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Nowhere scattered multiplier algebras

Eduard Vilalta
Fields Institute for Research in Mathematical Sciences, Toronto M5T 3J1, Canada
Previous institution: Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra, Barcelona 08193, Spain
(eduardvilaltavila@gmail.com)

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We study sufficient conditions under which a nowhere scattered C*-algebra A has a nowhere scattered multiplier algebra $\mathcal{M}(A)$, that is, we study when $\mathcal{M}(A)$ has no nonzero, elementary ideal-quotients. In particular, we prove that a σ-unital C*-algebra A of

(i) finite nuclear dimension, or

(ii) real rank zero, or

(iii) stable rank one with k-comparison,

is nowhere scattered if and only if $\mathcal{M}(A)$ is.

Keywords: C*-algebra; multiplier algebra; Cuntz semigroup

1. Introduction

The study of regularity properties of multiplier algebras appears throughout the literature; see, for example, [16, 22, 29, 36, 42]. One notable instance of this is the study of pure infiniteness and, more concretely, of when a C*-algebra has a purely infinite multiplier algebra; see [23, 27, 28, 30, 32]. When this condition is relaxed to weak pure infiniteness (in the sense of [25]) it was shown in [25, Proposition 4.11] that a C*-algebra is weakly purely infinite if and only if its multiplier algebra is. In general, given a certain property $\mathcal{P}$, one can ask: If a C*-algebra A satisfies $\mathcal{P}$, when does $\mathcal{M}(A)$ satisfy $\mathcal{P}$? For example, Brown and Pedersen conjecture in [8] that the multiplier algebra of a real rank zero C*-algebra with trivial K_1-group is again of real rank zero; see also [31] and [56].

In this paper, we study the question above for the property of being *nowhere scattered* (see paragraph 2.1). This notion ensures sufficient noncommutativity of the algebra, and can be characterized in a number of ways. As shown in [49, Theorem 3.1], a C*-algebra A is nowhere scattered if and only if it has no nonzero elementary ideal-quotients. This is in turn equivalent to no hereditary sub-C*-algebra of A admitting a one-dimensional representation. Every weakly purely infinite C*-algebra is nowhere scattered ([49, Example 3.3]) but, in contrast to the weakly purely infinite case, the multiplier algebra of a nowhere scattered C*-algebra need not be nowhere scattered; see examples 5.2 and 5.3.

Question 1.1. Let A be nowhere scattered. When is $\mathcal{M}(A)$ nowhere scattered?

One of the motivations behind question 1.1 is the study of when a multiplier algebra has no characters, since such a property leads to important structure results. For instance, it follows from [**39**, Theorem 3.2] that every element in a nowhere scattered multiplier algebra can be written as the finite sum of commutators and products of two commutators. Moreover, nowhere scatteredness for unital C*-algebras is related to the existence of full, square-zero elements [**51**, Theorem 3.6(3)]. Thus, knowing that $\mathcal{M}(A)$ is nowhere scattered would have implications on its unitary group; see [**11**, Section 3].

Nowhere scatteredness of a C*-algebra A can also be characterized in terms of its Cuntz semigroup $\mathrm{Cu}(A)$, a powerful invariant for C*-algebras introduced in [**15**] and further developed in [**14**]; see also [**2**, **3**, **21**, **25**, **43**, **50**, **53**]. Explicitly, it was shown in [**49**, Theorem 8.9] that a C*-algebra is nowhere scattered if and only if its Cuntz semigroup is *weakly* $(2, \omega)$*-divisible*, a notion defined by Robert and Rørdam in [**40**] to study when certain C*-algebras have characters; see paragraph 2.3. Consequently, the study of question 1.1 leads naturally to the study of divisibility properties of $\mathrm{Cu}(\mathcal{M}(A))$. The main divisibility properties at play in this case are weak $(2, \omega)$-divisibility and its bounded counterpart, known as *weak* (m, n)*-divisibility*; see paragraph 2.3 for the definitions.

The nowhere scattered C*-algebras from examples 5.2 and 5.3, which fail to have a nowhere scattered multiplier algebra, have a weakly $(2, \omega)$-divisible Cuntz semigroup. However, they both have *unbounded divisibility*, that is, for every pair $m, n \in \mathbb{N}$ there exists a Cuntz class that is not weakly (m, n')-divisible for any $n' \leqslant n$. Thus, we ask:

Question 1.2. Let A be a nowhere scattered C*-algebra. When does there exist $n \in \mathbb{N}$ such that $[a]$ is weakly $(2, n)$-divisible for every $a \in A_+$?

More generally, when can one find $m, n \in \mathbb{N}$ such that $[a \otimes 1_m]$ is weakly $(2m, n)$-divisible for every $a \in A_+$?

This question is studied in detail in § 4, where we provide a number of examples where one can answer question 1.2 affirmatively. Further, we also show that having bounded divisibility is characterized by the soft part of the monoid; see theorem 4.11.

We prove in § 5 that question 1.1 has a positive answer whenever question 1.2 does.

Theorem 1.3 5.11. *Let A be a σ-unital* C***-algebra. Assume that for every orthogonal sequence $(a_i)_i$ of positive elements in A there exist $m, n \in \mathbb{N}$ such that $[a_i \otimes 1_m]$ is weakly $(2\,m, n)$-divisible for every i. Then $\mathcal{M}(A)$ is nowhere scattered.*

Using the study of bounded divisibility from § 4, and that nowhere scatteredness passes to ideals ([**49**, Proposition 4.2]), one obtains:

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THEOREM 1.4 5.12. *Let A be a σ-unital* C*-*algebra. Assume that A is of*

(i) *real rank zero, or*

(ii) *finite nuclear dimension, or*

(iii) *stable rank one with k-comparison.*

Then A is nowhere scattered if and only if $\mathcal{M}(A)$ is nowhere scattered.

Further, one can generalize (iii) above by changing stable rank one for the condition of having a *surjective rank map*; see proposition 4.3 and remark 4.4.

We also prove a weak converse of theorem 1.3 for stable C*-algebras:

THEOREM 1.5 5.18. *Let A be a σ-unital, stable* C*-*algebra. Assume that $\mathcal{M}(A)$ is nowhere scattered. Then, for every $a \in A_+$ and $m \in \mathbb{N}$ there exists n such that $[a]$ is weakly (m, n)-divisible.*

Most of the results in § 3 and 4 can be translated to the more general setting of abstract Cuntz semigroups, or Cu-semigroups for short; see for example [**3**] and [**21**]. However, since in this paper we focus on multiplier algebras (which have no known Cu-counterpart), we state all the results in the language of C*-algebras.

2. Preliminaries

2.1. Nowhere scatteredness and the Global Glimm Property

As defined in [**49**, Definition A], a C*-algebra is *nowhere scattered* if none of its quotients contains a minimal open projection, where recall that an *open projection* in a C*-algebra B is a projection $p \in B^{**}$ that can be written as the strong limit of an increasing sequence of positive elements in B. As noted in [**49**, Paragraph 2.1], an open projection $p \in B^{**}$ is *minimal* with respect to the order if and only if $p \in B$ and $pBp = \mathbb{C}p$. By [**49**, Theorem 3.1], a C*-algebra A is nowhere scattered if and only if no nonzero ideal-quotient of A is elementary.

One also says that a C*-algebra A has the *Global Glimm Property* ([**25**, Definition 4.12]) if for every $a \in A_+$ and $\varepsilon > 0$ there exists a *-homomorphism $\varphi\colon M_2(C_0(0, 1]) \to \overline{aAa}$ such that the image of φ contains $(a - \varepsilon)_+$.

A C*-algebra is nowhere scattered whenever it has the Global Glimm Property. The converse remains open, and is known as the *Global Glimm Problem*; see [**2**], [**18**] and [**51**].

Examples of C*-algebras with the Global Glimm Property include simple, non-elementary C*-algebras, $\mathcal{Z}$-stable C*-algebras and traceless C*-algebras (which include purely infinite C*-algebras).

2.2. The Cuntz semigroup

For any pair of positive elements a, b in a C*-algebra A one writes $a \precsim b$ if $a = \lim_n r_n b r_n^*$ for some sequence $(r_n)_n$ in A. One also says that a is *Cuntz equivalent* to b, in symbols $a \sim b$, whenever $a \precsim b$ and $b \precsim a$.

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The *Cuntz semigroup* $\mathrm{Cu}(A)$ is defined as the quotient $(A\otimes\mathcal{K}_+)/\sim$ equipped with the order induced by $\precsim$ and the addition induced by diagonal addition; see [15] and [14] for details.

Given elements $x, y\in\mathrm{Cu}(A)$, we write $x\ll y$ whenever there exists $a\in(A\otimes\mathcal{K})_+$ and $\varepsilon>0$ such that $x\leqslant[(a-\varepsilon)_+]$ and $y=[a]$. As shown in [14], every increasing sequence in $\mathrm{Cu}(A)$ has a supremum, and every element can be written as the supremum of a $\ll$-increasing sequence.

