 have identities e_1 and e_2, and if $t(M_3) = (M_1, M_2)$, then $M_2 = M_3 \cap (e_1 \otimes A_2)$ and $M_1 = M_3 \cap (A_1 \otimes e_2)$.*

Proof. Ad(i) By an "approximate identity" in A_i is meant an A_i-valued function v_{ip}, on a directed set P_i such that

$$\lim_{P_i} v_{ip} a_i = a_i$$

for any a_i in A_i, $(i = 1, 2)$. We have observed that $\mathfrak{M}_1 \times \mathfrak{M}_2$ is always naturally embedded in $\mathfrak{M}_3$. On the other hand, for a given M_3 in $\mathfrak{M}_3$ the construction of the naturally associated pair (M_1, M_2) can begin with the mappings G_1, G_2 as above. This time, however, the proof of the regularity and maximality of the relevant ideals proceeds differently.

First, recognizing that A_3 is an A_i-module, we remark that

$$\lim_{P_i} v_{ip}\, q = q, \qquad i = 1, 2,$$

for any q in A_3. Thus, if u is an identity modulo M_3,

$$\lim_{P_i} E_3(v_{ip} u) = E_3(u) = e,$$

the identity of A_3/M_3. This means that $G_i(A_i)$ has e as a point of closure. On the other hand, $G_i(A_i)$ is a complete normed space, and thus e is in $G_i(A_i)$, whence $G_i^{-1}(0) \equiv M_i$ is regular, $i = 1, 2$. The maximality of M_i can be established as in the previous case, once the regularity is known. Of course, our observations on the ambiguity of the epimorphisms can be repeated.

Ad (ii) The existence of t is assured by i. $e_1 \otimes A_2$ and A_2 are isomorphic. Clearly $M_3 \cap (e_1 \otimes A_2)$ is isomorphic to some ideal N_2 in A_2. Let E_3, G_2 have meanings as given earlier. Then, since $e_1 \otimes e_2$ is an identity modulo M_3,

$$G_2(a_2) = E_3((e_1 \otimes e_2)a_2) = E_3(e_1 \otimes a_2)$$

and $G_2(a_2) = 0$ if and only if $E_3(e_1 \otimes a_2) = 0$, that is, if and only if $e_1 \otimes a_2$ is in M_3. Since $e_1 \otimes a_2$ is in $e_1 \otimes A_2$ we see $G_2(a_2) = 0$ if and only if $e_1 \otimes a_2$ is in $M_3 \cap (e_1 \otimes A_2) = N_2$. Thus $N_2 = M_2 = G_2^{-1}(0)$.

3. Group Representations. In the particular case where $A_1 = L^1(G)$, G is a locally compact abelian group, and A_2 is a commutative Banach algebra with an involution and an identity, there are some interesting group representations which can be found.

If $\alpha(x)$ is in G^+ (the character group of G), then for $f(x)$ in $A_3 = B^1(G, A_2)$, the mapping $\pi_\alpha: A_3 \to A_2$ defined by

$$\pi_\alpha(f(x)) = \int_G f(x)\overline{\alpha(x)}\, dx$$

is a homomorphism. If we define an "involution" in A_3 by the formula $f^\dagger(x) = (f(x^{-1}))^*$ where * is the involution in A_2, then $\pi_\alpha(f^\dagger(x)) = (\pi_\alpha(f(x))^*$. Clearly π_α is continuous, and actually π_α is an epimorphism (which commutes with multiplication by elements of A_2), since $\pi_\alpha(\lambda(x)e_1) = e_1$, if $\lambda(x)$ is in $L^1(G)$ and $\lambda^+(\alpha) = 1$.

On the other hand, let $\mathfrak{J}$ be the non-empty set of inverses in A_2 and let π be a $\dagger{*}A_2$-epimorphism: $\pi: A_3 \to A_2$, that is, π commutes with multiplication by elements of A_2 and $\pi(f^\dagger) = (\pi f)^*$. For arbitrary f in $\pi^{-1}(\mathfrak{J})$ define $\alpha_\pi(x)$ by the formula $(\pi(f_x))(\pi f)^{-1}$. Then, in the usual fashion, one can show: $\alpha_\pi(xy) = \alpha_\pi(x)\alpha_\pi(y)$; $\alpha_\pi(x^{-1}) = (\alpha_\pi(x))^*$; $\alpha_\pi(e) = e_2$; $\alpha_\pi(x)$ is f-free, bounded, continuous; $\pi(u_x) \to \alpha_\pi(x)$ for any approximate identity $\{u\}$ in $L^1(G)$, where e is the identity of G, e_2 is the identity of A. We call $\alpha_\pi(x)$ a *unitary representation* of G into $\mathfrak{J}$. The direct computation which follows shows that

$$\pi(f(x)) = \int_G f(x)(\alpha_\pi(x))^*\, dx.$$

If we let g be in $\pi^{-1}(\mathfrak{J})$, then

$$\int_G f(x)(\alpha_\pi(x))^* dx = \left(\int_G f(x)\pi(g_{x^{-1}})dx\right)(\pi(g))^{-1} = \pi(f*g)(\pi(g))^{-1} = \pi(f).$$

Hence there is a $1-1$ *correspondence between* $\dagger{*}A_2$*-epimorphisms* π *of* A_3 *onto* A_2 *and unitary representatives* α_π *of* G *into* $\mathfrak{J}$.

