

## THE PHILLIP ISLAND PENGUIN PARADE (A MATHEMATICAL TREATMENT)

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### Abstract

Penguins are flightless, so they are forced to walk while on land. In particular, they show rather specific behaviours in their homecoming, which are interesting to observe and to describe analytically. We observed that penguins have the tendency to waddle back and forth on the shore to create a sufficiently large group, and then walk home compactly together. The mathematical framework that we introduce describes this phenomenon, by taking into account “natural parameters”, such as the eyesight of the penguins and their cruising speed. The model that we propose favours the formation of conglomerates of penguins that gather together, but, on the other hand, it also allows the possibility of isolated and exposed individuals.

The model that we propose is based on a set of ordinary differential equations. Due to the discontinuous behaviour of the speed of the penguins, the mathematical treatment (to get existence and uniqueness of the solution) is based on a “stop-and-go” procedure. We use this setting to provide rigorous examples in which at least some penguins manage to safely return home (there are also cases in which some penguins remain isolated). To facilitate the intuition of the model, we also present some simple numerical simulations that can be compared with the actual movement of the penguin parade.

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## 1. Introduction

The goal of this paper is to provide a simple but rigorous mathematical model which describes the formation of groups of penguins on the shore at sunset. The results that we obtain are the following. First, we construct a mathematical model to describe the formation of groups of penguins on the shore and their march towards their burrows; this model is based on systems of ordinary differential equations with a number of degrees of freedom which is variable in time. We show that the model admits a unique solution, which needs to be appropriately defined. Then, we give some rigorous mathematical results which provide sufficient conditions for a group of penguins to reach the burrows. In addition, we provide some numerical simulations which show that the mathematical model well predicts, at least at a qualitative level, the formation of clusters of penguins and their march towards the burrows; these simulations are easily implemented by images and videos.

It would be desirable to have empirical data about the formation of penguin clusters on the shore and their movements, in order to compare and adapt the model to experimental data, and possibly give a quantitative description of concrete scenarios. The methodology used here is based on direct observations on site, strict interactions with experts in biology and penguin ecology, mathematical formulation of the problem and rigorous deductive arguments, and numerical simulations.

Here, we describe the elements which lead to the construction of the model, presenting its basic features and also its limitations. Given the interdisciplinary flavour of the subject, it is not possible to completely split the biological discussion from the mathematical formulation. Note that the main mathematical equation is given in formula (1.1), and the main information coming from live observations is presented prior to this formula. Afterwards, the mathematical quantities involved in the equation are discussed and elucidated. The existence and uniqueness theory for equation (1.1) is presented in Section 2, and some rigorous mathematical results about this equation are given in Section 3. Roughly speaking, these are the results which give sufficient conditions on the initial data of the system and on the external environment for the successful homecoming of the penguins, and their precise formulation requires the development of the mathematical framework in (1.1). In Section 4 we present numerics, images and videos which support our intuition and set the mathematical model of (1.1) into a concrete framework that is easily comparable with the real-world phenomenon.

Prior to this, we think it is important to describe our experience of the penguin parade on Phillip Island, both to allow the reader who is not familiar with the event to concretely take part in it, and to describe some peculiar environmental aspects which are crucial in understanding our description (for instance, the weather on Phillip Island is completely different from that in the Antarctic, so many of our considerations are meant to be limited to this particular habitat). Also, our personal experience in this bio-mathematical adventure is a crucial point, in our opinion, to describe how scientific curiosity can trigger academic activities.

**1.1. Description of the penguin parade** An extraordinary event in the state of Victoria, Australia, is the march of the little penguins (*Eudyptula minor*) who live on Phillip Island. At sunset, when it gets too dark for the little penguins to hunt for their food in the sea, they return to their homes (which are small cavities in the terrain, located some tens of metres from the water's edge). What follows is the mathematical description that came out of the observations on site at Phillip Island, enriched by the scientific discussions we later had with penguin ecologists.

Watching the penguin parade on Phillip Island, it seemed to us that some simple features appeared in a very unusual pattern followed by the little penguins. First of all, they have a strong tendency to gather together in sufficiently large numbers before starting their march home. They also have a tendency to march in a straight line, compactly arranged in a cluster or group. To form this group, they move back and forth waiting for others to join them, even going back to the sea if no other mate is around.

If a little penguin remains isolated, some parameters in the model proposed may lead to the individual coming to a complete stop. More precisely, in the model that we propose, there is a term which makes the velocity vanish. In practice, this interruption in the penguin's movement is not due to physical impediments, but rather to the fact that there is no other penguin in a sufficiently small neighbourhood: in this sense, at a mathematical level, a quantified version of the notion of "isolation" leads the penguin to stop.

Of course, from the point of view of ethology, it would be desirable to have further noninvasive tests to measure how the situation that we describe is felt by the penguin at an emotional level (at the moment, we are not aware of experiments like this in the literature). Also, it would be highly desirable to have some precise experiments to determine how many penguins do not manage to return to their burrows within a certain time after dusk and stay either in the water or in the vicinity of the shore.

On the one hand, in our opinion, it is likely that rigorous experiments on site will demonstrate that the phenomenon for which an isolated penguin stops is rather uncommon, but not completely exceptional, in nature. On the other hand, our model is general enough to take into account the possibility that a penguin stops its march, and, at a quantitative level, we emphasize this feature in the figures of Section 4 to make the situation visible. The reader who does not want to take into account the stopping function in the model can just set this function to be identically equal to 1 (the mathematical formulation of this remark will be given after formula (1.9)). In this particular case, our model still exhibits the formation of groups of penguins moving together.

Though no experimental test has been run on the emotive feelings of penguins during their homecoming, in the parade that we have seen live, it indeed happened that one little penguin remained isolated from the others. Even though it was quite fit, and no concrete obstacle was obstructing its motion, it became completely stuck for half an hour, and the staff of the Nature Park had to go and provide assistance. We stress again that the fact that the penguin stopped did not seem to be caused by

any physical impediment (as confirmed to us by the Ranger on site), since no extreme environmental condition was occurring, the animal was not underweight, and was able to come out of the water and move effortlessly on the shore autonomously for about 15 metres, before suddenly stopping. For a short video (courtesy of Phillip Island Nature Parks) of the little penguin parade, in which the formation of groups is rather evident; see, for example, the file `Penguins1.MOV`, available at <https://youtu.be/x488k4n3ip8>.

The simple features listed above are likely to be a consequence of the morphological structure of the little penguins and the natural environment. Little penguins are a marine-terrestrial species. They are highly efficient swimmers, but possess a rather inefficient form of locomotion on land (indeed, flightless penguins, like the ones on Phillip Island, waddle rather than walk). At dusk, about 80 minutes after sunset, according to the data collected by Rodríguez et al. [22], little penguins return ashore after their fishing activity in the sea. Since their bipedal locomotion is slow and rather goofy (not only from the human subjective point of view, but also in comparison to the velocity or agility that is well known to be typical of predators in nature), and since the easily recognizable countershading of the penguins is likely to make them visible to predators, the transition between the marine and the terrestrial environment may be particularly stressful for the penguins [17]. This fact is probably related to the formation of penguin groups (see, for example, [7]). Thus, in our opinion, the rules that we have listed may be seen as the outcome of the difficulty of the little penguins in performing their transition from a more favourable environment to a habitat in which their morphology turns out to be suboptimal.

At the moment, there seems to be no complete experimental evidence measuring the subjective perceptions of the penguins with respect to their environments. Nevertheless, given the swimming ability of the penguins and the environmental conditions, one may well conjecture that an area of high potential danger for a penguin is that adjacent to the shoreline, since this is a habitat which provides little or no shelter, and it is also an area of reduced visibility. In fact, to protect the penguins in this critical area next to the water's edge, the rangers on Phillip Island implemented a control on the presence of the foxes in the proximity of the shore, with the aim of limiting the number of possible predators.

**1.2. Comparison with the existing literature** To the best of our knowledge, there has been no specific mathematical attempt to provide a concise description of the penguin parade. The mathematical literature on penguins has mostly focused on the description of the heat flow in the penguins' feathers [9], the numerical analysis to mark animals for later identification [23], the statistics of the Magellanic penguins at sea [24], the hunting strategies of fishing penguins [13], and the isoperimetric arrangement of the Antarctic penguins to prevent the heat dispersion caused by the polar wind and the crystal structures and solitary waves produced by such arrangements [12, 20]. We remark that the climatic situation on Phillip Island is rather different from that of the Antarctic, and given the very mild temperatures of the area, we do not think that heat considerations should affect the behaviour and the moving

strategies of the Victorian little penguins too much; their tendency to cluster seems more likely to be a defensive strategy against possible predators.

Though no mathematical formulation of the little penguin parade has hitherto been given, a series of experimental analyses has recently been performed on the specific environment of Phillip Island. We recall, in particular, the work of Daniel et al. [7] in which the association of the little penguins in groups was described by collecting data spanning over several years, and the article of Chiaradia et al. [5] in which there was a description of the effect of fog on the orientation of the little penguins (who may actually not come back home in conditions of poor visibility). Further, Macintosh et al. [18] and Reynolds et al. [21] presented a data analysis to show the fractal structure in space and time for the foraging of the little penguins, in relation to Lévy flights and fractional Brownian motions.

