intersection of a cylinder with two equal and similarly situated spheres, the diameter of the cylinder being the length of the bar, and the radius of each sphere the length of either cord.

The proof, then, may be given concisely as follows :—The point of attachment of the moving bar traces out the intersection of a cylinder with a sphere, this—in practical cases—is very approximately that of a cylinder with an inclined plane; the steepness in this case is as the sine of the angle through which the bar is turned. This is directly as the magnitude of the force tending to turn it.

The ordinary trigonometrical treatment may be found in Wiedemann's Galvanismus, B. II., Th. I., s. 289.

Abstract of one of Euler's papers.

By J. S. Mackay, M.A.

The paper is entitled *Solutio facilis problematum quorumdam geometricorum difficillimorum*, and is printed in *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, Tom. xi., pp. 103-123. The volume is for the year 1765; the title-page is dated 1767.

The investigation is concerned with a plane triangle and its four points the orthocentre, the centroid, the inscribed centre, and the circumscribed centre.

Not to complicate a single figure with too many lines, four figures are exhibited. In fig. 40, AM, BN, CP are the perpendiculars from A, B, C on the opposite sides, and E is the orthocentre. In fig. 41, Aa, Bb, Cc are the medians from A, B, C, and F is the centroid; FQ and CP are perpendicular to AB. In fig. 42, Aα, Bβ, Cγ are the bisectors of angles A, B, C, and G is the inscribed centre; GR is perpendicular to AB. In fig. 43, S, T, V are the middle points of AB, BC, CA, and SH, TH, VH respectively perpendicular to those sides meet in H the circumscribed centre; AM, CP are perpendicular to BC, AB, and AH is joined.

The sides of the triangle BC, CA, AB are denoted respectively by a, b, c, [see the notation for triangle FGH on p. 54] and the area by A.

Orthocentre E.

By application of Euclid II. 13, and from the similar triangles ABM, AEP it is found that

$$\mathrm{AP}=\frac{c^2+b^2-a^2}{2c},\ \mathrm{PE}=\frac{(c^2+b^2-a^2)(a^2+c^2-b^2)}{8c\mathrm{A}}.$$

Centroid F.

From the similar triangles FcQ, CcP and the preceding result it is found that $\mathrm{AQ}=\dfrac{3c^2+b^2-a^2}{6c}$, $\mathrm{QF}=\dfrac{2\mathrm{A}}{3c}$.

Inscribed Centre G.

Since R is the point of contact of the inscribed circle with AB it is found that $\mathrm{AR}=\dfrac{c+b-a}{2}$, $\mathrm{RG}=\dfrac{2\mathrm{A}}{a+b+c}$.

Circumscribed Centre H.

From the similar triangles AHS, ACM it is found that

$$\mathrm{AS}=\tfrac{1}{2}c,\ \mathrm{SH}=\frac{c(a^2+b^2-c^2)}{8\mathrm{A}}.$$

Hence when these four points are taken on the same figure, the expressions for their mutual distances will be

$$\begin{aligned}
\mathrm{EF}^2&=(\mathrm{AP}-\mathrm{AQ})^2+(\mathrm{PE}-\mathrm{QF})^2\\
\mathrm{EG}^2&=(\mathrm{AP}-\mathrm{AR})^2+(\mathrm{PE}-\mathrm{RG})^2\\
\mathrm{EH}^2&=(\mathrm{AP}-\mathrm{AS})^2+(\mathrm{PE}-\mathrm{SH})^2\\
\mathrm{FG}^2&=(\mathrm{AQ}-\mathrm{AR})^2+(\mathrm{QF}-\mathrm{RG})^2\\
\mathrm{FH}^2&=(\mathrm{AQ}-\mathrm{AS})^2+(\mathrm{QF}-\mathrm{SH})^2\\
\mathrm{GH}^2&=(\mathrm{AR}-\mathrm{AS})^2+(\mathrm{RG}-\mathrm{SH})^2.
\end{aligned}$$