Let $\tilde{G}$ denote the group of all such unitary representations $\alpha_\pi(x)$. The compact-open and weak* topologies (for mappings of A_3 into A_2) are identical for $\tilde{G}$. In general, $\tilde{G}$ is not locally compact.

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The proofs of the last two statements are straightforward and are therefore omitted.

If M_2 in $\mathfrak{M}_2$ is fixed, then $(\alpha_\pi(x))^+(M_2)$, as a function on G is a member of G^+. Hence, for each M_2 in $\mathfrak{M}_2$, there is an epimorphism

$$E_{M_2}: \tilde{G} \to G^+,$$

given by:

$$E_{M_2}(\alpha_\pi(x)) = (\alpha_\pi(x))^+(M_2).$$

If π is fixed, then $(\alpha_\pi(x))^+(M_2)$ is a G^+-valued function on $\mathfrak{M}_2$, and actually $(\alpha_\pi(x))^+(M_2)$ is in $C(\mathfrak{M}_2, G^+)$.

If A_2 and $A_2{}^+$ are isomorphic, if $A_2{}^+ = C(\mathfrak{M}_2)$, that is, if A_2 and $C(\mathfrak{M}_2)$ are equivalent, and if β is in $C(\mathfrak{M}_2, G^+)$, define π_β in A_3 by:

$$\pi_\beta(f(x)) = \left(\int_G f(x)\overline{\beta(x; M_2)}\, dx\right)^+ (M_2).$$

Then

$$(\alpha_{\pi_\beta}(x))^+(M_2) = \beta(x; M_2).$$

We have thus far shown that there is a natural mapping γ of $\tilde{G}$ into $C(\mathfrak{M}_2, G^+)$ given by

$$\gamma(\alpha_\pi(x)) = \alpha_\pi(x)^+(M_2)$$

and that if A_2 and $C(\mathfrak{M}_2)$ are equivalent, then the natural mapping γ carries $\tilde{G}$ *onto* $C(\mathfrak{M}_2, \tilde{G}^+)$.

Before stating the next theorem we shall require the following discussion.

If $\alpha(x)$ is in G^+, then for each x there is a unique real number $\beta(x)$, $0 \leqslant \beta(x) < 2\pi$ such that $\alpha(x) = \exp(i\beta(x))$. For example, if G is the circle group (the reals reduced modulo 2π), and if $\alpha(x)$ is in G^+, then there is an integer n such that $\alpha(x) = \exp(i\{nx\})$ where $\{nx\}$ is the residue of nx modulo 2π. Although $\exp(i\{x\})$ is a character in this case, $\exp(i\{\frac{1}{2}\{x\}\})$ is not (since, for example, $\exp(i\{\frac{1}{2}\{2\pi - y\}\}) = \exp(i(\frac{1}{2})(2\pi - y)) \to \exp(i\pi) = -1$ as $y \downarrow 0$, whereas $\exp(i\{\frac{1}{2}\{2\pi - y\}\})$ should approach 1 as $y \downarrow 0$ if $\exp(i\{\frac{1}{2}\{2\pi - y\}\})$ is a character). Hence, in general, even if $\exp(i\beta(x))$ is a character, $\exp(i\{s\beta(x)\})$ is not necessarily a character for all real s.

On the other hand, if G is the additive group of real numbers, and $\alpha(x)$ is in G^+, then there is a real number t such that $\alpha(x) = \exp(i\{tx\})$, $\{tx\}$ the residue of tx modulo 2π. In this case, for any real s, $\exp(i\{s\{tx\}\})$ is again a character.

If a group G has the property that $\exp(i\beta(x))$ is a character implies $\exp(i\{s\beta(x)\})$ is a character, for all real s, we shall call G *real-closed*.