For an exhaustive list of publications focused on the behaviour of the little penguins of Phillip Island, we refer to the website <https://www.penguins.org.au/conservation/research/publications/>. This contains more than 160 publications related to the environment of Phillip Island, with special emphasis on the biology of little penguins.

We recall that there is also a wide literature from the point of view of biology and ethology focused on collective mathematical behaviours, also in terms of formation of groups and hierarchies (see, for example, [3, 10, 19]). The mathematical literature studying the collective behaviour of animal groups is also rather broad. We mention, in particular, some works which analysed the local rules of interaction of individual birds in airborne flocks [1], the self-organization from a microscopic to a macroscopic scale [6], movements with a speed depending on an additional variable [4], and different models on opinion formation within an interacting group [15].

We remark that our model is specifically tailored to the Phillip Island penguins. Other colonies of penguins, such as those on St Kilda, exhibit behaviours different from those on Phillip Island, due to the different environmental conditions (see, for example, the scientific report by [14] for additional information on the penguin colony on the St Kilda breakwater).

**1.3. Mathematical formulation** In this section we provide a mathematical description of the penguin parade which was described in Section 1.1. The idea of providing an equation for this parade is to prescribe that the velocity of a group of penguins travelling in a line is influenced by the natural environment and by the position of the other visible groups. Whenever a group is formed, the equation needs to be modified to encode the formation of this new structure. The main mathematical notation is described in Table 1.

Further, in order to translate the simple observations on the penguins' behaviour given in Section 1.1 into a mathematical framework, we propose the following equation:

$$\dot{p}_i(t) = \mathfrak{F}_i(p(t), w(t); t) (\varepsilon + \mathcal{V}_i(p(t), w(t); t)) + f(p_i(t), t). \quad (1.1)$$

The variable  $t \geq 0$  represents time, and  $p(t)$  is a vector-valued function of time that takes into account the positions of the different groups of penguins. Roughly speaking,

TABLE 1. Notation.

$p_i(t)$	one-dimensional position of the $i$ th group of penguins at time $t$
$w_i(t)$	number of penguins belonging to the $i$ th group of penguins at time $t$
$f$	function describing the environment (sea, shore, presence of predators, etc.)
$\mathfrak{F}_i$	stopping function
$\varepsilon$	speed of a solitary penguin in a neutral condition (may be zero)
$\mathcal{V}_i$	strategic speed of the $i$ th group of penguins (depending on the position of the penguins, on the size of the group and on time)
$v$	speed of “large” groups of penguins
$m_i$	influence of the “visible” penguins ahead and behind on the speed of the $i$ th group
$s$	eyesight of the penguins

at time  $t$ , there are  $n(t)$  groups of penguins, therefore  $p(t)$  is an array with  $n(t)$  components, and so we write

$$p(t) = (p_1(t), \dots, p_{n(t)}(t)). \quad (1.2)$$

We stress that  $n(t)$  may vary in time (in fact, it will be taken to be piecewise constant), hence the spatial dimension of the image of  $p$  is also a function of time. For any  $i \in \{1, \dots, n(t)\}$ , the  $i$ th group of penguins contains a number of penguins denoted by  $w_i(t)$  (thus, the number of penguins belonging to each group is also a function of time). In further detail, the following notation is used. The function  $n : [0, +\infty) \rightarrow \mathbb{N}_0$ , where  $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$ , is piecewise constant and nonincreasing, that is, there exist a (possibly finite) sequence  $0 = t_0 < t_1 < \dots < t_j < \dots$  and integers  $n_1 > \dots > n_j > \dots$ , such that

$$n(t) = n_j \in \mathbb{N}_0 \quad \text{for any } t \in (t_{j-1}, t_j). \quad (1.3)$$

In this model, for simplicity, the spatial occupancy of a cluster of penguins coincides with that of a single penguin: of course, in reality, there is a small repulsion playing among the penguins, which cannot stay too close to one another. This additional complication may also be taken into account in our model, by enlarging the spatial size of the cluster in dependence of the numerosness of the penguins in the group. In any case, for practical purposes, we think it is not too inaccurate to identify a group of penguins with just a single element, since the scale at which the parade occurs (several tens of metres) is much larger than the size of a single penguin (little penguins are only about 30 cm tall).

We also consider the array  $w(t) = (w_1(t), \dots, w_{n(t)}(t))$ . We assume that  $w_i$  is piecewise constant,  $w_i(t) = \bar{w}_{i,j}$  for any  $t \in (t_{j-1}, t_j)$ , for some  $\bar{w}_{i,j} \in \mathbb{N}_0$ , that is, the number of little penguins in each group remains constant, until the next penguins join the group at time  $t_j$  (if, for the sake of simplicity, one wishes to think that initially all the little penguins are separated from one another, then one may also suppose that  $w_i(t) = 1$  for all  $i \in \{1, \dots, n_1\}$  and  $t \in [0, t_1)$ ).

By possibly renaming the variables, we suppose that the initial position of the groups is increasing with respect to the index, that is,

$$p_1(0) < \dots < p_{n_1}(0). \tag{1.4}$$

The parameter  $\varepsilon \geq 0$  represents the drift velocity of the penguins towards their home, which is located at the point  $H \in (0, +\infty)$ . The parameter  $\varepsilon$ , from the biological point of view, represents the fact that each penguin, in a neutral situation, has a natural tendency to move towards its burrow. We can also allow  $\varepsilon = 0$  in our treatment; in this case, the existence and uniqueness theory in Section 2 remains unchanged if  $\varepsilon = 0$ , and the rigorous results in Section 3 present cases in which they still hold true when  $\varepsilon = 0$ , compared, in particular, with assumptions (3.1) and (3.3). For concreteness, if  $p_i(T) = H$  for some  $T \geq 0$ , we can set  $p_i(t) = H$  for all  $t \geq T$  and remove  $p_i$  from the equation of motion, that is, the penguin has safely returned home.

For any  $i \in \{1, \dots, n(t)\}$ , the quantity  $\mathcal{V}_i(p(t), w(t); t)$  represents the strategic velocity of the  $i$ th group of penguins, and it can be considered as a function with domain varying in time,  $\mathcal{V}_i(\cdot, \cdot; t) : \mathbb{R}^{n(t)} \times \mathbb{N}^{n(t)} \rightarrow \mathbb{R}$ , that is,

$$\mathcal{V}_i(\cdot, \cdot; t) : \mathbb{R}^{n_j} \times \mathbb{N}^{n_j} \rightarrow \mathbb{R} \quad \text{for any } t \in (t_{j-1}, t_j),$$

and for any  $(p, w) = (p_1, \dots, p_{n(t)}, w_1, \dots, w_{n(t)}) \in \mathbb{R}^{n(t)} \times \mathbb{N}^{n(t)}$ , it is of the form

$$\mathcal{V}_i(p, w; t) = (1 - \mu(w_i)) m_i(p, w; t) + v\mu(w_i). \tag{1.5}$$

In this setting, for any  $(p, w) = (p_1, \dots, p_{n(t)}, w_1, \dots, w_{n(t)}) \in \mathbb{R}^{n(t)} \times \mathbb{N}^{n(t)}$ ,

$$m_i(p, w; t) = \sum_{j \in \{1, \dots, n(t)\}} \text{sign}(p_j - p_i) w_j \mathfrak{s}(|p_i - p_j|), \tag{1.6}$$

where  $\mathfrak{s} \in \text{Lip}([0, +\infty))$  is nonnegative and nonincreasing, and as usual, we denote the “sign function” as

$$\mathbb{R} \ni r \mapsto \text{sign}(r) = \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r = 0, \\ -1 & \text{if } r < 0. \end{cases}$$

Also, for any  $\ell \in \mathbb{N}$ , we set

$$\mu(\ell) = \begin{cases} 1 & \text{if } \ell \geq \kappa, \\ 0 & \text{if } \ell \leq \kappa - 1, \end{cases} \tag{1.7}$$

for a fixed  $\kappa \in \mathbb{N}$  with  $\kappa \geq 2$  and  $v > \varepsilon$ .

In our framework, the meaning of the strategic velocity of the  $i$ th group of penguins is the following. When the group of penguins is too small, that is, it contains fewer than  $\kappa$  little penguins, the term involving  $\mu$  vanishes. Thus the strategic velocity reduces to the term given by  $m_i$ ; this term, in turn, takes into account the position of the other groups of penguins. That is, each penguin is endowed with an “eyesight” (i.e., the capacity of seeing the other penguins that are “sufficiently close” to them), which is modelled by the function  $\mathfrak{s}$ . For instance, if  $\mathfrak{s}$  is identically equal to 1 then the penguin

has a “perfect eyesight”, if  $\varsigma(r) = e^{-r^2}$  then the penguin sees close objects much better than distant ones, and if  $\varsigma$  is compactly supported then the penguin does not see objects that are too far, etc. Based on the position of the other mates that it sees, the penguin has the tendency to move either forward or backward: the more penguins it sees ahead, the more inclined it is to move forward; the more penguins it sees behind, the more inclined it is to move backward; nearby penguins are weighted more than distant ones, due to the monotonicity of the function  $\varsigma$ . This strategic tension coming from the position of the other penguins is encoded by the function  $m_i$ .