In recent years, the Cuntz semigroup has benefited from the study of abstract Cuntz semigroups, or Cu-*semigroups* for short. Good references for this include [5], [3] and [21]. In this paper, we will not use the language of Cu-semigroups (except in corollary 4.13), but we will still make use of some of the abstract properties that the Cuntz semigroup of a C*-algebra always satisfies. These are:

(O5) Given $x', x, y', y, z\in\mathrm{Cu}(A)$ such that $x+y\leqslant z$ with $x'\ll x$ and $y'\ll y$, there exists $c\in\mathrm{Cu}(A)$ such that $y'\ll c$ and $x'+c\leqslant y\leqslant x+c$.

(O6) Given $x'\ll x\ll y+z$, there exist $v\leqslant x, y$ and $w\leqslant x, z$ such that $x'\leqslant v+w$.

The reader is referred to [3, Proposition 4.6] (also [44]) and [38] for the respective proofs.

2.3. Divisibility in the Cuntz semigroup

Let A be a C*-algebra, let $x\in\mathrm{Cu}(A)$ and take $n\geqslant 1$ and $m\geqslant 2$. Following [40], we will say that x is *weakly* (m, n)*-divisible* if, whenever $x'\ll x$, there exist elements $y_1, \ldots, y_n$ such that $x'\leqslant y_1+\ldots+y_n$ and $my_j\leqslant x$ for each $j\leqslant n$. Similarly, x is *weakly* (m, ω)*-divisible* if the previous condition holds without any bound on n, that is, allowing n to depend on x'.

The element x is said to be (m, n)*-divisible* (resp. (m, ω)-divisible) if one can always set $y_1=\ldots=y_n$.

A C*-algebra is nowhere scattered if and only if every element in its Cuntz semigroup is weakly (m, ω)-divisible for each m, whilst a C*-algebra has the Global Glimm Property if and only if every element is (m, ω)-divisible for each m; see [49, Theorem 8.9] and [51, Theorem 3.6] respectively.

3. Finite divisibility

In this section, we recall the notions of finite divisibility introduced in [40]; see definition 3.1. We prove their main properties (lemma 3.3), and study some situations where $\mathrm{Cu}(A)$ contains a sup-dense subset of elements with finite divisibility. We also find sufficient conditions for finite weak divisibility to imply finite divisibility; see theorem 3.15.

DEFINITION 3.1 [40]. *Let a be a positive element of a C*-algebra A, and let $m\geqslant 2$. We let $\mathrm{div}_m([a])$ and $\mathrm{Div}_m([a])$ be the least positive integers n, n' such that $[a]\in\mathrm{Cu}(A)$ is weakly (m, n)-divisible and (m, n')-divisible respectively.*

If no such n or n' exist, we set $\mathrm{div}_m([a])=\infty$ or $\mathrm{Div}_m([a])=\infty$.

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REMARK 3.2. The class of a positive element $a \in A_+$ in $\mathrm{Cu}(A)$ is said to be *compact* if $[a] \ll [a]$. Every projection gives rise to a compact Cuntz class and, in some cases, these are the only compact classes; see [**10**].

Given a compact element $[a] \in \mathrm{Cu}(A)$, then $[a]$ is weakly (m, ω)-divisible if and only if $\mathrm{div}_m([a]) < \infty$.

However, for a non-compact element $[a]$, $\mathrm{div}_m([a]) < \infty$ is not equivalent to $[a]$ being weakly (m, ω)-divisible; see example 3.6 and remark 5.10.

Let us first summarize the main properties of $\mathrm{div}_m()$. Part (iii) of the following lemma is in analogy to [**25**, Lemma 4.9], while (iv) is a Cu-analogue of [**40**, Proposition 3.6].

Recall that, given $x, y \in \mathrm{Cu}(A)$, the infimum $x \wedge \infty y$ always exists; see [**1**, Remark 2.6]. Here, ∞y denotes the element $\sup_n ny$.

LEMMA 3.3. *Let A be a* C*-*algebra, and let $x = [a] \in \mathrm{Cu}(A)$ and $m \in \mathbb{N}$. Then,*

(i) $\mathrm{div}_m(x) \leqslant \mathrm{div}_{m'}(x)$ *whenever* $m \leqslant m'$.

(ii) $\mathrm{div}_m([a]) \leqslant N(\mathrm{div}_m([b_1]) + \ldots + \mathrm{div}_m([b_r]))$ *whenever*

$$[b_1], \ldots, [b_r] \leqslant [a] \leqslant N([b_1] + \ldots + [b_r]).$$

(iii) $\mathrm{div}_m([a + b]) \leqslant \mathrm{div}_m([a]) + \mathrm{div}_m([b])$.

(iv) *Given* $x, y \in \mathrm{Cu}(A)$ *such that* $y \leqslant x$ *and* $x \wedge \infty y = y$, *then* $\mathrm{div}_m(y) \leqslant \mathrm{div}_m(x)$.

(v) $\mathrm{div}_m(x) \leqslant \sup_k \mathrm{div}_m(x_k)$ *for every increasing sequence* $(x_k)_k$ *in* $\mathrm{Cu}(A)$ *with supremum* x.

Proof. (i) follows directly from the definition of $\mathrm{div}_m([a])$.

To see (ii), let $n_j = \mathrm{div}_m([b_j])$ for $j = 1, \ldots, r$. We may assume that all quantities are finite, since otherwise we are done.

Take $x \ll [a]$. Since $[a] \leqslant N[b_1] + \ldots + N[b_r]$, we can find elements x_j such that $x_j \ll [b_j]$ and $x \leqslant Nx_1 + \ldots + Nx_r$. The element $[b_j]$ is weakly (m, n_j)-divisible, so we can find elements $y_{1,j}, \ldots, y_{n_j,j}$ such that

$$my_{i,j} \leqslant [b_j], \text{ and } x_j \leqslant y_{1,j} + \ldots + y_{n_j,j}$$

for each i and j.

Using that $[b_j] \leqslant [a]$, one has $my_{i,j} \leqslant [a]$. Further, since we also have $x \leqslant N\sum_j x_j$, we deduce that $x \leqslant N\sum_{i,j} y_{i,j}$. It follows that $\mathrm{Div}_m([a]) \leqslant N(n_1 + \ldots + n_r)$, as desired.

Note that, given $a, b \in (A \otimes \mathcal{K})_+$, we have $[a + b] \leqslant [a] + [b]$ and $[a], [b] \leqslant [a + b]$. Thus, (iii) follows directly from (ii).

For (iv), let $x, y \in \mathrm{Cu}(A)$ be as stated. As before, we may assume that $\mathrm{div}_m(x) = n$ for some $n \in \mathbb{N}$. Take $y' \ll y$, and apply the weak (m, n)-divisibility of x to obtain elements $y_1, \ldots, y_n$ satisfying the properties from paragraph 2.3 for y' and x.

Since $my_j \leqslant x$ for every j, and $x \wedge \infty y = y$, one obtains $m(y_j \wedge \infty y) \leqslant x \wedge \infty y = y$. To see that $y' \leqslant (y_1 \wedge \infty y) + \ldots + (y_n \wedge \infty y)$, note that y' is bounded by y (and

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thus ∞y) and by $y_1 + \ldots + y_n$. Using that the map $t \mapsto t \wedge \infty y$ preserves addition [1, Theorem 2.5(i)], one has

$$y' \leqslant (y_1 + \ldots + y_n) \wedge \infty y = (y_1 \wedge \infty y) + \ldots + (y_n \wedge \infty y),$$

which shows that the elements $y_j \wedge \infty y$ satisfy the conditions from paragraph 2.3 for y' and y, as required.

To prove (v), assume that the supremum $\sup_k \mathrm{div}_m(x_k)$ is finite, since otherwise there is nothing to prove. Let $n \in \mathbb{N}$ be such that $\sup_k \mathrm{div}_m(x_k) \leqslant n$, and take $x' \in \mathrm{Cu}(A)$ such that $x' \ll x$. Since $x = \sup_k x_k$, it follows that $x' \ll x_k$ for some $k \in \mathbb{N}$. Thus, using that $\mathrm{div}_m(x_k) \leqslant n$, we obtain elements $y_1, \ldots, y_n$ with $x' \leqslant y_1 + \ldots + y_n$ and $my_j \leqslant x_k \leqslant x$ for each j. We get that $\mathrm{div}_m(x) \leqslant n$, as desired. □

Recall that a C*-algebra A is said to have *strict comparison* if $[a] \leqslant [b]$ in $\mathrm{Cu}(A)$ whenever there exists $\delta > 0$ such that $\lim_n \tau(a^{1/n}) \leqslant (1-\delta) \lim_n \tau(b^{1/n})$ for every 2-quasitrace τ; see [17] for more details.

LEMMA 3.4. *Let A be a C*-algebra, let $[a] \in \mathrm{Cu}(A)$ and $m \in \mathbb{N}$. Then,*

$$\mathrm{div}_{km}(k[a]) \leqslant \mathrm{div}_m([a])$$

for every $k \in \mathbb{N}$.

If A has strict comparison, one has $\mathrm{div}_m([a]) \leqslant \mathrm{div}_{k(m+1)}(k[a])$.

Proof. Let $n = \mathrm{div}_m([a])$, which we may assume to be finite, and take $x \ll k[a]$. Then, there exists $\varepsilon > 0$ such that $x \ll k[(a-\varepsilon)_+]$. Since $[(a-\varepsilon)_+] \ll [a]$, there exist $y_1, \ldots, y_n \in \mathrm{Cu}(A)$ such that

$$[(a-\varepsilon)_+] \leqslant y_1 + \ldots + y_n, \text{ and } my_j \leqslant [a]$$

for each j.