Now here it is particularly to be observed that those distances between our four points ought necessarily to be so expressed that the three sides of the triangle may enter equally into the expressions, since no precedence can be attributed to any one side over the rest in respect of these distances. Accordingly let

$$a+b+c=p,\ ab+ac+bc=q,\ abc=r;\ \text{whence}$$

$$a^2+b^2+c^2=p^2-2q,\ a^2b^2+a^2c^2+b^2c^2=q^2-2pr,$$

$$a^4+b^4+c^4=p^4-4p^2q+2q^2+4pr,$$

$$\mathrm{A}^2=\tfrac{1}{16}p(-p^3+4pq-3r).$$

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Hence, by substitution and after considerable calculation, the expressions for the six distances become

$$EF^2 = \frac{r^2}{4A^2} - \frac{4}{9}(p^2 - 2q)$$

$$EG^2 = \frac{r^2}{4A^2} - p^2 + 3q - \frac{4r}{p}$$

$$EH^2 = \frac{9r^2}{16A^2} - (p^2 - 2q)$$

$$FG^2 = -\frac{1}{9}p^2 + \frac{5}{9}q - \frac{2r}{p}$$

$$FH^2 = \frac{r^2}{16A^2} - \frac{1}{9}(p^2 - 2q)$$

$$GH^2 = \frac{r^2}{16A^2} - \frac{r}{p}.$$

Here it is evident that $EH = \frac{3}{2}EF$, and $FH = \frac{1}{2}EF$, and thus the point H is determined by the points E, F. That is, if the three points E, F, G form a triangle EFG, then the fourth point H will be situated in EF produced so that $FH = \frac{1}{2}EF$, and therefore $EH = \frac{3}{2}EF$.
Hence is deduced $4GH^2 + 2EG^2 = 3EF^2 + 6FG^2$.
[Euler gives no hint of how he obtains this last result. It is got from a theorem which he afterwards (*Acta Academiae Scientiarum Imperialis Petropolitanae, pro Anno* 1780. *Pars Prior*, pp. 92-93) proved as a lemma in order to inscribe in a circle a triangle whose three sides pass through three given points. The theorem is Prop. II. of Matthew Stewart's *Some General Theorems of considerable use in the higher parts of Mathematics,* (Edinburgh, 1746) and may be thus enunciated :—If three collinear points A, C, B be joined to any point D, then $DA^2 \cdot BC + DB^2 \cdot AC - DC^2 \cdot AB = AB \cdot AC \cdot BC$.]

To give greater simplicity to these formulae, assume

$$P = p^2,\ Q = \frac{r}{p},\ R = \frac{r^2}{4A^2},$$

so that P, Q, R are quantities involving two dimensions.
Thence are obtained

$$p = \sqrt{P},\ q = \frac{1}{4}P + 2Q + \frac{Q^2}{R},\ r = Q\sqrt{P},$$

and the following expressions :—

$$EF^2 = R - \frac{2}{9}P + \frac{16}{9}Q + \frac{8Q^2}{9R}$$

$$EG^2 = R - \frac{1}{4}P + 2Q + \frac{3Q^2}{R}$$

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$$EH^2 = \frac{9}{4}R - \frac{1}{2}P + 4Q + \frac{2Q^2}{R}$$

$$FG^2 = \quad + \frac{1}{36}P - \frac{8}{9}Q + \frac{5Q^2}{9R}$$

$$FH^2 = \frac{1}{4}R - \frac{1}{18}P + \frac{4}{9}Q + \frac{2Q^2}{9R}$$

$$GH^2 = \frac{1}{4}R - Q.$$

Accordingly since of these four points three (unless these three E, F, and H are taken) contain the determination of the fourth, there results the single problem :—

These four points, E, F, G, H, connected with any triangle being given in position, to construct the triangle. (See fig. 44).