The eyesight function can be also considered as a modification of the interaction model based simply on metric distance. Another interesting feature which has been observed in several animal groups (see, for example, [1]), is the so-called “topological interaction” model, in which every agent interacts only with a fixed number of agents among the ones which are closer. A modification of the function  $\varsigma$  can also take this possibility into account. It is of course very interesting to investigate by direct observations how much topological, quantitative and metric considerations influence the formation and the movement of little penguin clusters.

When the group of penguins is sufficiently large (that is, it contains at least  $\kappa$  little penguins), then the term involving  $\mu$  is equal to 1. In this case, the strategic velocity is  $v$ , that is, when the group of penguins is sufficiently rich in population, its strategy is to move forward with cruising speed equal to  $v$ .

The function  $\mathfrak{F}_i(p(t), w(t); t)$  describes the case of extreme isolation of the  $i$ th individual from the rest of the herd. Here, we take  $\bar{d} > \underline{d} > 0$ , a nonincreasing function  $\varphi \in \text{Lip}(\mathbb{R}, [0, 1])$  with  $\varphi(r) = 1$  if  $r \leq \underline{d}$  and  $\varphi(r) = 0$  if  $r \geq \bar{d}$ , and for any  $\ell \in \mathbb{N}_0$ ,

$$w(\ell) = \begin{cases} 1 & \text{if } \ell \geq 2, \\ 0 & \text{if } \ell = 1. \end{cases} \tag{1.8}$$

As our stopping function with variable domain we take

$$\mathfrak{F}_i(\cdot, \cdot; t) : \mathbb{R}^{n(t)} \times \mathbb{N}^{n(t)} \rightarrow [0, 1],$$

that is,

$$\mathfrak{F}_i(\cdot, \cdot; t) : \mathbb{R}^{n_j} \times \mathbb{N}^{n_j} \rightarrow [0, 1] \quad \text{for any } t \in (t_{j-1}, t_j),$$

given by

$$\mathfrak{F}_i(p, w; t) = \max \left\{ w(w_i), \max_{\substack{j \in \{1, \dots, n(t)\} \\ j \neq i}} \varphi(|p_i - p_j|) \right\} \tag{1.9}$$

for any  $(p, w) = (p_1, \dots, p_{n(t)}, w_1, \dots, w_{n(t)}) \in \mathbb{R}^{n(t)} \times \mathbb{N}^{n(t)}$ . Here the notation “Lip” stands for bounded and Lipschitz continuous functions. The case of  $\varphi$  being identically equal to 1 can also be contained in our setting. In this case,  $\mathfrak{F}_i$  is also identically equal to 1, corresponding to the case where the stopping function has no effect.

The stopping function describes the fact that the group may have a tendency to suddenly stop. This happens when the group contains only one element (that is,  $w_i = 0$ ), and the other groups are far apart (that is, at distance larger than  $\bar{d}$ ). Conversely, if the group contains at least two little penguins, or if there is at least one



other group sufficiently close (say, at distance smaller than  $d$ ), then the group is self-confident, namely, the function  $\mathfrak{B}_i(p(t), w(t); t)$  is equal to 1 and the total intentional velocity of the group coincides with the strategic velocity.

Interestingly, the stopping function  $\mathfrak{B}_i$  may be independent of the eyesight function  $\mathfrak{s}$ , namely, a little penguin can stop if it feels too exposed, even if it can see other little penguins (for instance, if  $\mathfrak{s}$  is identically equal to 1, the little penguin always sees the other members of the herd, and can stop if they are too far away).

The function  $f \in \text{Lip}(\mathbb{R} \times [0, +\infty))$  takes into account the environment. For a neutral environment, this term vanishes (here “neutral” means that the environment does not favour or penalize the homecoming of the penguins). In practice, it may take into account the ebb and flow of the sea on the foreshore where the little penguin parade starts, the possible ruggedness of the terrain, the presence of predators, etc. (as a variation, one can consider also a stochastic version of this term). This environment function can take into account several characteristics at the same time. For example, a possible situation that we wish to model is that the sea occupies the spatial region  $(-\infty, 0)$ , producing waves that are periodic in time with frequency  $\varpi$  and amplitude  $\delta$ . Suppose also that the shore is located in the spatial region  $(-\infty, 0)$ , presenting a steep hill in the region  $(1, 2)$  which can slow down the motion of the penguins, whose burrows are located at the point 4. In this setting, a possible choice of the environment function  $f$  is

$$\mathbb{R} \times [0, +\infty) \ni (p, t) \mapsto f(p, t) = \delta \sin(\varpi t + \phi) \chi_{(-\infty, 0)}(p) - h \chi_{(1, 2)}(p).$$

In this notation,  $h > 0$  is a constant that takes into account “how steep” the hill located in the region  $(1, 2)$  is,  $\phi \in \mathbb{R}$  is an initial phase of the wave in the sea, and  $\chi_E$  is the characteristic function of a set  $E$ , namely,

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Given the interpretations above, equation (1.1) comprises the pattern that we described in words and sets the scheme of motion of the little penguins into a mathematical framework.

**1.4. Preliminary presentation of the mathematical results** In this paper, three main mathematical results are presented. First of all, in Section 2, we provide an existence and uniqueness theory for the solutions of equation (1.1). From a mathematical viewpoint, we remark that (1.1) does not fall into the classical framework of the standard Cauchy initial value problem for ordinary differential equations (for example, compare with formula (2.3) and [2, Theorem 2.1]), since the right-hand side of the equation is not Lipschitz continuous (and, in fact, is not even continuous). This mathematical complication is indeed the counterpart of the real motion of the little penguins in the parade, which have the tendency to change their speed rather abruptly to maintain contact with the other elements of the herd. That is, in our view, it does not seem unreasonable to model, as a simplification, the speed of the

penguin as a discontinuous function, to take into account the sudden modifications of the waddling according to the position of the other penguins with the conclusive aim of gathering together a sufficient number of penguins in a group which eventually will march concurrently in the direction of their burrows.

Then, in Section 3, we provide two rigorous results which guarantee suitable conditions under which all the penguins, or some of them, safely return to their burrows. In Theorem 3.1 we establish that if the sum of the drift velocity and the environmental function is strictly positive, and if there is a time (which can be the initial time or a subsequent one) for which the group at the end of the line consists of at least two penguins, then all the penguins reach their burrows in a finite time, which can be explicitly estimated.

Also, in Theorem 3.2 we prove that if the sum of the drift and cruise velocities and the environmental function is strictly positive, and if there is a time for which one of the penguin groups is sufficiently numerous, then all the penguins in this group and in the groups ahead safely return home in a finite time, which can be explicitly estimated. Rigorous statements and proofs will be given in Sections 2 and 3.

**1.5. Detailed organization of the paper** The mathematical treatment of equation (1.1) that we provide in this paper is the following. In Section 2 we provide a notion of solution for which (1.1) is uniquely solvable in the appropriate setting. This notion of solution will be obtained by a “stop-and-go” procedure, which is compatible with the idea that when two (or more) groups of penguins meet, they form a new, bigger group which moves coherently with the rest of the march.

In Section 3 we discuss a couple of concrete examples in which the penguins are able to safely return home, that is, we show that there are “nice” conditions in which the strategy of the penguins allows a successful homecoming. In Section 4 we present a series of numerical simulations to compare our mathematical model with the real-world experience. This part also contains some figures produced by the numerics. Several possible structural generalizations of the model proposed are presented in Section 5. Furthermore, the model that we propose can be easily generalized to a multi-dimensional setting, as discussed in Section 6. Section 7 concludes our discussion.

## 2. Existence and uniqueness theory for equation (1.1)

We stress that equation (1.1) does not lie within the setting of ordinary differential equations, since the right-hand side is not Lipschitz continuous (due to the discontinuity of the functions  $w$  and  $m_i$ , and in fact the right-hand side also involves functions with domain varying in time). As far as we know, the weak formulations of ordinary differential equations as the ones treated by [8] do not take into consideration the setting of equation (1.1), so we briefly discuss here a direct approach to the existence and uniqueness theory for such equation. To this end, and to clarify our direct approach, we present two illustrative examples [11].