In particular, one gets

$$x \ll k[(a-\varepsilon)_+] \leqslant ky_1 + \ldots + ky_n, \text{ and } m(ky_j) \leqslant k[a],$$

which implies $\mathrm{div}_{km}(k[a]) \leqslant n$, as desired.

Now assume that A has strict comparison of positive elements, and let $n = \mathrm{div}_{k(m+1)}(k[a])$. As before, we may assume n to be finite. Let $x \ll [a]$, which implies $kx \ll k[a]$. Then, there exist $z_1, \ldots, z_n$ such that $x \leqslant kx \leqslant z_1 + \ldots + z_n$ and $k(m+1)z_j \leqslant k[a]$. Thus, one has $(km+1)mz_j \leqslant km(m+1)z_j \leqslant (km)[a]$.

Using [17, Proposition 6.2], we get $mz_j \leqslant [a]$. This implies $\mathrm{div}_m([a]) \leqslant \mathrm{div}_{k(m+1)}(k[a])$. □

REMARK 3.5. Let A be a C*-algebra and let $m \in \mathbb{N}$. Assume that $\mathrm{div}_m([a]) < \infty$ for every $a \in A_+$. Lemma 3.3 (ii) implies, in particular, that $\mathrm{div}_m([b]) < \infty$ for every $b \in M_n(A)_+$ and $n \in \mathbb{N}$.

Indeed, given $b \in M_n(A)_+$ there exist elements $b_1, \ldots, b_n \in A_+$ and $N \in \mathbb{N}$ such that $[b_i] \leqslant [b] \leqslant N([b_1] + \ldots + [b_n])$ for each $i \leqslant n$; see [4, 4.2] and [51, Lemma 3.3].

By lemma 3.3 (ii), we have that $\mathrm{div}_m(b) < \infty$, as desired.

EXAMPLE 3.6. Let $(A_k)_k$ be a family of unital C*-algebras such that

$$k \leqslant \mathrm{div}_2([1_{A_k}]) \leqslant 7k$$

for each k.

By [**40**, Theorem 7.9], such C*-algebras can be taken to be simple, unital, infinite dimensional AH-algebras. Using [**49**, Proposition 4.13], one sees that $A := \oplus_k A_k$ is nowhere scattered. Thus, every element in $\mathrm{Cu}(A)$ is weakly $(2, \omega)$-divisible.

Consider the element $x = \sup_n \sum_{k=1}^n [1_{A_k}] \in \mathrm{Cu}(\oplus_k A_k)$, and denote by ι_k the induced inclusion $\mathrm{Cu}(A_k) \to \mathrm{Cu}(\oplus_k A_k)$. Then,

$$\iota_k([1_{A_k}]) \leqslant x, \text{ and } x \wedge \infty \iota_k([1_{A_k}]) = \iota_k([1_{A_k}]).$$

Thus, lemma 3.3 (iv) implies that

$$k \leqslant \mathrm{div}_2(\iota_k([1_{A_k}])) \leqslant \mathrm{div}_2(x).$$

This shows that x is weakly $(2, \omega)$-divisible, but $\mathrm{div}_2(x) = \infty$; see also corollary 5.16.

We now move our attention to C*-algebras whose Cuntz semigroup $\mathrm{Cu}(A)$ satisfies that every element (or, at least, a finite multiple of it) has finite weak divisibility. Particularly, there will be some situations where we do not know if an element $a \in A_+$ satisfies $\mathrm{div}_2([a]) < \infty$, but where we do know that $\mathrm{div}_{2m}([a \otimes 1_m]) < \infty$ for some $m \in \mathbb{N}$. This is the case, for example, when a is a positive element in a nowhere scattered C*-algebra of finite nuclear dimension (see proposition 4.1). Keeping track of such elements will be important in our investigations, and so we provide a precise definition for them:

DEFINITION 3.7. *Let A be a* C*-*algebra, and let $[a] \in \mathrm{Cu}(A)$. We will say that*

(i) $[a]$ *is of finite weak divisibility if* $\mathrm{div}_{2m}([a]) < \infty$ *for some* $m \in \mathbb{N}$.

(ii) $[a]$ *has a multiple of finite weak divisibility if* $\mathrm{div}_{2m}(m[a]) < \infty$ *for some* $m \in \mathbb{N}$.

(iii) $[a]$ *has a multiple of finite divisibility if* $\mathrm{Div}_{2m}(m[a]) < \infty$ *for some* $m \in \mathbb{N}$.

We will denote by $\mathrm{Cu}(A)_{\mathrm{div}}$ *(resp.* $\mathrm{Cu}(A)_{\mathrm{mdiv}}$ *and* $\mathrm{Cu}(A)_{\mathrm{mDiv}}$*) the subset of* $\mathrm{Cu}(A)$ *consisting of the elements* $[a]$ *of finite weak divisibility (resp. with a multiple of finite weak divisibility, and with a multiple of finite divisibility).*

REMARK 3.8. Note that one always has $\mathrm{Cu}(A)_{\mathrm{div}} \subseteq \mathrm{Cu}(A)_{\mathrm{mdiv}}$. Further, $\mathrm{Cu}(A)_{\mathrm{div}}$ and $\mathrm{Cu}(A)_{\mathrm{mdiv}}$ agree whenever $\mathrm{Cu}(A)$ is unperforated. Indeed, let $[a] \in \mathrm{Cu}(A)$ and $m \in \mathbb{N}$ be such that $\mathrm{div}_{2m}(m[a]) = n < \infty$. For any $\varepsilon > 0$, we have $m[(a-\varepsilon)_+] \ll m[a]$. Thus, there exist $y_1, \ldots, y_n \in \mathrm{Cu}(A)$ such that

$$[(a-\varepsilon)_+] \leqslant m[(a-\varepsilon)_+] \leqslant y_1 + \ldots + y_n, \text{ and } 2my_j \leqslant m[a].$$

Since $\mathrm{Cu}(A)$ is unperforated, we get $2y_j \leqslant [a]$. This shows that $\mathrm{div}_2([a]) \leqslant n < \infty$, as desired.

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Example 3.9. If $A = \mathbb{C}$, its Cuntz semigroup is isomorphic to $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. It is readily checked that $\mathrm{div}_m(x) < \infty$ if and only if $x \geqslant m$. Thus, one has $\mathrm{Cu}(A)_{\mathrm{div}} = \mathrm{Cu}(A)_{\mathrm{mdiv}} = \{0, 2, 3, \ldots, \infty\}$.

Example 3.10. Every $\mathcal{Z}$-stable C*-algebra A satisfies $\mathrm{Cu}(A) = \mathrm{Cu}(A)_{\mathrm{div}}$ by [**54**, Proposition 3.7]. More generally, recall that a C*-algebra is said to be N-*almost divisible* (see [**54**]) if for each pair x', $x \in \mathrm{Cu}(A)$ satisfying $x' \ll x$, and $k \in \mathbb{N}$, there exists $y \in \mathrm{Cu}(A)$ such that

$$ky \leqslant x, \text{ and } x' \leqslant (k+1)(N+1)y.$$

Thus, if A is N-almost divisible for some $N \in \mathbb{N}$, one also has $\mathrm{Cu}(A) = \mathrm{Cu}(A)_{\mathrm{div}}$.

By [**41**, Theorem 3.1], every nowhere scattered C*-algebra of nuclear dimension N that has no nonzero, simple, purely infinite ideal-quotients is N-almost divisible. This includes residually stably finite, nowhere scattered C*-algebras of finite nuclear dimension, and nowhere scattered C*-algebras of finite decomposition rank. Consequently, one has $\mathrm{Cu}(A)_{\mathrm{div}} = \mathrm{Cu}(A)$.

As we will see in proposition 4.1, $\mathrm{Cu}(A)_{\mathrm{mdiv}} = \mathrm{Cu}(A)$ for any nowhere scattered C*-algebra of finite nuclear dimension. In fact, one has more: $\mathrm{Cu}(\oplus_{i=1}^{\infty} A)_{\mathrm{mDiv}} = \mathrm{Cu}(\oplus_{i=1}^{\infty} A)$.

Example 3.11. As shown in [**49**, Theorem 9.1], every element x in the Cuntz semigroup of a nowhere scattered, real rank zero C*-algebra is *weakly divisible*, that is, there exist y, z such that $x = 2y + 3z$. In particular, one gets

$$2(y+z) \leqslant x \leqslant 3(y+z).$$

Thus, it follows that $\mathrm{div}_2(x) \leqslant 3$ for every x and, consequently, that $\mathrm{Cu}(A)_{\mathrm{div}} = \mathrm{Cu}(A)$.

As noted in example 3.11 above, every element in the Cuntz semigroup of a nowhere scattered, real rank zero C*-algebra has finite divisibility. Lemmas 3.12 and 3.14 below show that the same is true for a sup-dense subset of (non-elementary) simple C*-algebras and nowhere scattered C*-algebras of topological dimension zero. However, this may not imply that every element in such a C*-algebra has finite weak divisibility.

Recall that an element $x \in \mathrm{Cu}(A)$ is *idempotent* if $x = 2x$. In the Cuntz semigroup of a simple C*-algebra there is only one nonzero idempotent element, which we denote by ∞.