Assume $GH = f$, $FH = g$, and $FG = h$,

whence $\quad EF = 2g$, $EH = 3g$, $EG = \sqrt{6g^2 + 3h^2 - 2f^2}$,
and the three following equations are obtained

$$f^2 = \frac{1}{4}R - Q$$

$$g^2 = \frac{1}{4}R - \frac{1}{18}P + \frac{4}{9}Q + \frac{2Q^2}{9R}$$

$$h^2 = \frac{1}{36}P - \frac{8}{9}Q + \frac{5Q^2}{9R}.$$

From these the values of P, Q, R, and consequently of p, q, r may be found. The sides of the triangle then will be the three roots of the cubic equation

$$z^3 - pz^2 + qz - r = 0.$$

In the case when these four points are in a straight line (see fig. 45) EG becomes $= 2g - h$, and $g = f - h$.
When substitution is made it appears that the cubic equation for determining the three sides of the triangle has two equal roots, and that consequently the triangle is isosceles.

This case is easy to resolve independently, since the straight line in which the given points are must necessarily cut the triangle into two similar parts, and accordingly the triangle is isosceles. Starting with the assumption that $b = a$, the following values are obtained

$$A = \tfrac{1}{4}c\sqrt{4a^2 - c^2},\ AP = AQ = AR = AS = \tfrac{1}{2}c,$$

$$PE = \frac{c^3}{8A},\ QF = \frac{2A}{3c},\ RG = \frac{2A}{2a + c},\ SH = \frac{c(2a^2 - c^2)}{8A}.$$

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If O be the middle point of the base, P, Q, R, S coincide at O, and the six distances between the four points are

$$\mathrm{OF}-\mathrm{OE}=\frac{2(a^2-c^2)}{3\sqrt{4a^2-c^2}}\qquad \mathrm{OG}-\mathrm{OE}=\frac{c(a-c)}{\sqrt{4a^2-c^2}}$$

$$\mathrm{OH}-\mathrm{OE}=\frac{a^2-c^2}{\sqrt{4a^2-c^2}}\qquad \mathrm{OF}-\mathrm{OG}=\frac{(a-c)(2a-c)}{3\sqrt{4a^2-c^2}}$$

$$\mathrm{OH}-\mathrm{OF}=\frac{a^2-c^2}{3\sqrt{4a^2-c^2}}\qquad \mathrm{OH}-\mathrm{OG}=\frac{a(a-c)}{\sqrt{4a^2-c^2}}.$$

Here two cases are to be considered, when $a>c$ and when $a<c$; for if $a=c$, the triangle is equilateral, and the four points coalesce in one.

If $a>c$, the points will be situated as in fig 46 where $\mathrm{HF}=\frac{1}{3}\mathrm{EH}$, or $\mathrm{EF}=\frac{2}{3}\mathrm{EH}$, and $\mathrm{EG}<\frac{1}{2}\mathrm{EH}$.

In this case O falls in HE produced beyond E so that $\mathrm{OE}=\dfrac{c^2}{2\sqrt{4a^2-c^2}}$.

If $a<c$, the points will be situated as in fig. 47 where $\mathrm{HF}=\frac{1}{3}\mathrm{EH}$ or $\mathrm{EF}=\frac{2}{3}\mathrm{EH}$, but $\mathrm{EG}>\frac{1}{2}\mathrm{EH}$.

In this case O falls in EH produced beyond H so that $\mathrm{HO}=\dfrac{2a^2-c^2}{2\sqrt{4a^2-c^2}}$; whence if $2a^2<c^2$, the point O falls between H and E.

Sixth Meeting, April 8th, 1886.

DR FERGUSON, F.R.S.E., President, in the Chair.

On the Divisibility of certain Numbers.

By J. S. MACKAY, M.A.

Of the following properties the first two are obvious; the third was communicated to the Royal Society of London by the Rev. Dr James Booth* in 1854; the others, obtained many years ago, have not as far as I know been remarked. They seem to be more curious than useful.

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* See *Proceedings of the Royal Society of London*, (London, 1856) vol. vii. pp. 42-43.