**EXAMPLE 2.1.** Setting  $x : [0, +\infty) \rightarrow \mathbb{R}$ , the ordinary differential equation

$$\dot{x}(t) = \begin{cases} -1 & \text{if } x(t) \geq 0, \\ 1 & \text{if } x(t) < 0, \end{cases} \tag{2.1}$$

is not well posed. Indeed, taking an initial datum  $x(0) < 0$ , it will evolve with the formula  $x(t) = t + x(0)$  for any  $t \in [0, -x(0)]$  till it hits the zero value. At that point, equation (2.1) would prescribe a negative velocity, which becomes contradictory with the positive velocity prescribed for the negative coordinates.  $\square$

**EXAMPLE 2.2.** The ordinary differential equation

$$\dot{x}(t) = \begin{cases} -1 & \text{if } x(t) > 0, \\ 0 & \text{if } x(t) = 0, \\ 1 & \text{if } x(t) < 0, \end{cases} \tag{2.2}$$

is similar to the one in (2.1), in the sense that it does not fit into the standard theory of ordinary differential equations, due to the lack of continuity of the right-hand side. But, unlike the one in (2.1), it can be set into an existence and uniqueness theory by a simple “reset” algorithm.

Namely, taking an initial datum  $x(0) < 0$ , the solution evolves with the formula  $x(t) = t + x(0)$  for any  $t \in [0, -x(0)]$  till it hits the zero value. At that point, equation (2.2) would prescribe a zero velocity, thus a natural way to continue the solution is to take  $x(t) = 0$  for any  $t \in [-x(0), +\infty)$  (similarly, in the case of positive initial datum  $x(0) > 0$ , a natural way to continue the solution is  $x(t) = -t + x(0)$  for any  $t \in [0, x(0)]$  and  $x(t) = 0$  for any  $t \in [x(0), +\infty)$ ). The basic idea for this continuation method is to flow the equation according to the standard Cauchy theory of ordinary differential equations for as long as possible, and then, when the classical theory breaks, “reset” the equation with respect of the datum at the break time. This method is not universal, and indeed it does not work for (2.1), but it produces a natural global solution for (2.2).  $\square$

In the light of Example 2.2, we now present a framework in which equation (1.1) possesses a unique solution (in a suitable “reset” setting). To this end, we first notice that the initial number of groups of penguins is fixed to be equal to  $n_1$  and each group is given by a fixed number of little penguins packed together, that is, the number of little penguins in the  $i$ th initial group being equal to  $\bar{w}_{i,1}$  and  $i$  ranges from 1 to  $n_1$ ). So we set  $\bar{w}_1 = (\bar{w}_{1,1}, \dots, \bar{w}_{n_1,1})$  and  $\bar{\omega}_{i,1} = \omega(\bar{w}_{i,1})$ , where  $\omega$  is as defined in (1.8). Also, for any  $p = (p_1, \dots, p_{n_1}) \in \mathbb{R}^{n_1}$ , let

$$\mathfrak{P}_{i,1}(p) = \max \left\{ \bar{\omega}_{i,1}, \max_{\substack{j \in \{1, \dots, n_1\} \\ j \neq i}} \varphi(|p_i - p_j|) \right\}. \tag{2.3}$$

The reader may compare this definition with the one in (1.9). For any  $i \in \{1, \dots, n_1\}$ , we also set

$$\bar{\mu}_{i,1} = \mu(\bar{w}_{i,1}),$$

where  $\mu$  is the function defined in (1.7), and, for any  $p = (p_1, \dots, p_{n_1}) \in \mathbb{R}^{n_1}$ ,

$$\bar{m}_{i,1}(p) = \sum_{j \in \{1, \dots, n_1\}} \text{sign}(p_j - p_i) \bar{w}_{j,1} \mathfrak{s}(|p_i - p_j|).$$

Compare this definition with (1.6). Recalling equation (1.4), we also set

$$\mathcal{D}_1 = \{p = (p_1, \dots, p_{n_1}) \in \mathbb{R}^{n_1} \text{ s.t. } p_1 < \dots < p_{n_1}\}.$$

We remark that if  $p \in \mathcal{D}_1$  then

$$\bar{m}_{i,1}(p) = \sum_{j \in \{i+1, \dots, n_1\}} \bar{w}_{j,1} \mathfrak{s}(|p_i - p_j|) - \sum_{j \in \{1, \dots, i-1\}} \bar{w}_{j,1} \mathfrak{s}(|p_i - p_j|),$$

and therefore,

$$\bar{m}_{i,1}(p) \text{ is bounded and Lipschitz for any } p \in \mathcal{D}_1. \tag{2.4}$$

Then we set

$$\mathcal{V}_{i,1}(p) = (1 - \bar{\mu}_{i,1}) \bar{m}_{i,1}(p) + v \bar{\mu}_{i,1}.$$

Compare this definition with the one in (1.5). Notice that in view of (2.4),

$$\mathcal{V}_{i,1}(p) \text{ is bounded and Lipschitz for any } p \in \mathcal{D}_1. \tag{2.5}$$

So we set

$$G_{i,1}(p, t) = \mathfrak{F}_{i,1}(p) (\varepsilon + \mathcal{V}_{i,1}(p)) + f(p_i, t).$$

From (2.3) and (2.5),  $G_{i,1}$  is bounded and Lipschitz in  $\mathcal{D}_1 \times [0, +\infty)$ . Consequently, from the global existence and uniqueness of solutions of ordinary differential equations, there exist  $t_1 \in (0, +\infty]$  and a solution  $p^{(1)}(t) = (p_1^{(1)}(t), \dots, p_{n_1}^{(1)}(t)) \in \mathcal{D}_1$  of the Cauchy problem

$$\begin{cases} \dot{p}_i^{(1)}(t) = G_{i,1}(p^{(1)}(t), t) & \text{for } t \in (0, t_1), \\ p^{(1)}(0) & \text{given in } \mathcal{D}_1, \end{cases}$$

and

$$p^{(1)}(t_1) \in \partial \mathcal{D}_1 \tag{2.6}$$

(see, for example, [16, Theorem 1.4.1]).

Note that, as is customary in the mathematical literature, we denoted by  $\partial$  the “topological boundary” of a set. In particular,

$$\begin{aligned} \partial \mathcal{D}_1 &= \{p = (p_1, \dots, p_{n_1}) \in \mathbb{R}^{n_1} \mid p_1 \leq \dots \leq p_{n_1} \\ &\text{and there exists } i \in \{1, \dots, n_1 - 1\} \text{ such that } p_i = p_{i+1}\}. \end{aligned}$$

Thus the idea for studying the Cauchy problem in our framework is that as long as the trajectory of the system stays in the interior of the domain  $\mathcal{D}_1$ , the forcing term remains uniformly Lipschitz, so that the flow does not develop any singularity. Hence, the trajectory exists, and it is defined up to the time (if any) at which it meets the

boundary of the domain  $\mathcal{D}_1$ , which in the biological framework corresponds to the situation in which two or more penguins meet (that is, they occupy the same position at the same time). In this case, the standard flow procedure of the ordinary differential equation is stopped, we will merge the coincident penguins into a common cluster, and then repeat the argument.

In further detail, the solution of equation (1.1) will be taken to be  $p^{(1)}$  in  $[0, t_1)$ , that is, we set  $p(t) = p^{(1)}(t)$  for any  $t \in [0, t_1)$ . We also set  $n(t) = n_1$  and  $w(t) = (\bar{w}_{1,1}, \dots, \bar{w}_{n_1,1})$ . With this setting, we have that  $p$  is a solution of equation (1.1) in the time range  $t \in (0, t_1)$  with prescribed initial datum  $p(0)$ . Condition (2.6) allows us to perform our “stop-and-go” reset procedure as follows. We denote by  $n_2$  the number of distinct points in the set  $\{p_1^{(1)}(t_1), \dots, p_{n_1}^{(1)}(t_1)\}$ . Notice that (2.6) says that if  $t_1$  is finite then  $n_2 \leq n_1 - 1$  (namely, at least two penguins have reached the same position). In this way, the set of points  $\{p_1^{(1)}(t_1), \dots, p_{n_1}^{(1)}(t_1)\}$  can be identified by the set of  $n_2$  distinct points, which we denote by  $\{p_1^{(2)}(t_1), \dots, p_{n_2}^{(2)}(t_1)\}$  with the convention that

$$p_1^{(2)}(t_1) < \dots < p_{n_2}^{(2)}(t_1).$$

For any  $i \in \{1, \dots, n_2\}$ , we also set

$$\bar{w}_{i,2} = \sum_{\substack{j \in \{1, \dots, n_1\} \\ p_j^{(1)}(t_1) = p_i^{(2)}(t_1)}} \bar{w}_{j,1}.$$

This says that the new group of penguins indexed by  $i$  contains all the penguins that have reached that position at time  $t_1$ .

Thus, having the “new number of groups”,  $n_2$ , the “new number of little penguins in each group”,  $\bar{w}_2 = (\bar{w}_{1,2}, \dots, \bar{w}_{n_2,2})$ , and the “new initial datum”,  $p^{(2)}(t_1) = (p_1^{(2)}(t_1), \dots, p_{n_2}^{(2)}(t_1))$ , we can solve a new differential equation with these new parameters, in exactly the same way as before, and keep iterating this process.