Lemma 3.12. *Let A be a simple, non-elementary* C*-*algebra. Then,* $\mathrm{Cu}(A)_{\mathrm{div}}$ *is sup-dense in* $\mathrm{Cu}(A)$.

If A is not stably finite, then $\mathrm{Cu}(A)_{\mathrm{div}} = \mathrm{Cu}(A)$.

Proof. Let $m \in \mathbb{N}$. If A is simple and non-elementary, it follows from [**38**, Proposition 5.2.1] that $\mathrm{Cu}(A)$ is (m, ω)-divisible.

Let $x \in \mathrm{Cu}(A)$ be such that $x \ll \infty$. Then, using that x is (m, ω)-divisible, there exists a nonzero element $y \in \mathrm{Cu}(A)$ such that $my \leqslant x$. Using that $x \ll \infty = \infty y$,

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we find $n \in \mathbb{N}$ such that $x \leqslant ny$. This implies that $\mathrm{div}_m(x) \leqslant n$. Since the subset of elements compactly contained in ∞ is sup-dense in $\mathrm{Cu}(A)$, the result follows.

Now assume that A is not stably finite. Then, the element ∞ in $\mathrm{Cu}(A)$ is compact; see, for example, [**46**, Lemma 6.21] or the proof of [**7**, Theorem 2.6].

It follows that every element in $\mathrm{Cu}(A)$ is compactly contained in ∞. Using the argument above, we see that $\mathrm{Cu}(A)_{\mathrm{div}} = \mathrm{Cu}(A)$. □

Remark 3.13. An inspection of the proof shows that lemma 3.12 is slightly more general: For a C*-algebra A, one has $\mathrm{Cu}(A)_{\mathrm{div}} = \mathrm{Cu}(A)$ whenever every idempotent element in $\mathrm{Cu}(A)$ is compact.

As defined in [**9**, Remark 2.5 (vi)], a C*-algebra has *topological dimension zero* whenever its primitive ideal space has a basis of compact-open subsets. Examples include all C*-algebras with the ideal property.

Lemma 3.14. *Let A be a separable nowhere scattered* C*-*algebra of topological dimension zero. Then $\mathrm{Cu}(A)_{\mathrm{div}}$ is dense in $\mathrm{Cu}(A)$.*

Proof. Take $y \in \mathrm{Cu}(A)$, and assume that there exists $y' \in \mathrm{Cu}(A)$ such that $y' \ll y \ll \infty y'$. Let $M \in \mathbb{N}$ be such that $y \leqslant My'$. Using that y is weakly (m, ω)-divisible, one finds $y_1, \ldots, y_n$ such that $my_i \leqslant y$ and $y' \leqslant y_1 + \ldots y_n$. Thus, one has $y \leqslant My_1 + \ldots + My_n$, and we get $\mathrm{div}_m(y) \leqslant Mn$.

Since A is separable and of topological dimension zero, it follows from [**51**, Proposition 4.18] that the Cu-semigroup $\mathrm{Cu}(A) \otimes \{0, \infty\}$ is algebraic. Thus, we know from [**51**, Lemma 4.16] that for every pair $x', x \in \mathrm{Cu}(A)$ such that $x' \ll x$ there exist $y', y \in \mathrm{Cu}(A)$ satisfying $x' \ll y \ll x$ and $y' \ll y \ll \infty y'$. The first part of the proof shows that $\mathrm{div}_m(y) < \infty$ and, consequently, that $\mathrm{Cu}(A)_{\mathrm{div}}$ is dense in $\mathrm{Cu}(A)$. □

3.1. The Global Glimm Problem

We finish this section by studying when $\mathrm{Cu}(A)_{\mathrm{div}} = \mathrm{Cu}(A)$ implies $\mathrm{Cu}(A)_{\mathrm{Div}} = \mathrm{Cu}(A)$. More concretely, we study when an element of finite weak divisibility has finite divisibility. Given its similarities with the Global Glimm Problem (see Paragraph 2.1), one could call this the *discrete Global Glimm Problem.*

Following the ideas from [**51**, Section 6], we obtain:

Theorem 3.15. *Let A be a* C*-*algebra satisfying $\mathrm{Cu}(A)_{\mathrm{div}} = \mathrm{Cu}(A)$. Assume that there exist $k \in \mathbb{N}$ and maps $N, M\colon \mathbb{N} \to \mathbb{N}$ such that*

(i) *whenever $x' \ll x \leqslant ny, nz$, there exists $t \in \mathrm{Cu}(A)$ such that $x' \ll N(n)t$ and $t \ll y, z$;*

(ii) *whenever $x' \ll x$ and $2x \ll y + 2nz$, there exists $g \in \mathrm{Cu}(A)$ such that $2g \ll y$ and $x' \ll g + M(n)z$;*

(iii) *whenever $x_1 + f, x_2 + f \ll y$ and $x_i' \ll x_i \ll f$ for $i = 1, 2$, one can find $z_1, z_2 \in \mathrm{Cu}(A)$ such that $z_1 + z_2 \ll y$ and $x_1' + x_2' \ll kz_1, kz_2$.*

Then, $\mathrm{Cu}(A)_{\mathrm{Div}} = \mathrm{Cu}(A)$.

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Proof. Assume that (i)–(iii) are satisfied, and let $x \in \mathrm{Cu}(A)$. By assumption, we have $\mathrm{div}_m(x) < \infty$, which implies $\mathrm{div}_2(x) < \infty$. Set $n := \mathrm{div}_2(x)$. We will show that $\mathrm{Div}_2(x) < \infty$ by proving that

$$\mathrm{Div}_2(x) \leqslant \Big(N\big(\max\{N(2N_{2,n}), M(n)\}\big) + N\big(N(2N_{2,n})\big)\Big) N(k),$$

where $N_{2,n} := (N\circ \overset{n}{\cdots} \circ N)(2)$.

Let $x' \in \mathrm{Cu}(A)$ be such that $x' \ll x$, and take y', y such that $x' \ll y' \ll y \ll x$. Then, since x is $(2, n)$-divisible, we obtain $y_1, \ldots, y_n$ satisfying $y \ll \sum y_j$ and $2y_j \leqslant x$ for each j. Take $y_j' \ll y_j$ such that $y \ll \sum y_j'$. By [**51**, Lemma 2.2], for each j we obtain an element r_j such that $y_j + r_j \leqslant x \leqslant 2r_j$ and $y_j' \leqslant r_j$.

Using (i), we find $r \in \mathrm{Cu}(A)$ satisfying $y \ll N_{2,n} r$ and $r \leqslant r_j$ for every j. Take $r' \ll r$ such that $y \ll N_{2,n} r'$. Using (O5) at $r' \ll r \leqslant x$, we obtain $c \in \mathrm{Cu}(A)$ such that $r' + c \leqslant x \leqslant r + c$.

In particular, since one has $y_j + r \leqslant x \leqslant r + c$, we obtain

$$\begin{aligned} 2y &\leqslant 2y_1 + \ldots + 2y_n \leqslant (2y_1 + \ldots + 2y_n) + r \\ &\leqslant (y_1 + 2y_2 + \ldots + 2y_n) + r + c \leqslant \ldots \leqslant r + (2n)c. \end{aligned}$$

Using (ii), we find elements g', $g \in \mathrm{Cu}(A)$ such that

$$2g \ll r, \quad y' \ll g' + M(n)c, \text{ and } g' \ll g.$$

Applying [**51**, Lemma 2.2] at $2g \ll r$, we find $d \in \mathrm{Cu}(A)$ such that $g' + d \leqslant r \leqslant 2d$. Thus, we have $y' \ll y \leqslant N_{2,n} r' \leqslant 2N_{2,n} d$. By (i), we find $f \in \mathrm{Cu}(A)$ such that

$$y' \leqslant N(2N_{2,n}) f, \text{ and } f \ll r', d.$$

By (O6) applied at $x' \ll y' \ll g' + M(n)c$, one finds elements s', s, t', $t \in \mathrm{Cu}(A)$ such that

$$x' \ll s' + t', \quad s' \ll s \leqslant y', g', \text{ and } t' \ll t \leqslant y', M(n)c.$$

Set $N_1 := N(\max\{N(2N_{2,n}), M(n)\})$ and $N_2 := N(N(2N_{2,n}))$. Applying (i) at $t' \ll t \leqslant M(n)c$ and $t' \ll t \leqslant y' \leqslant N(2N_{2,n})f$, we obtain $x_1 \in \mathrm{Cu}(A)$ such that $x_1 \ll c$, f and $t' \ll N_1 x_1$. Take x_1', $x_1'' \in \mathrm{Cu}(A)$ such that $x_1' \ll x_1'' \ll x_1$ and $t' \leqslant N_1 x_1'$.

Using (i) again, but this time at $s' \ll s \leqslant y' \leqslant N(2N_{2,n})f$, we obtain $x_2 \in \mathrm{Cu}(A)$ such that $x_2 \ll f$, s and $s' \ll N_2 x_2$. As before, take x_2', $x_2'' \in \mathrm{Cu}(A)$ such that $x_2' \ll x_2'' \ll x_2$ and $s' \leqslant N_2 x_2'$.