THEOREM 1. *If A_2 is semisimple γ is 1–1; the converse is false. If A_2 and $C(\mathfrak{M}_2)$ are equivalent and G is real-closed, then $\gamma: \tilde{G} \to C(\mathfrak{M}_2, G^+)$ is an isomorphism and conversely, if G is real-closed and γ is an isomorphism, then $A_2 = C(\mathfrak{M}_2)$.*

Proof. Assume A_2 is a semisimple and assume

$$\gamma(\alpha_{\pi_1}) = \gamma(\alpha_{\pi_2}).$$

If

$$\alpha_{\pi_1} \neq \alpha_{\pi_2}, \quad \text{then} \quad \pi_1 \neq \pi_2$$

and there is an f in A such that $\pi_1(f) = a_1 \neq a_2 = \pi_2(f)$. But

$$(a_1 - a_2)^+(M_2) = \left(\int_G f(x)(\alpha_{\pi_1}(x) - \alpha_{\pi_2}(x))^* dx\right)^+(M_2) = 0$$

for all M_2, a contradiction of the semisimplicity of A_2.

Assume A_2 has a radical, R_2. Then R_2 is a non-trivial group $R_2{}^o$ relative to the multiplication: $r_1 \, o \, r_2 = r_1 + r_2 - r_1 r_2$.

Now

$$\gamma(\alpha_{\pi_1}) = \gamma(\alpha_{\pi_2}) \text{ if, and only if, } \alpha_{\pi_1}(x) - \alpha_{\pi_2}(x)$$

is in R_2 for all x. But

$$\alpha_{\pi_1}(x) - \alpha_{\pi_2}(x) \in R_2 \text{ for all } x \text{ if, and only if, } 1 - \alpha_{\pi_1}(x)^*\alpha_{\pi_2}(x) = r(x) \in R_2$$

for all x. Clearly $r(x)$ is a representation of G into $R_2{}^o$ (R_2 as a group *re o*). Hence γ is not 1–1 if, and only if, there is a non-trivial representation $r(x)$ of G into $R_2{}^o$.

However, $R_2{}^o$ contains no elements (different from 0) of finite order. For

$$\overbrace{(r)o(r)o \ldots o(r)}^{n} \equiv r^{n(o)} = 1 - (1 - r)^n.$$

If $r^{n(o)} = 0$, r in $R_2{}^o$, $r \neq 0$, then $(-1)^{N+1}r^N = P_N$ where P_N is a polynomial of degree $N-n$ in n, with coefficients which are polynomials of degree not more than $n-1$ in r. Thus $||r^N|| \geqslant n^{N-n}||Q_o||$, where

$$Q_o = \sum_{k=1}^{n-1} (-1)^{k+1} {}_nC_k r^k.$$

Hence $||r^N||^{(1/N)} \geqslant n^{(1-n/N)}||Q_o||^{(1/N)} \to n$, as $N \to \infty$, a contradiction of the fact that r is in R_2.

Thus if G has only elements of finite order, $r(x)$ cannot be non-trivial. On the other hand, if $G = R_2{}^o$ (with discrete topology), then $r(x) \equiv x$ serves. Hence the monomorphy of γ depends both on the presence or absence of the radical in A_2 and on the nature of G.

If A_2 and $C(\mathfrak{M}_2)$ are equivalent, then, of course, A_2 is semisimple and hence γ is 1–1, and, as we have shown, an epimorphism. Thus γ is an isomorphism.

On the other hand, if γ is an isomorphism and if $A_2{}^+$ is a proper subset of $C(\mathfrak{M}_2)$ let $z(M_2)$ be in the complement of $A_2{}^+$. Let $z = u + iv$. Then one of u, v is not in $A_2{}^+$, whence we may assume z is real-valued. Since 1 is in $A_2{}^+$, $A_2{}^+$ contains all constants and hence for some constant c, $z(M_2) + c > 0$, all M_2. Hence we assume for any $w, 0 < w < 1$, there is a $z(M_2)$ in $C(\mathfrak{M}_2) - A_2^+$ such that

$$w = \inf \{ z(M_2) | M_2 \text{ in } \mathfrak{M}_2\} < \sup \{z(M_2) | M_2 \text{ in } \mathfrak{M}_2\} = 1.$$

Choose $g(x)$ in $L^1(G)$ so that: $g = 0$ outside some compact neighbourhood N of the identity e in G; $g(x) \geqslant 0$; $g^+(\alpha) \geqslant 0$; $||g||_1 = 1$; $g^+(\alpha)$ takes on at least three values. Clearly $1 = g^+(e^+) \leqslant ||g^+||_\infty \leqslant ||g||_1 = 1$. Let $g^+(\alpha_0) = w \neq 0, 1$. Then $0 < w < 1$ and we now assume $z(M_2)$ and w are related as indicated earlier. If $\alpha_0(x) = \exp\,(i\beta_0(x))$, let

$$h(s) = \int_G g(x) \exp(i\,\{s\beta_0(x)\})\,dx = \int_G g(x) \exp(is\beta_0(x))\,dx.$$

Then we see that for real s:

(a) $h(s)$ is in $C^\infty(-\infty, \infty)$;

(b) $h(s)$ is real;

(c) $|h^{(n)}(s)| \leqslant K^n$ *where* $K = sup\,\{|\beta_0(x)|\,|x$ *in* $N\}$.