Indeed, recursively, we suppose that we have found  $t_1 < t_2 < \dots < t_k$ ,  $p^{(1)} : [0, t_1] \rightarrow \mathbb{R}^{n_1}, \dots, p^{(k)} : [0, t_k] \rightarrow \mathbb{R}^{n_k}$  and  $\bar{w}_1 \in \mathbb{N}_0^{n_1}, \dots, \bar{w}_k \in \mathbb{N}_0^{n_k}$ , such that setting

$$p(t) = p^{(j)}(t) \in \mathcal{D}_j, \quad n(t) = n_j$$

and  $w(t) = \bar{w}_j,$

for  $t \in [t_{j-1}, t_j)$  and  $j \in \{1, \dots, k\}$ , yields that  $p$  solves (1.1) in each interval  $(t_{j-1}, t_j)$  for  $j \in \{1, \dots, k\}$ , with the “stop condition”

$$p^{(j)}(t_j) \in \partial\mathcal{D}_j,$$

where

$$\mathcal{D}_j = \{p = (p_1, \dots, p_{n_j}) \in \mathbb{R}^{n_j} \mid p_1 < \dots < p_{n_j}\}.$$

Then, since  $p^{(k)}(t_k) \in \partial\mathcal{D}_k$ , if  $t_k$  is finite, we find  $n_{k+1} \leq n_k - 1$  such that the set of points  $\{p_1^{(k)}(t_k), \dots, p_{n_k}^{(k)}(t_k)\}$  coincides with a set of  $n_{k+1}$  distinct points, which we denote by  $\{p_1^{(k+1)}(t_k), \dots, p_{n_k}^{(k+1)}(t_k)\}$ , with the convention that

$$p_1^{(k+1)}(t_k) < \dots < p_{n_k}^{(k+1)}(t_k).$$

For any  $i \in \{1, \dots, n_{k+1}\}$ , we set

$$\bar{w}_{i,k+1} = \sum_{\substack{j \in \{1, \dots, n_k\} \\ p_j^{(k)}(t_k) = p_i^{(k+1)}(t_k)}} \bar{w}_{j,k}. \tag{2.7}$$

It is useful to observe that in light of (2.7),

$$\sum_{i \in \{1, \dots, n_{k+1}\}} \bar{w}_{i,k+1} = \sum_{i \in \{1, \dots, n_k\}} \bar{w}_{i,k},$$

which says that the total number of little penguins always remains the same; more precisely, the sum of all the little penguins in all groups is constant in time.

Let also  $\bar{w}_{i,k+1} = w(\bar{w}_{i,k+1})$ . Then, for any  $i \in \{1, \dots, n_{k+1}\}$  and any  $p = (p_1, \dots, p_{n_{k+1}}) \in \mathbb{R}^{n_{k+1}}$ , we set

$$\mathfrak{P}_{i,k+1}(p) = \max \left\{ \bar{w}_{i,k+1}, \max_{\substack{j \in \{1, \dots, n_{k+1}\} \\ j \neq i}} \varphi(|p_i - p_j|) \right\}.$$

For any  $i \in \{1, \dots, n_{k+1}\}$  we also define  $\bar{\mu}_{i,k+1} = \mu(\bar{w}_{i,k+1})$ , where  $\mu$  is the function defined in (1.7), and, for any  $p \in \mathbb{R}^{n_{k+1}}$ ,

$$\bar{m}_{i,k+1}(p) = \sum_{j \in \{1, \dots, n_{k+1}\}} \text{sign}(p_j - p_i) \bar{w}_{j,k+1} s(|p_i - p_j|).$$

We notice that  $\bar{m}_{i,k+1}(p)$  is bounded and Lipschitz for any  $p \in \mathcal{D}_{k+1}$  with  $\{p = (p_1, \dots, p_{n_{k+1}}) \in \mathbb{R}^{n_{k+1}} \mid p_1 < \dots < p_{n_{k+1}}\}$ .

We also define

$$\mathcal{V}_{i,k+1}(p) = (1 - \bar{\mu}_{i,k+1}) \bar{m}_{i,k+1}(p) + v \bar{\mu}_{i,k+1}$$

and

$$G_{i,k+1}(p, t) = \mathfrak{P}_{i,k+1}(p) (\varepsilon + \mathcal{V}_{i,k+1}(p)) + f(p_i, t).$$

In this way, we have that  $G_{i,k+1}$  is bounded and Lipschitz in  $\mathcal{D}_{k+1} \times [0, +\infty)$  and so we find the next solution  $p^{(k+1)}(t) = (p_1^{(k+1)}(t), \dots, p_{n_{k+1}}^{(k+1)}(t)) \in \mathcal{D}_{k+1}$  in the interval  $(t_k, t_{k+1})$ , with  $p^{(k+1)}(t_{k+1}) \in \partial \mathcal{D}_{k+1}$ , by solving the ordinary differential equation

$$\dot{p}_i^{(k+1)}(t) = G_{i,k+1}(p^{(k+1)}(t), t).$$

This completes the iteration argument and provides the desired notion of solution for equation (1.1).

### 3. Examples of safe return home

Here we provide some sufficient conditions for the penguins to reach their home, located at the point  $H$ , which is taken to be “far away with respect to the initial position of the penguins”, that is, we assume that

$$H > \max_{i \in \{1, \dots, n(0)\}} p_i(0),$$

and  $\varepsilon$  has to be taken sufficiently small. Recall that in the live parade that we saw, one little penguin remained stuck and did not manage to return home. So giving a mathematical treatment of the case in which the strategy of the penguins turns out to be successful reassured us as to the fate of the species.

To give a mathematical framework of the notion of homecoming, we introduce the function

$$[0, +\infty) \ni t \mapsto \mathcal{N}(t) = \sum_{\substack{j \in \{1, \dots, n(t)\} \\ p_j(t) = H}} w_j(t).$$

In the setting of Section 1.3, the function  $\mathcal{N}(t)$  represents the number of penguins that have safely returned home at time  $t$ . For counting reasons, we also point out that the total number of penguins is constant and is given by

$$\mathcal{M} = \sum_{j \in \{1, \dots, n(0)\}} w_j(0) = \sum_{j \in \{1, \dots, n(t)\}} w_j(t) \quad \text{for any } t \geq 0.$$

Our first result shows that if at some time the group of penguins that stays further behind gathers into a group of at least two elements, then all the penguins will manage to return home eventually. The mathematical setting is as follows.

**THEOREM 3.1.** *Let  $t_0 \geq 0$ , and assume that*

$$\varepsilon + \inf_{(r,t) \in \mathbb{R} \times [t_0, +\infty)} f(r, t) \geq \iota, \tag{3.1}$$

for some  $\iota > 0$ , and

$$w_1(t_0) \geq 2. \tag{3.2}$$

Then there exists  $T \in [t_0, t_0 + (H - p_1(t_0))/\iota]$  such that  $\mathcal{N}(T) = \mathcal{M}$ .

**PROOF.** We observe that  $w_1(t)$  is nondecreasing in  $t$ , by (2.7), and therefore (3.2) implies that  $w_1(t) \geq 2$  for any  $t \geq t_0$ . Consequently, from (1.8) we obtain  $w(w_1(t)) = 1$  for any  $t \geq t_0$ . This and (1.9) yield  $\mathfrak{F}_1(p, w(t); t) = 1$  for any  $t \geq t_0$  and  $p \in \mathbb{R}^{n(t)}$ . Accordingly, from the equation of motion in (1.1),

$$\begin{aligned} \dot{p}_1(t) &= \varepsilon + \mathcal{V}_1(p(t), w(t); t) + f(p_1(t), t) \\ &\geq \varepsilon + f(p_1(t), t) \\ &\geq \iota \end{aligned}$$

for any  $t \geq t_0$ , by (3.1). That is, for any  $j \in \{1, \dots, n(t)\}$ ,

$$p_j(t) \geq p_1(t) \geq \min\{H, p_1(t_0) + \iota(t - t_0)\},$$

which gives the desired result. □

A simple variation of Theorem 3.1 says that if at some time, a group of little penguins reaches a sufficiently large size, then all the penguins in this group (as well as the ones ahead) safely reach their home. The precise statement (whose proof is similar to that of Theorem 3.1, up to some technical modifications, and is therefore omitted) is as follows.

**THEOREM 3.2.** *Let  $t_o \geq 0$ , and assume that*

$$\varepsilon + \nu + \inf_{(r,t) \in \mathbb{R} \times [t_o, +\infty)} f(r,t) \geq \iota, \quad (3.3)$$

for some  $\iota > 0$ , and

$$w_{j_o}(t_o) \geq \kappa,$$

for some  $j_o \in \{1, \dots, n(t_o)\}$ , where  $\kappa$  is defined in (1.7). Then there exists  $T \in [t_o, t_o + (H - p_{j_o}(t_o))/\iota]$  such that

$$\mathcal{N}(T) \geq \sum_{j \in \{j_o, \dots, n(t_o)\}} w_j(t_o).$$

#### 4. Pictures, videos and numerics

Here, we present some simple numerical experiments to facilitate the intuition behind the model presented in (1.1). These simulations may actually show some of the typical traits of the little penguin parade, such as the oscillations and sudden change of direction, the gathering of the penguins into clusters and the possibility that some elements of the herd remain isolated, either on the land or in the sea.