One has

$$x_1 + f \ll c + r' \leqslant x, \quad x_2 + f \ll s + d \leqslant g' + d \leqslant r \leqslant x, \text{ and } x_1, x_2 \ll f.$$

By (iii), we find z_1, $z_2 \in \mathrm{Cu}(A)$ such that

$$z_1 + z_2 \leqslant x, \quad \text{and} \quad x_1'' + x_2'' \ll kz_1, kz_2.$$

Applying (i) one last time, we find $z \in \mathrm{Cu}(A)$ such that

$$z \leqslant z_1, z_2, \quad \text{and} \quad x_1' + x_2' \leqslant N(k)z.$$

The element z satisfies $2z \leqslant z_1 + z_2 \leqslant x$, and

$$x' \leqslant t' + s' \leqslant (N_1 + N_2)(x'_1 + x'_2) \leqslant (N_1 + N_2)N(k)z,$$

as desired. □

One can check that real rank zero, or stable rank one C*-algebras satisfy (i)–(iii) in theorem 3.15 above. In particular, this recovers the discrete part of [**2**, Theorem 5.5].

Corollary 3.16. *Let A be a* C*-*algebra. Assume that A is either of stable rank one, or real rank zero. Then,* $\mathrm{Cu}(A)_{\mathrm{div}} = \mathrm{Cu}(A)$ *if and only if* $\mathrm{Cu}(A)_{\mathrm{Div}} = \mathrm{Cu}(A)$.

Proof. If A has real rank zero, use example 3.11. For stable rank one C*-algebras, this just amounts to an inspection of a number of proofs:

[**2**, Lemma 5.3] implies that (i) is satisfied with $N = 2n - 1$.

The proof of [**51**, Lemma 5.5], but using weak cancellation instead of residually stably finiteness, gives (ii) in theorem 3.15 with $k = 2$.

Finally, (iii) follows from an inspection of [**49**, 7.6–7.8] using that, when A has stable rank one, (O8) can be used without the assumption of an element being idempotent; see Definition 7.2 and Proposition 7.5 in [**49**] for more details. □

Remark 3.17. One can actually show that $\mathrm{Cu}(A)_{\mathrm{mdiv}} = \mathrm{Cu}(A)$ if and only if $\mathrm{Cu}(A)_{\mathrm{mDiv}} = \mathrm{Cu}(A)$ whenever A is of stable rank one. Indeed, given $x \in \mathrm{Cu}(A)$ such that $\mathrm{div}_{2m}(mx) < \infty$, it follows directly from [**2**, Theorem 5.5] that $\mathrm{Div}_{2m}(mx) < \infty$.

The same result holds for real rank zero C*-algebras by example 3.11.

Question 3.18. Let A be a C*-algebra such that $\mathrm{Cu}(A) = \mathrm{Cu}_{\mathrm{mdiv}}(A)$. When does A satisfy $\mathrm{Cu}(A) = \mathrm{Cu}_{\mathrm{mDiv}}(A)$?

4. Bounded divisibility

We focus in this section on C*-algebras where there is a global bound for the divisibility of the elements. That is, we look at those C*-algebras A such that there exist n, $m \in \mathbb{N}$ with $\mathrm{div}_m([a]) \leqslant n$ for every $a \in A_+$. More generally, we focus on those that satisfy $\sup_{a\in A_+} \mathrm{div}_{2m}(m[a]) < \infty$ for some $m \in \mathbb{N}$. As we will see in theorem 5.11, these algebras will have a nowhere scattered multiplier algebra.

We begin the section by providing sufficient conditions for this bounded divisibility to occur (propositions 4.1 and 4.3), and by showing that C*-algebras with this property are closed under extensions (proposition 4.5). Note that we have already seen some examples, such as tracially $\mathcal{Z}$-stable C*-algebras, or C*-algebras of real rank zero; see examples 3.10 and 3.11.

In theorem 4.11, we prove that the set of soft elements characterizes when a C*-algebra has multiples of bounded divisibility, and we use this to deduce the property for simple, weakly cancellative C*-algebras of Cuntz covering dimension zero; see corollary 4.13.

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As mentioned in [41], it is unclear if $\dim_{\mathrm{nuc}}(A) < \infty$ (with A nowhere scattered) implies $\mathrm{div}(a) < \infty$ for every $a \in A_+$. However, one still gets the following result, which is a direct consequence of [41, Proposition 3.2].

PROPOSITION 4.1. *Let A be a nowhere scattered* C*-*algebra of finite nuclear dimension at most k. Then, for every $[a] \in \mathrm{Cu}(A)$, one has*

$$\mathrm{div}_{4(k+1)}(2(k+1)[a]) \leqslant 8(k+1)^2.$$

In particular, one has $\mathrm{Cu}(A) = \mathrm{Cu}(A)_{\mathrm{mdiv}}$.

Proof. If A is nowhere scattered and of nuclear dimension at most k, so is $A \otimes \mathcal{K}$.

Let $[a] \in \mathrm{Cu}(A)$. Upon passing to the hereditary C*-algebra $\overline{a(A \otimes \mathcal{K})a}$, we may assume that a is full in $A \otimes \mathcal{K}$; note that $\overline{a(A \otimes \mathcal{K})a}$ is still nowhere scattered by [49, Proposition 4.1], and has nuclear dimension at most k by [55, Proposition 2.5].

Recall from [49, Theorem 3.1] that $A \otimes \mathcal{K}$ is nowhere scattered if and only if it has no finite-dimensional irreducible representations. By [41, Proposition 3.2], there exists $[b] \in \mathrm{Cu}(A)$ such that

$$[(a-\varepsilon)_+] \leqslant 4(k+1)[b], \text{ and } 4(k+1)[b] \leqslant 2(k+1)[a].$$

Thus, we obtain that $2(k+1)[(a-\varepsilon)_+] \leqslant 8(k+1)^2[b]$. Since $2(k+1)[a]$ can be written as the supremum of the elements $2(k+1)[(a-\varepsilon)_+]$ with $\varepsilon \to 0$, it follows that $\mathrm{div}_{4(k+1)}(2(k+1)[a]) < 8(k+1)^2$, as desired. □

Let us denote by $F(A)$ the central sequence algebra, as defined in [24, Definition 1.1]. Given a separable, unital C*-algebra A, it is not known if A being $\mathcal{Z}$-stable is equivalent to $F(A)$ admitting no characters; see [26]. Although this question remains open, one does have the following result from [12, Part II, Article B, Proposition 2.8].

PROPOSITION 4.2 [12]. *Let A be a separable* C*-*algebra. Assume that $F(A)$ admits no characters. Then, for every $m \in \mathbb{N}$ there exists n such that $\mathrm{div}_m([a]) \leqslant n$ for all $a \in A_+$.*

Proof. It follows from [40, Corollary 5.6] that $F(A)$ has no characters if and only if $\mathrm{div}_2([1]) < \infty$ in $\mathrm{Cu}(F(A))$. Let $n_0 = \mathrm{div}_2([1])$. [12, Part II, B, Proposition 2.8] implies that $\mathrm{div}_2([a]) \leqslant n_0$ for every $a \in A_+$.

An inductive argument now shows that $\sup_{a \in A_+} \mathrm{div}_m([a]) < \infty$ for every $m \in \mathbb{N}$; see, for example, [51, Lemma 3.4]. □

Given $[a], [b] \in \mathrm{Cu}(A)$, recall that we write $[a] <_s [b]$ whenever there exists $\gamma < 1$ such that $\lambda([a]) \leqslant \gamma\lambda([b])$ for every $\lambda \in F(\mathrm{Cu}(A)) \cong \mathrm{QT}(A)$; see [41, 2.1].

Let $k \in \mathbb{N}$. As defined in [54, Definition 2.1], a C*-algebra A is said to have k-*comparison* if for every $[a], [b_0], \ldots, [b_k] \in \mathrm{Cu}(A)$ such that $[a] <_s [b_i]$ for all i, we have $[a] \leqslant \sum_i [b_i]$.

Also recall that A has a *surjective rank map* if every element in $L(F(\mathrm{Cu}(A)))$ can be realized as a rank function; see [2, Section 7] for details.

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Proposition 4.3. *Let A be a nowhere scattered* C*-*algebra with k-comparison. Assume that the family of separable sub-*C*-*algebras of A with a surjective rank map is σ-complete and cofinal. Then,* $\mathrm{Div}_{2(k+1)}((k+1)[a]) \leqslant 4(k+1)^3$ *for every* $[a] \in \mathrm{Cu}(A)$.

Proof. Let $x := [a]$. By [48, Proposition 6.1] and [20, Proposition 3.8.1], there exists a separable sub-C*-algebra $B \subseteq A$ that has a surjective rank map, is nowhere scattered, contains x and is such that the induced inclusion map $\mathrm{Cu}(B) \to \mathrm{Cu}(A)$ is an order-embedding.