Hence $h(s)$ is entire. Since $h(0) = 1$, $h(1) = w < 1$, we see $h(s)$ is not constant. Hence there is an interval (s', s''), $0 < s' < s'' < 1$ where $h'(s) < 0$, and on (s', s'') $h(s)$ has a continuous real-valued inverse: $s = h^{-1}(t)$, where $h(s'') = t'' < t < t' = h(s')$. Let $y(M_2) = az(M_2) + b$ be such that $t'' < y(M_2) < t'$. Then $y(M_2)$ is not in A_2^+ and if $k(M_2) = h^{-1}(y(M_2))$, then $k(M_2)$ is in $C(\mathfrak{M}_2)$, and $k(M_2)$ is real-valued. For $f(x) = g(x)e_2$ (e_2 the identity of A_2) consider

$$\int_G f(x) \exp(ik(M_2)\beta_0(x))\,dx = \int_G f(x) \exp(i\,\{k(M_2)\beta_0(x)\})\,dx$$
$$= h(k(M_2)) = y(M_2).$$

Clearly $\exp\,(i\{k(M_2)\beta_0(x)\})$ is in $C(\mathfrak{M}_2, G^+)$ and by hypothesis (G is real-closed) there is an α in G such that $\gamma(\alpha_\pi) = \exp\,(i\{k(M_2)\beta_0(x)\})$. But then $(\pi(f))^+(M_2) = y(M_2)$, contradicting the fact that $y(M_2)$ is not in A_2^+. The proof of the theorem is complete.

4. Miscellany. If G is locally compact abelian, $A_1 = L^1(G)$, and if A_2 has an involution (A_2 is assumed to have no identity and is not assumed to be commutative) easily verified extensions of the above read as follows:

1. *Suppose A_2 is extended to A_{2e} by the adjunction of an identity. Let A_{3e} be the completion of the tensor product of A_1 and A_{2e}. Call an epimorphism $\pi: A_3 \to A_2$ extendable if there is an epimorphism $\pi_e: A_{3e} \to A_{2e}$ which coincides with π on A_3 (naturally embedded in A_{3e}). Then the extendable epimorphisms $\pi: A_3 \to A_2$ are in $1-1$ correspondence with the unitary representations of G into the multiplicative group of A_{2e}.*

2. *If $v(x)$ is a continuous homomorphism*

$$v: G \to G^o_{A_2}$$

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(the multiplicative group of A_2 relative to o), then the formula:

$$\pi(f) = \int_G f(x)dx - \int_G f(x)(v(x))^*dx$$

defines an extendable epimorphism $\pi: A_3 \to A_2$. The extension of π is given by the formula:

$$\pi_e(\lambda(x)e + f(x)) = \int_G (\lambda(x)e + f(x))(e - (v(x))^*)dx.$$

More generally, if $a(x)$ is a numerical function, $a_2(x)$ an A_2-valued function such that $(1 - a(x))(e - a_2(x)) = u(x)$ is a unitary representation of G into the multiplicative group of A_{2e}, then the formula

$$\pi(f) = \int_G f(x)(1 - a(x))dx - \int_G f(x)(a_2(x))^*dx$$

defines an extendable epimorphism $\pi: A_3 \to A_2$. The extension π_e is given by the formula

$$\pi_e(\lambda(x)e + f(x)) = \int_G (\lambda(x)e + f(x))(u(x))^*dx.$$

Conversely, an extendable epimorphism $\pi: A_3 \to A_2$ serves to define two functions $a(x)$, $a_2(x)$ such that $(1 - a(x))(e - a_2(x)) = u(x)$ is a unitary representation of G into the multiplicative group of A_{2e}.

The last result stems from defining T_x on A_{2e} as follows: If a_{2e} is in A_{2e}, and $a_{2e} = \pi_e(\lambda(x)e + f(x))$ then let $T_x(a_{2e}) = \pi_e(\lambda e + f) - \pi_e(\lambda_x e + f_x)$. This definition of T_x is $(\lambda e + f)$-free. T_x satisfies the classic *criterion*:

$$T_x(ab) = (T_x a)b$$

for membership in A_{2e} considered as a subalgebra of the ring $\mathrm{E}(A_e)$ of endomorphisms of A_e. Hence $T_x = a(x)e + a_2(x)$ and the verification of the result follows immediately.

Remark. The *criterion* mentioned above is not valid for algebras having no identity. For example, if $A_2 = L^1(-\infty, \infty)$, $Tf = f_x$, then $T(f^*g) = (Tf)^*g$. But there is no h in A_2 such that $Tf = h^*f$, as is well known.

The standard techniques also show that $a(x)e + a_2(x) = \lim_u \pi_e(u_x e)$ for any approximate identity $\{u\}$ in A.

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