The possibility that a penguin remains isolated in the sea is a very real one, as demonstrated by the last penguin in the herd on the video (courtesy of Phillip Island Nature Parks) named Penguins2.MOV available online at [https://youtu.be/dVkluYbH\\_Xc](https://youtu.be/dVkluYbH_Xc).

In our simulations, for the sake of simplicity, we considered 20 penguins returning to their burrows from the shore – some of the penguins may start their trip from the sea (which occupies the region below level 0 in the simulations) in which waves and currents may affect the movements of the animals. All the figures presented here have the time variable on the horizontal axis and the space variable on the vertical axis (with the burrows of the penguin community set at level 4 for definiteness). The figures are self-explanatory. For instance, in Figure 1, we present a case in which, fortunately, all the little penguins manage to safely return home, after having gathered into groups: as a matter of fact, in the first of these pictures all the penguins safely reach home together at the same time (after having rescued the first penguin, who stayed still for a long period due to isolation). The second of these pictures shows that a first group of penguins, which originated from the animals that were on the land at the initial time, reaches home slightly before the second group of penguins, which originated from the animals that were in the sea at the initial time. Also notice that the motion of the penguins in the sea appears to be affected by waves and currents.

We also observe a different scenario depicted in Figure 4 with two different functions to represent the currents in the sea; in this situation, a large group of 18 penguins gathers together, collecting also penguins who were initially in the water, and safely returns home. Two penguins remain isolated in the water, and they keep slowly moving towards their final destination; they reach home after a longer time.



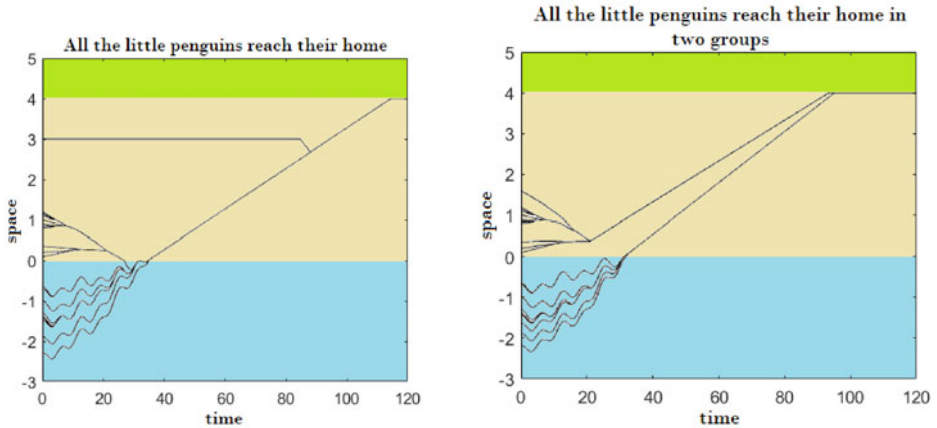


FIGURE 1. All the little penguins safely return home.

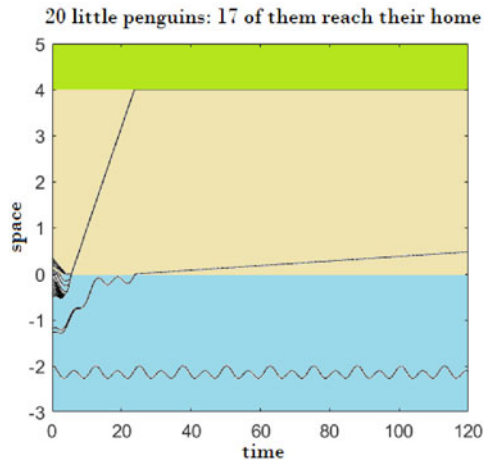


FIGURE 2. One penguin remains in the water.

Similarly, in Figure 2, almost all the penguins gather into a single group and reach home, while two penguins get together in the sea, come ashore and slowly waddle towards their final destination, and one single penguin remains isolated in the water, moved by the currents.

The situation in Figure 3 is slightly different, since the last penguin at the beginning moves towards the others, but does not manage to join the forming group by the time the other penguins decide to move consistently towards their burrows – so unfortunately this last penguin, in spite of its initial effort, finally remains in the water.

With simple modifications of the function  $f$ , one can also consider the case in which the waves of the sea change with time and their influence may become more (or less)

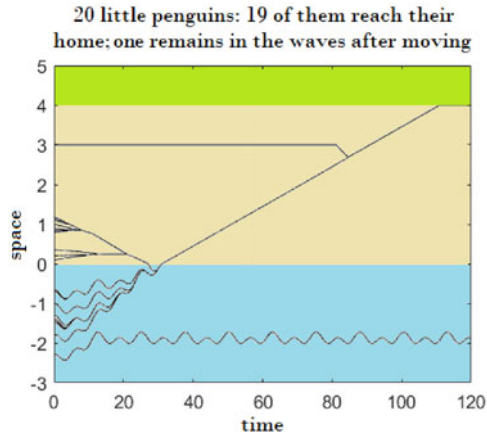


FIGURE 3. One penguin moves towards the others but remains in the water.

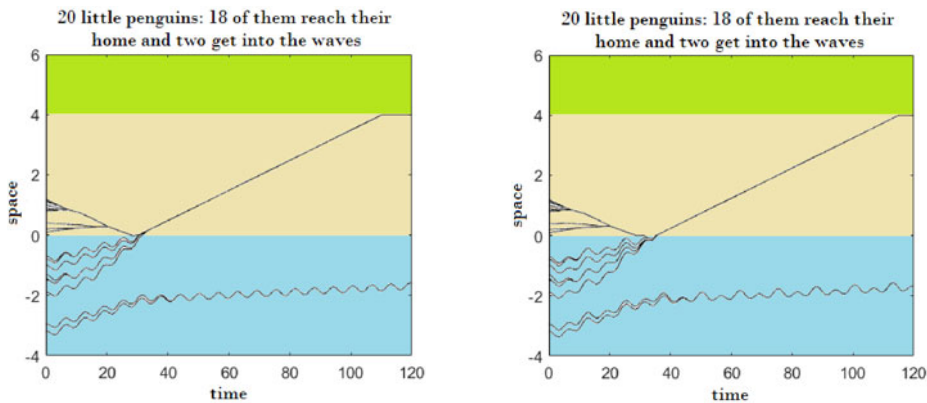


FIGURE 4. Two penguins are still in the water after a long time.

relevant for the swimming of the little penguins (for an example of this feature, see Figures 5 and 6).

In Figures 7 and 8 we give some examples of what happens when varying the parameters that we used in the numerics of the other figures. For example, we consider different values of  $\kappa$ , the parameter which encodes when a group of penguins is big enough to be self-confident and waddle home without being influenced by the other groups of penguins in sight.

By considering small values of  $\kappa$ , we represent a strong preference of the penguins to go straight towards their home, instead of first trying to form a large group. This situation is depicted in the second picture in Figure 8, where we see that after a short time the penguins form two distinct small groups and go towards home without trying to form a unique large group together. In contrast, considering a large value

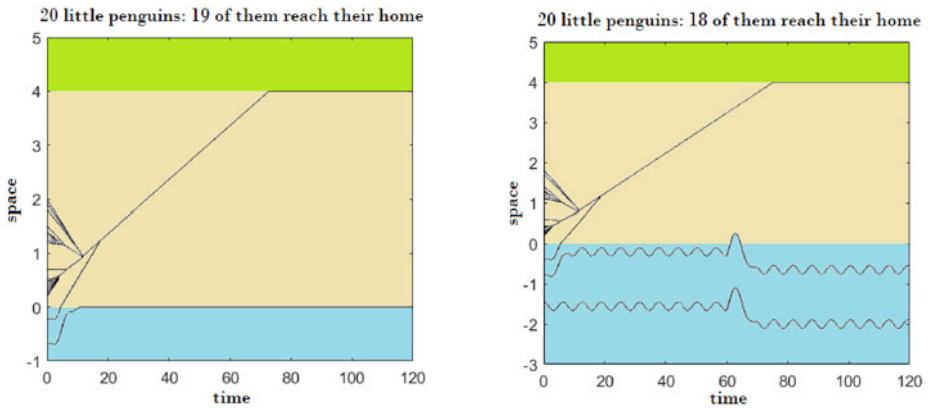


FIGURE 5. Effect of the waves on the movement of the penguins in the sea.