By definition, every element in $L(F(\mathrm{Cu}(B)))$ can be realized as a rank function. Thus, there exists $y \in \mathrm{Cu}(B)$ such that $\widehat{y} = \frac{1}{3(k+1)}\widehat{x}$. This implies, in particular, that

$$2(k+1)\widehat{y} \leqslant \frac{2}{3}\widehat{x}, \quad \text{and} \quad \widehat{x} \leqslant 3(k+1)\widehat{y} = \frac{3}{4}(4(k+1))\widehat{y}.$$

Applying k-comparison, one obtains $2(k+1)y \leqslant (k+1)x$ and $x \leqslant 4(k+1)^2 y$ in $\mathrm{Cu}(B)$. Using that the map $\mathrm{Cu}(B) \to \mathrm{Cu}(A)$ is an order-embedding, the same is true in $\mathrm{Cu}(A)$. Thus, we have

$$2(k+1)y \leqslant (k+1)x \leqslant 4(k+1)^3 y,$$

as desired. □

Remark 4.4. As shown in [2, Theorem 7.14], a separable, nowhere scattered C*-algebra of stable rank one always has a surjective rank map. Thus, proposition 4.3 above applies to all stable rank one C*-algebras with k-comparison.

In fact, it was shown in [2, Theorem 8.12] that a separable, nowhere scattered C*-algebra of stable rank one has k-comparison if and only if it has strict comparison. Consequently, we get that $\mathrm{div}_2(x) \leqslant 4$ for every $x \in \mathrm{Cu}(A)$. Further, a small modification of the proof shows that $\mathrm{Cu}(A)$ is almost divisible (i.e. 0-almost divisible). Indeed, following the notation of the proof, given any $x \in \mathrm{Cu}(A)$ and any $k \in \mathbb{N}$ find $y \in \mathrm{Cu}(B)$ such that $\widehat{y} = \frac{2}{2k+1}\widehat{x}$. A direct computation shows that

$$k\widehat{y} = \frac{2k}{2k+1}\widehat{x}, \quad \text{and} \quad \widehat{x} \leqslant \left(\frac{2k+1}{2k+2}\right)(k+1)\widehat{y}.$$

Using strict comparison, one gets $ky \leqslant x \leqslant (k+1)y$, as desired.

The following is a generalization of [49, Proposition 4.2].

Proposition 4.5. *Let I be an ideal of a* C*-*algebra A. Assume that there exist m_0, m_1, n_0, $n_1 \in \mathbb{N}$ such that*

$$\mathrm{div}_{2m_0}(m_0[a]) \leqslant n_0, \quad \textit{and} \quad \mathrm{div}_{2m_1}(m_1[b]) \leqslant n_1$$

for every $a \in (A/I)_+$ and every $b \in I_+$. Then,

$$\mathrm{div}_{2m_0 m_1}((m_0 m_1)[c]) \leqslant m_0(m_1 n_0 + n_1)$$

for every $c \in A_+$.

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Conversely, $\mathrm{div}_{2m}(m[a]), \mathrm{div}_{2m}(m[b]) \leqslant n$ *for every* $a \in (A/I)_+$ *and* $b \in I_+$ *whenever* $\sup_{c\in A_+} \mathrm{div}_{2m}(m[c]) \leqslant n$ *for some* $m, n \in \mathbb{N}$.

Proof. It follows from [**13**] that $\mathrm{Cu}(I)$ can be identified with an ideal of $\mathrm{Cu}(A)$, and that $\mathrm{Cu}(A/I)$ is naturally isomorphic to the quotient $\mathrm{Cu}(A)/\mathrm{Cu}(I)$. We will make these identifications without explicitly writing the isomorphisms.

Let $c \in A_+$, and set $x := [c]$. Denote by $\mathrm{Cu}(\pi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(A/I)$ the induced quotient map. Given x', x'' in $\mathrm{Cu}(A)$ such that $x' \ll x'' \ll x$, we can apply the bounded divisibility of $\mathrm{Cu}(A/I)$ to obtain elements $y_j'', y_j', y_j \in \mathrm{Cu}(A)$ for $j = 1, \ldots, n_0$ such that

$$(2m_0)\,\mathrm{Cu}(\pi)(y_j) \leqslant \mathrm{Cu}(\pi)(m_0x), \quad \mathrm{Cu}(\pi)(m_0x'') \leqslant \mathrm{Cu}(\pi)(y_1'') + \ldots + \mathrm{Cu}(\pi)(y_{n_0}''),$$

and $y_j'' \ll y_j' \ll y_j$ for each j. Note that this is possible because $\mathrm{Cu}(\pi)(x) = [\pi(c)]$ and $\mathrm{Cu}(\pi)(x)$ has a representative in $(A/I)_+$.

Take $w \in \mathrm{Cu}(I)$ such that $w = 2w$, and

$$(2m_0)y_j' \ll (2m_0)y_j \leqslant m_0x + w, \text{ and } m_0x'' \leqslant y_1'' + \ldots + {y_{n_0}}' + w.$$

Using [**49**, Proposition 7.8], we find elements $z_j', z_j \in \mathrm{Cu}(A)$ satisfying

$$(2m_0)z_j \ll m_0x, \quad y_j'' \ll z_j' + w, \quad \text{and} \quad z_j' \ll z_j$$

for every $j = 1, \ldots, n_0$.

In particular, we have

$$x' \ll x'' \leqslant m_0x'' \leqslant z_1' + \ldots + z_{n_0}' + w.$$

Applying (O6), we obtain an element $\tilde{r} \in \mathrm{Cu}(A)$ such that

$$x' \ll z_1' + \ldots + z_{n_0}' + \tilde{r}, \quad \text{and} \quad \tilde{r} \leqslant x'', w.$$

Note that, since $\tilde{r} \leqslant x'' \ll [c]$ with $c \in A_+$, there exists $b \in A_+$ such that $r := [b]$ satisfies $r \leqslant \tilde{r}$ and $x' \ll z_1' + \ldots + z_{n_0}' + r$; see, for example, [**49**, Lemma 3.3(1)]. Further, since we also have $[b] \leqslant w$, one gets $b \in I_+$. Take $r' \ll r$ such that $x' \leqslant z_1' + \ldots + z_{n_0}' + r'$.

Now, using that $\mathrm{div}_{2m_1}(m_1[b]) \leqslant n_1$, one gets elements $r_1, \ldots, r_{n_1}$ such that

$$(2m_1)r_i \ll m_1r, \quad \text{and} \quad m_1r' \ll r_1 + \ldots + r_{n_1}$$

for each $i \leqslant n_1$.

Thus, it follows that

$$(m_0m_1)x' \leqslant (m_0m_1)z_1' + \ldots + (m_0m_1)z_{n_0}' + m_0r_1 + \ldots + m_0r_{n_1}$$

and

$$(2m_0m_1)z_j' \leqslant (m_0m_1)x, \quad (2m_0m_1)r_i \leqslant (m_0m_1)r \leqslant (m_0m_1)x$$

for each i and j.

This proves that

$$\mathrm{div}_{2m_0m_1}((m_0m_1)x) \leqslant m_0(m_1n_0 + n_1),$$

as desired.

The converse follows from a standard argument; see [**51**, Proposition 3.9]. □

The previous result also works, mutatis-mutandis, with Div() instead of div(). In this case, an inspection of [**51**, Theorem 3.10] gives the following:

THEOREM 4.6. *Let I be an ideal of a C*-algebra A. Assume that there exist m_0, m_1, n_0, $n_1 \in \mathbb{N}$ such that*

$$\mathrm{Div}_{4m_0}(m_0[a]) \leqslant n_0, \quad \textit{and} \quad \mathrm{Div}_{4m_1}(m_1[b]) \leqslant n_1$$

for every $a \in I_+$ and every $b \in (A/I)_+$. Then, there exist M, $N \in \mathbb{N}$ depending only on m_0, m_1, n_0, n_1 such that

$$\mathrm{Div}_{2M}(M[c]) \leqslant N$$

for every $c \in A_+$.

Conversely, $\mathrm{Div}_{2m}(m[a])$, $\mathrm{Div}_{2m}(m[b]) \leqslant n$ *for every $a \in (A/I)_+$ and $b \in I_+$ whenever* $\sup_{c\in A_+} \mathrm{Div}_{2m}(m[c]) \leqslant n$ *for some m, $n \in \mathbb{N}$.*

Proof. Assume first that I and A/I satisfy the stated conditions. Let $c \in A_+$ and take $\varepsilon > 0$. Set

$$x := [c], \quad x'' := [(c-\varepsilon)_+], \quad \text{and} \quad x' := [(c-2\varepsilon)_+].$$

As before, denote by π the induced quotient map $\mathrm{Cu}(A) \to \mathrm{Cu}(A/I)$. Applying $\mathrm{Div}_{4m_1}(m_1\pi(x)) \leqslant n_1$ to $\pi(x'') \ll \pi(x)$, we find $y \in \mathrm{Cu}(A)$ such that

$$4m_1\pi(y) \leqslant m_1\pi(x), \text{ and } m_1\pi(x'') \ll n_1\pi(y).$$

Take y', $y'' \in \mathrm{Cu}(A)$ such that $y' \ll y'' \ll y$ and $m_1\pi(x'') \leqslant n_1\pi(y')$. Find $w \in \mathrm{Cu}(I)$ such that

$$4m_1y'' \ll 4m_1y \leqslant m_1x + w, \quad m_1x'' \leqslant n_1y' + w, \quad \text{and} \quad w = 2w.$$

Applying [**49**, Proposition 7.8] at the first inequality and the pair $y' \ll y''$, we find $z \in \mathrm{Cu}(A)$ satisfying

$$4m_1z \ll m_1x, \quad \text{and} \quad y' \ll z + w.$$

Choose $z' \in \mathrm{Cu}(A)$ such that $z' \ll z$ and $y' \ll z' + w$. Note that we have

$$4m_1z + m_1x \leqslant 2m_1x. \tag{4.1}$$

Further, using that $y' \ll z' + w$, we obtain

$$x' \ll x'' \leqslant m_1x'' \leqslant n_1y' + w \leqslant n_1z' + w.$$

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Applying (O6), we find $\tilde{r} \in S$ such that

$$x' \ll n_1 z' + \tilde{r}, \quad \text{and} \quad \tilde{r} \leqslant x', w.$$

Proceeding as in the previous proof, note that, since $x'' = [(c-\varepsilon)_+]$ and $\tilde{r} \leqslant w \in \mathrm{Cu}(I)$, we can find $a \in I_+$ such that $r := [a]$ satisfies $r \leqslant \tilde{r}$ and $x' \ll n_1 z' + r$. Take r' with $r' \ll r$ and $x' \ll n_1 z' + r'$.

Applying $\mathrm{Div}_{4m_0}(m_0 r) \leqslant n_0$ to $m_0 r' \ll m_0 r$, we get $t \in \mathrm{Cu}(I)$ such that

$$4m_0 t \leqslant m_0 r, \quad \text{and} \quad m_0 r' \leqslant n_0 t. \tag{4.2}$$

Recall that $r \leqslant x$. Combining (4.1) and (4.2) we obtain

$$4m_0 m_1 (z+t) \leqslant 4m_0 m_1 z + m_0 m_1 x \leqslant 2m_0 m_1 x.$$

One also gets

$$2m_0 m_1 x' \leqslant 2m_0 m_1 (n_1 z' + r') \leqslant 2m_1 (m_0 n_1 z' + n_0 t) \leqslant 2m_0 m_1 n_0 n_1 (z' + t),$$

which shows that $\mathrm{Div}_{4m_0 m_1}(2m_0 m_1 x) \leqslant 2m_0 m_1 n_0 n_1$, as desired.

As in the proof of the previous proposition, we note that the converse follows from similar arguments to those in [**51**, Proposition 3.9]. □

As noted in lemma 3.12, having a sup-dense subset of elements with finite weak divisibility does not imply that every element has such property. However, proposition 4.8 below shows that it is enough to check finite weak divisibility for strongly soft elements (which, in general, are not sup-dense).

4.7. Strongly soft elements and retracts

Let A be a C*-algebra. As defined in [**52**], we denote by $\mathrm{Cu}(A)_{\mathrm{soft}}$ the set of *strongly soft elements*, that is, those elements $x \in \mathrm{Cu}(A)$ such that for every $x' \ll x$ there exists $t \in \mathrm{Cu}(A)$ satisfying $x' + t \leqslant x \leqslant \infty t$. When A is residually stably finite, $[a] \in \mathrm{Cu}(A)$ is strongly soft if and only if $\overline{aAa}$ has no nonzero, unital quotients; see [**52**, Proposition 4.16].

Under certain assumptions, given any $x \in \mathrm{Cu}(A)$ one can find the largest strongly soft element below x. When this is the case, we denote by $\sigma\colon \mathrm{Cu}(A) \to \mathrm{Cu}(A)_{\mathrm{soft}}$ the map that sends an element x to the largest strongly soft element dominated by x.

In [**6**, Theorem 5.6], it is shown that σ can always be defined whenever A is separable, has the Global Glimm Property and its Cuntz semigroup is *left-soft separative*, that is, if for any triplet of elements $y, t \in S$ and $x \in S_{\mathrm{soft}}$ satisfying

$$x + t \ll y + t, \quad t \ll \infty y, \quad \text{and} \quad t \ll \infty x,$$

we have $x \ll y$; see [**6**, Definition 3.2]. As explained in [**6**, Section 3], C*-algebras with strict comparison or stable rank one have a left-soft separative Cuntz semigroup.

In this case, σ is an order- and suprema-preserving superadditive map that satisfies $x \leqslant \sigma(x) + t$ whenever $x \leqslant \infty t$. Recall that a map ϕ is *superadditive* if $\phi(x+y) \leqslant \phi(x) + \phi(y)$.

PROPOSITION 4.8. *Let A be a separable C^*-algebra with the Global Glimm Property such that $\mathrm{Cu}(A)$ is left-soft separative, and let $x \in \mathrm{Cu}(A)$ and $m \in \mathbb{N}$. Then*

$$\mathrm{div}_m(\sigma(x)) - 1 \leqslant \mathrm{div}_m(x) \leqslant \mathrm{div}_m(\sigma(x)) + 1.$$

Proof. First, to see that $\mathrm{div}_m(\sigma(x)) \leqslant \mathrm{div}_m(x) + 1$, assume that $\mathrm{div}_m(x) = n$ for some $n \in \mathbb{N}$, since otherwise there is nothing to prove.

Let $s \ll \sigma(x)$. Using that σ preserves suprema of increasing sequences, we can find $x' \ll x$ such that $s \ll \sigma(x')$.

Since A satisfies the Global Glimm Property, one can use [**52**, Proposition 7.7] to deduce that there exists $t \in \mathrm{Cu}(A)_{\mathrm{soft}}$ such that $(nm)t \leqslant x \leqslant \infty t$. Note that $(nm)t \leqslant \sigma(x)$ because t is strongly soft. Using that x is weakly (m, n)-divisible, one finds $z_1, \ldots, z_n \in \mathrm{Cu}(A)$ such that

$$\sigma(x') \leqslant z_1 + \ldots + z_n, \quad \text{and} \quad mz_j \leqslant x$$

for every j.

Now, using that $z_i \leqslant \infty t$ for every i and that $\mathrm{Cu}(A)$ is left-soft separative, we obtain

$$\sigma(x') \leqslant z_1 + \ldots + z_n \leqslant (\sigma(z_1) + t) + z_2 + \ldots + z_n \leqslant \sigma(z_1) + \ldots + \sigma(z_n) + nt.$$

Further, using that σ is superadditive, we also get $m\sigma(z_j) \leqslant \sigma(mz_j) \leqslant \sigma(x)$. Since $n(mt) \leqslant \sigma(x)$, we deduce that $\mathrm{div}_n(\sigma(x)) \leqslant n + 1$, as desired.

To prove that $\mathrm{div}_m(x) \leqslant \mathrm{div}_m(\sigma(x)) + 1$, take $t \in \mathrm{Cu}(A)$ such that $mt \leqslant x \leqslant \infty t$. Note that this can be done because A has the Global Glimm Property and, consequently, $\mathrm{Cu}(A)$ is (m, ω)-divisible; see paragraph 2.3. Using that $x \leqslant \infty t$, we have $x \leqslant \sigma(x) + t$. Let $x' \ll x$. Since $x' \ll \sigma(x) + t$, we can use (O6) to obtain an element y such that $x' \ll y + t$ with $y \ll \sigma(x)$.

Let n be such that $\sigma(x)$ is weakly (m, n)-divisible. Then, there exist $z_1, \ldots, z_n$ satisfying

$$y \leqslant z_1 + \ldots + z_n, \quad \text{and} \quad mz_j \leqslant \sigma(x)$$

for each j.

We obtain

$$x' \leqslant z_1 + \ldots + z_n + t, \quad mt \leqslant x, \quad \text{and} \quad mz_j \leqslant \sigma(x) \leqslant x$$

for every j, which implies that $\mathrm{div}_m(x) \leqslant n + 1$. □

LEMMA 4.9. *Let A be a C^*-algebra, and let $s, t \in \mathrm{Cu}(A)$. Assume that s is strongly soft. Then, the element $s \wedge \infty t$ is strongly soft in $\mathrm{Cu}(A)$.*

Proof. Let $x', x \in \mathrm{Cu}(A)$ be such that $x' \ll x \ll s \wedge \infty t$. By definition, this implies that $x' \ll s$ and $x' \ll \infty t$.

Since s is strongly soft, we find $y \in \mathrm{Cu}(A)$ such that $x' + y \leqslant s \leqslant \infty y$. Thus, using that taking infima with an idempotent is a monoid morphism ([**1**, Section 2]), we have

$$x' + y \wedge \infty t = (x' + y) \wedge \infty t \leqslant s \wedge \infty t \leqslant \infty y \wedge \infty t = \infty(y \wedge \infty t).$$

This shows that $s \wedge \infty t$ is strongly soft. □

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LEMMA 4.10. *Let A be a* C*-*algebra with the Global Glimm Property. Then, for* $x', x \in \mathrm{Cu}(A)$ *such that* $x' \ll x$, *there exist* $s_1, s_2 \in \mathrm{Cu}(A)_{\mathrm{soft}}$ *such that* $s_1, s_2 \leqslant x$ *and* $x' \leqslant s_1 + s_2$.

Proof. Let $x', x \in \mathrm{Cu}(A)$ satisfy $x' \ll x$. It follows from [**52**, Proposition 7.7] that there exists $y \in \mathrm{Cu}(A)_{\mathrm{soft}}$ satisfying $y \leqslant x \leqslant \infty y$. In particular, one has $x' \ll \infty y$.