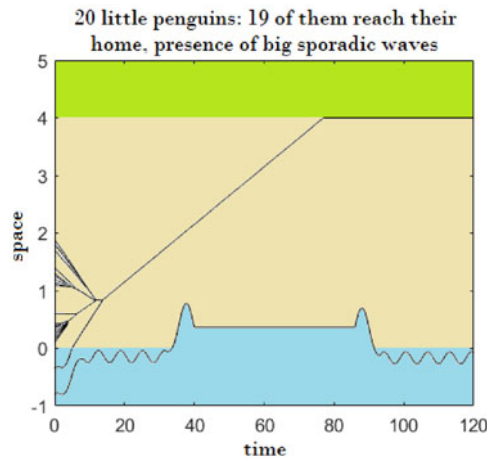


FIGURE 6. Effect of the waves on the movement of the penguins in the sea.

of  $\kappa$  represents the preference of the penguins to gather in a very large group before starting their march towards home, as in the first picture in Figure 8. This situation could represent, for example, the penguins being timorous because of the presence of predators.

We think that the case in which one penguin, or a small number of penguins, remain(s) in water even after the return of the main group is worth further investigation by means of concrete experiments. One possible scenario is that the penguins in the water may just wait long enough for other penguins to get close to the shore and join them to form a new group. On the other hand, if all the other penguins have already returned, the few who remained in the water may have to accept the risk of returning

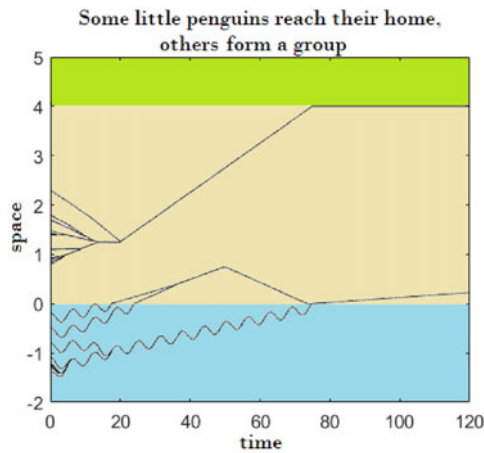


FIGURE 7. The penguins form smaller groups and move towards their home.

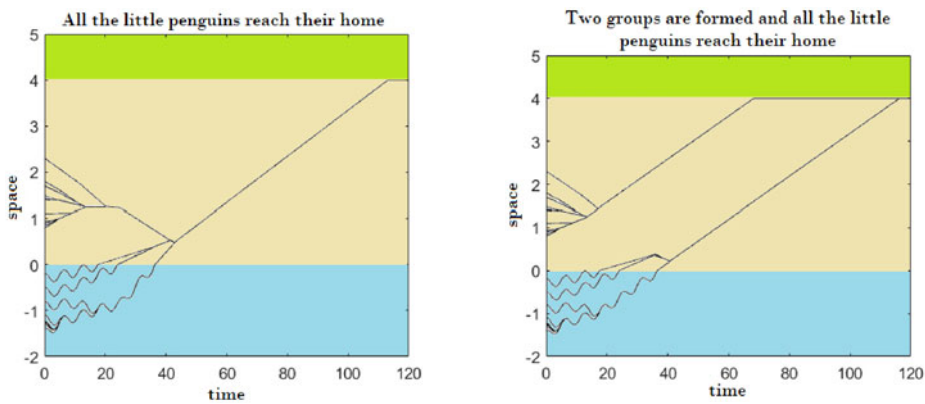


FIGURE 8. The penguins form groups of different sizes and reach their home.

home isolated from the other conspecifics and in an unprotected situation, and we think that interesting biological features could be detected in this case.

Finally, we recall that once a group of little penguins is created, it moves consistently altogether. This is of course a simplifying assumption, and it might happen in reality that one or a few penguins leave a large group after its formation – perhaps because one penguin is slower than the other penguins in the group, or perhaps because it gets distracted by other events on the beach, or simply because it feels too exposed being at the side of the group and may prefer to form a new group in which it finds a more central and protected position. We plan to describe this case in detail in a forthcoming project (also possibly in light of morphological and social considerations and taking into account a possible randomness in the system).

The situation in which one little penguin seems to think about leaving an already formed group can be observed in the video (courtesy of Phillip Island Nature Parks) named `Penguins2.MOV` and available online at [https://youtu.be/dVk1uYbH\\_Xc](https://youtu.be/dVk1uYbH_Xc) (see in particular the behaviour of the second penguin from the bottom, that is, the last penguin of the already formed large cluster). We point out that all these pictures have been easily obtained by short programs in MATLAB.

We describe the algorithm of the basic program here, with waves of constant size and standard behaviour of all the little penguins. The modified versions (periodic strong waves, tired little penguins and so on) can be easily inferred from it. We take into account  $N$  little penguins, we set their house at  $H = 4$  and the sea below the location 0. Strong waves can go beyond the location 0 in some cases, but in the standard program we just consider normal ones. We take a small  $\varepsilon$  to represent the natural predisposition of the little penguins to return home, and we define a constant  $\delta = (N + 1)\varepsilon$  that we need to define the velocity of the little penguins. We define the waves as  $WAVE = \delta \sin(T)$ , where  $T$  is the array of times. The speed of the animals is related to the one of waves in such a way that it becomes the strongest just when the little penguins form a group that is big enough.

The program starts with a “for” loop that counts all the animals in a range near the chosen little penguin. This “for” loop gives us two values: the indicator of the parameter PAN (short for “panic”) and the function  $W$ , which represents the number of animals in the same position as the one we are considering. We needed this function since we have seen that when the little penguins form a group that is big enough, they proceed towards their home with a cruise speed that is higher than it was before. We define this cruise speed as  $vc$  (short for “velocity”) in the program.

Then we start computing the speed  $V$  of the little penguin. If PAN is equal to zero, the little penguin freezes. His velocity is zero if he is on the shore (that is, his position is greater or equal than zero), or it is given by the waves if he is in the water. It is worth noting that at each value of time the “for” loop counts the value of PAN, hence a little penguin can leave the stopping condition, if it sees some mates and start moving again.

If PAN is not zero we have two cases, according to the fact that a big group is formed or not. If this has happened, that is,  $W > N/2$ , then the little penguin we are considering is in the group, so he goes towards home with a cruise speed  $vc$ , possibly modified by the presence of waves. If the group is not formed yet, the animal we are considering is surrounded by some mates, but there are not enough of them to proceed straight home. Its speed is positive or negative, that is, it moves forward or backward, depending on the number of little penguins it has ahead of or behind it. Its speed is given by

$$V = \varepsilon + M,$$

where  $M$  is the number of penguins ahead of it minus the number of animals behind it multiplied by  $\delta/N$ , and  $\varepsilon$  is as already defined. As in the other cases, the speed can be modified by the presence of waves, if the position is less than zero. Now that we have

computed the speed of the animal, we can obtain its position  $P$  after a discrete time interval  $t$  by considering  $P(k+1) = P(k) + Vt$ .

The purpose of the last “for” loop is to put in the same position two animals that are close enough. Then we reset the counting variables PAN, W and M, and we restart the loop. For completeness, we have made the source code for all the programs available at <https://www.dropbox.com/sh/odgic3a0ke5qp0q/AABIMaasAcTwZQ3qKRoB--xra?dl=0>.

An example of the code is given in the [Appendix](#). The simplicity of these programs shows that the model in (1.1) is indeed very simple to implement numerically, still producing sufficiently “realistic” results in terms of cluster formation and cruising speed of the groups. The parameters in the code are chosen as examples, producing simulations that show some features similar to those observed on site and in the videos. From one picture to another, what is varying are the initial conditions and the environment function (minor modifications in the code would allow us also to change the number of penguins, their eyesight, the drift and cruise velocities, the stopping function, and also to take into account multi-dimensional cases).

Also, these pictures can be easily translated into animations. Simple videos that we have obtained by these numerics are available at <https://www.youtube.com/playlist?list=PLASZVs0A5ReZgEinpnJFat66lo2kIkWTS>.

The source code of the animations is available online at <https://www.dropbox.com/s/11z5riqtc8jzxls/scatter.txt?dl=0>.

## 5. Discussion on the model proposed: simplifications, generalizations and further directions of investigation

We stress that the model proposed in (1.1) is of course a dramatic simplification of “reality”. As indeed often happens in science, several simplifications have been adopted in order to allow a rigorous mathematical treatment and convenient numerical computations. Nevertheless the model is already rich enough to detect some specific features of the little penguin parade, such as the formation of groups, the oscillatory waddling of the penguins and the possibility of isolated and exposed individuals. Moreover, our model is flexible enough to allow specific distinctions between the single penguins (for instance, with minor modifications, one can take into account the possibility that different penguins have different eyesight, that they have a different reaction to isolation, or that they exhibit some specific social behaviour that favours the formation of clusters selected by specific characteristics); similarly, the modelling of the habitat may also encode different possibilities (such as the burrows of the penguins being located in different places), and multi-dimensional models can be also constructed using similar ideas (see Section 6 for details). We observe that the quantities  $v, s, \mu, \kappa, \varphi$  can be replaced with  $v_i, s_i, \mu_i, \kappa_i, \varphi_i$ , if one wants to customize these features for every group.

Furthermore, natural modifications lead to the possibility that one or a few penguins may leave an already formed group, for instance, rather than forming one single group, the model can still consider the penguins of the cluster as separate elements, each one

with its own peculiar behaviour. At the moment, for simplicity, we consider here the basic model in which, once a cluster is formed, it keeps moving without losing any of its elements – we plan to address in detail in a future project the case of groups which may also decrease in size, possibly depending on random fluctuations or social considerations among the members of the group.

In addition, for simplicity, in this paper we modelled each group as located at a precise point: though this is not a completely unrealistic assumption (given that the scale of the individual penguin is much smaller than that of the beach), one can also easily modify this feature by locating a cluster in a region comparable to its size.

In future projects, we plan to introduce other more sophisticated models, also taking into account stochastic oscillations and optimization methods, and, in the long run, to use these models in a detailed experimental analysis taking advantage of the automated monitoring systems which is under development on Phillip Island.

The model that we propose here is also flexible enough to allow quantitative modifications of all the parameters involved. This is quite important, since these parameters may vary due to different conditions of the environment. For instance, the eyesight of the penguins can be reduced by fog [5] and by the effect of moonlight and artificial light [22].

Similarly, the number of penguins in each group and the velocity of the herd may vary due to structural changes to the beach. Roughly speaking, from the empirical data, penguins typically gather into groups of 5–10 individuals (but we have also observed much larger groups forming on the beach) within 40-second intervals [7], but the way these groups are built varies year by year and, for instance, the number of individuals which always gather in the same group changes year by year in strong dependence on the breeding success of the season [7]. Also, tidal phenomena may change the number of little penguins in each group and the velocity of the group, since the change to the beach width alters the penguins' perception of risk. For instance, a low tide produces a larger beach, with higher potential risk of predators, thus making the penguins gather in groups of larger size [17].

## 6. Multi-dimensional models

It is interesting to note that the model in (1.1) can be easily generalized to the multi-dimensional case. That is, for any  $i \in \{1, \dots, n(t)\}$ , the  $i$ th coordinate  $p_i$  can be taken to have image in some  $\mathbb{R}^d$ . More generally, the dimension of the target space can also vary in time, by allowing, for any  $i \in \{1, \dots, n(t)\}$ , the  $i$ th coordinate  $p_i$  to range in some  $\mathbb{R}^{d_i(t)}$ , with  $d_i(t)$  piecewise constant, namely  $d_i(t) = d_{i,j} \in \mathbb{N}_0$  for any  $t \in (t_{j-1}, t_j)$  (compare with (1.3)).

This modification just causes a small notational complication in (1.2), since each  $p_i(t)$  would now be a vector in  $\mathbb{R}^{d_i(t)}$  and the array  $p(t)$  would now be of dimension  $d_1(t) + \dots + d_{n(t)}$ . While we do not indulge in this generalization here, we observe that such a mathematical extension may be useful, in practice, to consider the specific location of the burrows and describe, for instance, the movements of the penguins on

the beach (say, a two-dimensional surface) which, as time flows, gather together in a single queue and then move in a one-dimensional line.

Of course, the rigorous results in Section 3 need to be structurally modified in higher dimension, since several notions of “proximity” of groups, “direction of march” and “orientation of the eyesight” can be considered.

## 7. Conclusions

As a result of our direct observation at Phillip Island and a series of scientific discussions with penguin ecologists, we provide a simple but rigorous mathematical model which aims to describe the formation of groups of penguins on the shore at sunset and the return to their burrows. This model is proved to possess existence and uniqueness of solutions, and quantitative results on the homecoming of the penguins are given. The framework is general enough to show the formation of groups of penguins marching together, as well as the possibility that some penguins remain isolated from the rest of the herd. The model is also numerically implemented in simple and explicit simulations.

We believe that the method proposed can be suitably compared with the real penguin parade, thus triggering specific fieldwork on this rather peculiar topic. Indeed, at the moment, a precise collection of data focused on the penguin parade seems to be still missing in the literature, and we think that a mathematical formulation provides the necessary setting for describing specific behaviours in ethology, such as the formation of groups and the possible isolation of penguins, in a rigorous and quantitative way.

Given the simple and quantitative mathematical setting that we introduced here, we also believe that our formulation can be easily modified and improved to capture possible additional details of the penguins’ march provided by the biological data which may be collected in future specialized fieldwork. We hope that this problem will also take advantage of statistically sound observations by ecologists, possibly taking into account the speed of the penguins in different environments, the formation of groups of different size, the velocity of each group depending on its size and the links between group formations motivated by homecoming and the social structures of the penguin population.

Due to the lack of available biological theories and precise experimental data, the form of some of the functions considered in this article should be viewed as an example. This applies, in particular, to the strategic velocity function, the eyesight function and the stopping function, and it would be ideal to run experiments to provide a better quantification of these notions.

Also, it would be interesting to detect how changes in the environment such as modified visibility or presence of predators influence the formation of groups, their size and speed. In general, we think that it is very important to provide precise conditions for clustering, and to explore these conditions systematically. In addition, it would be desirable to adapt models of this type to social studies, politics and



evolutionary biology, in order to describe and quantify the phenomenon of “front-runners” who “wait for the formation of groups of considerable size” in order to “more safely proceed towards their goal”.

## Appendix

### Example of a program listing

```
H=4; % Position of the burrow of the penguins community
S=-2; % The sea lies in the region  $(-\infty, 0]$ . For simplicity we assume
      that penguins start near the shore, that is, the initial position
      of each penguin is at least S
eps=0.005; % Drift velocity of the penguins
vc=0.05; % Cruising speed of a big enough raft of penguins
N=20; % Number of penguins
delta=(N+1)*eps; % This parameter is used to compute the strategic
                  velocity of a penguin.

% These parameters define the time interval
TMAX=(H-S)/(2*eps);
t=0.01;
T=(0:t:TMAX);
TG=T(1:1,1:12000);

P=zeros(N,length(T));
% The following is the array of the initial positions of the N penguins
P(:,1)=[-1.95 -1.5 -1.05 -0.6 -0.55 -0.4 -0.2 0.1 0.2 0.4 0.8 0.85 0.9
        1 1.1 1.15 1.2 1.65 3 3.4];

s=(H-S)/3; % The parameter encoding the eye-sight of the penguins
pgot=(H-S)/12; % The parameter representing the stopping function
M=zeros(1,N);
V=M;
PAN=-1;
W=0;

WAVE=sin(T)*delta; % The "environment function". In this case only
                   waves are taken into account

for k=1 : length(T)-1
  for i=1 : N
    if P(i,k)<H
      for j=1: N % This cycle checks if the ith penguin is in panic
        if -pgot<P(i,k)-P(j,k) & P(i,k)-P(j,k)<pgot
          PAN=PAN+1;
          if P(i,k)==P(j,k)
            W=W+1; % This counts the number of penguins in the same
                  position as the ith penguin, that is, the dimension
                  of the raft
          end
        end
      end
    end
  end
end
```

```

        end
    end
end
if PAN==0 % The ith penguin is stuck because of panic
    if -3.5<P(i,k) & P(i,k)<0
        V(i)=-WAVE(k);
        P(i,k+1)=P(i,k)+V(i)*t;
    else
        P(i,k+1)=P(i,k);
    end
else
    if W>N/2 % The ith penguin is a member of a big enough raft,
        so it tends to go home, forgetful of the other penguins
        if -3.5<P(i,k) & P(i,k)<0 % The environment can still affect
            the movement of the raft
            V(i)=vc-WAVE(k);
        else
            V(i)=vc; % If the environment does not affect the movement,
                the penguin moves at cruise velocity
        end
    else % The raft is not big enough, so the strategic velocity
        of the ith penguin is influenced by the other penguins in sight
        for j=1 : N
            if -s<P(i,k)-P(j,k) & P(i,k)-P(j,k)<0
                M(i)=M(i)+delta/N; % Each penguin in sight ahead adds a
                    delta/N to the strategic velocity of the
                    ith penguin
            else
                if 0<P(i,k)-P(j,k) & P(i,k)-P(j,k)<s
                    M(i)=M(i)-delta/N; % Each penguin in sight behind
                        subtracts a delta/N from the strategic
                        velocity of the ith penguin
                end
            end
        end
        if -3.5<P(i,k) & P(i,k)<0
            V(i)=eps+M(i)-WAVE(k);
        else
            V(i)=eps+M(i);
        end
    end
    P(i,k+1)=P(i,k)+V(i)*t;
end
else
    P(i,k+1)=H;
end
PAN=-1;
W=0;
end

```

```

M=zeros(1,N);
for i=2 : N
    for j=1 : i-1
        if -0.011<P(j,k+1)-P(i,k+1) & P(j,k+1)-P(i,k+1)<0.011
            P(j,k+1)=P(i,k+1); % For simplicity, we assume that penguins
                close enough occupy the same position, forming a raft
                and moving together
        end
    end
end
end
Q=P(1:N,1:length(TG));
plot(TG,Q)

```

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