By [**52**, Proposition 5.6], we can take $y', y'' \in \mathrm{Cu}(A)_{\mathrm{soft}}$ such that $y' \ll y'' \ll y$ and $x' \ll \infty y'$. Since y'' is soft, we find $t \in \mathrm{Cu}(A)_{\mathrm{soft}}$ such that $y' + t \leqslant y'' \leqslant \infty t$; see [**52**, Proposition 4.13]. Now, by (O5) applied to $y'' \ll y \leqslant x$, there exists $c \in \mathrm{Cu}(A)$ satisfying

$$y'' + c \leqslant x \leqslant y + c.$$

Thus, we have

$$y' + (c+t) \leqslant x \leqslant y + c \leqslant y + (c+t).$$

Using that $x' \leqslant \infty y'$, we obtain

$$x' \leqslant x \wedge \infty y'' \leqslant (y + (c+t)) \wedge \infty y'' = y \wedge \infty y' + (c+t) \wedge \infty y''$$

and

$$y \wedge \infty y', (c+t) \wedge \infty y'' \leqslant x.$$

By lemma 4.9, the elements $y \wedge \infty y''$ and $t \wedge \infty y''$ are strongly soft. Further, note that one gets

$$(c+t) \wedge \infty y' = c \wedge \infty y' + t \wedge \infty y''$$

with $c \wedge \infty y'' \leqslant \infty y'' \leqslant \infty(t \wedge \infty y'')$.

Using [**52**, Theorem 4.14(2)], we deduce that $(c+t) \wedge \infty y''$ is also strongly soft. Set $s_1 := y \wedge \infty y''$ and $s_2 := (c+t) \wedge \infty y''$. We have $s_1, s_2 \in \mathrm{Cu}(A)_{\mathrm{soft}}$,

$$s_1, \quad s_2 \leqslant x, \quad \text{and} \quad x' \leqslant s_1 + s_2,$$

as required. □

When $\mathrm{Cu}(A)$ is not left-soft separative, we still have the following:

THEOREM 4.11. *Let A be a* C*-*algebra with the Global Glimm Property, and let* $m \in \mathbb{N}$. *Then, the following are equivalent:*

(i) *there exists* $n \in \mathbb{N}$ *such that* $\mathrm{div}_{2m}(m[a]) \leqslant n$ *for every* $[a] \in \mathrm{Cu}(A)$;

(ii) *there exists* $n \in \mathbb{N}$ *such that* $\mathrm{div}_{2m}(m[a]) \leqslant n$ *for every* $[a] \in \mathrm{Cu}(A)_{\mathrm{soft}}$.

Proof. That (i) implies (ii) is trivial.

To prove the converse, let $x', x \in \mathrm{Cu}(A)$ be such that $x' \ll x$. Using lemma 4.10, one finds $s_1, s_2 \in \mathrm{Cu}(A)_{\mathrm{soft}}$ satisfying $s_1, s_2 \leqslant x$ and $x' \leqslant s_1 + s_2$. In particular, $ms_1, ms_2 \leqslant mx$ and $mx' \leqslant ms_1 + ms_2$.

By lemma 3.3 (ii), we have

$$\mathrm{div}_{2m}(mx) \leqslant \mathrm{div}_{2m}(ms_1) + \mathrm{div}_{2m}(ms_2) \leqslant 2n,$$

as desired. □

COROLLARY 4.12. *Let A be a* C^**-algebra with the Global Glimm Property. Assume that* $\mathrm{Cu}(A)_{\mathrm{soft}}$ *is a retract of a Cu-semigroup S satisfying* $\sup_{s\in S}\mathrm{div}_{2k}(ks) < \infty$ *for some $k \in \mathbb{N}$. Then, there exist n, $m \in \mathbb{N}$ such that* $\mathrm{div}_{2m}(m[a]) \leqslant n$ *for every* $[a] \in \mathrm{Cu}(A)$*.*

Proof. Using that $\mathrm{Cu}(A)_{\mathrm{soft}}$ is a retract of S, it follows that

$$\mathrm{div}_{2k}(ky) \leqslant \sup_{s\in S}\mathrm{div}_{2k}(ks) < \infty$$

for every $y \in \mathrm{Cu}(A)_{\mathrm{soft}}$.

Applying theorem 4.11, there exist n, $m \in \mathbb{N}$ such that $\mathrm{div}_{2m}(mx) \leqslant n$ for every $x \in \mathrm{Cu}(A)$. □

Recall from [**50**, Definition 3.1] that one says that $\mathrm{Cu}(A)$ has *covering dimension zero* if, whenever $x' \ll x \ll y_1 + y_2$, there exist z_1, $z_2 \in \mathrm{Cu}(A)$ such that $x' \leqslant z_1 + z_2 \leqslant x$. Examples of dimension zero include the Cuntz semigroup of any real rank zero C^*-algebra, as well as other semigroups, such as the Cuntz semigroup of the Jacelon–Razak algebra; see [**50**, Section 5].

We say that a Cuntz semigroup is *weakly cancellative* if $x \ll y$ whenever $x + z \ll y + z$ for some element z. Stable rank one C^*-algebras have a weakly cancellative Cuntz semigroup by [**44**, Theorem 4.3].

COROLLARY 4.13. *Let A be a simple, non-elementary* C^**-algebra of Cuntz covering dimension zero. Assume that* $\mathrm{Cu}(A)$ *is weakly cancellative. Then there exists $n \in \mathbb{N}$ such that* $\mathrm{div}_2([a]) \leqslant n$ *for every* $[a] \in \mathrm{Cu}(A)$*.*

Proof. Assume first that A is separable. [**50**, Theorem 7.10] shows that the submonoid $\mathrm{Cu}(A)_{\mathrm{soft}}$ is a retract of an almost divisible Cu-semigroup S; see also the proof of [**50**, Proposition 7.13].

As explained in example 3.11, almost divisibility implies $\mathrm{div}_2(s) \leqslant 3$ for every element $s \in S$. Thus, the result in this case follows from corollary 4.12 above.

If A is not separable, one can use the same techniques as in the proof of proposition 4.3 to get the desired result. □

5. Nowhere scattered corona and multiplier algebras

In this section, we study when a multiplier algebra of a σ-unital, nowhere scattered C^*-algebra A is nowhere scattered. Our main result is that this happens whenever $\sup_{a\in A_+}\mathrm{div}_{2m}(m[a]) < \infty$ for some $m \in \mathbb{N}$; see theorem 5.11. Note that the supremum is not taken over all the elements of the Cuntz semigroup (i.e. over $A \otimes \mathcal{K}$), but only over those with a representative in A_+. This set is called the *scale* of the Cuntz semigroup in [**4**, 4.2].

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As an application, we prove in theorem 5.12 that nowhere scattered C*-algebras of finite nuclear dimension, or real rank zero, or stable rank one and k-comparison all have a nowhere scattered multiplier algebra. We also show in theorem 5.18 that, for stable C*-algebras, the multiplier algebra being nowhere scattered implies that every element in the algebra has finite weak divisibility.

Let us begin the section with some examples:

Example 5.1. Let A be a weakly purely infinite C*-algebra. It was shown in [**25**, Proposition 4.11] that $\mathcal{M}(A)$ is weakly purely infinite, and thus nowhere scattered by [**49**, Example 3.3].

As a similar example, it follows from [**32**] that a simple, σ-unital, non-elementary (i.e. nowhere scattered) C*-algebra with continuous scale has a purely infinite corona algebra. This implies that $\mathcal{M}(A)$ is nowhere scattered by [**49**, Proposition 4.2]; see also [**23**] and [**30**].

The examples below appeared in [**49**, Examples 4.14, 4.15]. We recall them here for the convenience of the reader.

Example 5.2. Let A_k be the family of separable, simple, AH-algebras from [**40**] (see example 3.6). As shown in [**40**, Corollary 8.6], the product $\prod_k A_k$ has a one-dimensional, irreducible representation. By [**49**, Theorem 3.1], this implies that $\prod_k A_k$ is not nowhere scattered. Consequently, $A := \oplus A_k$ is a nowhere scattered C*-algebra with a multiplier algebra $\mathcal{M}(A) \cong \prod_k A_k$ that is not nowhere scattered.

Example 5.3. In [**45**, Theorem 1], Sakai constructs a simple C*-algebra A such that its derived algebra $D(A)$ satisfies $D(A)/A \cong \mathbb{C}$. Using that $D(A) \cong \mathcal{M}(A)$ for simple C*-algebras (see the remarks after [**35**, Proposition 2.6]), one sees that $\mathcal{M}(A)$ cannot be nowhere scattered.

The following lemma will play an important role in the proof of proposition 5.9.

Lemma 5.4. *Let A be a* C*-*algebra. The following are equivalent:*

(i) *A is nowhere scattered.*

(ii) *For every $a \in A_+$ and $n \in \mathbb{N}$, the class $[a \otimes 1_n] \in \mathrm{Cu}(A)$ is weakly $(2n, \omega)$-divisible.*

(iii) *For every $a \in A_+$, there exists $n = n(a) \in \mathbb{N}$ such that the class $[a \otimes 1_n] \in \mathrm{Cu}(A)$ is weakly $(n+1, \omega)$-divisible.*

Proof. We know from [**49**, Theorem 8.9] that A is nowhere scattered if and only if every element in $\mathrm{Cu}(A)$ is weakly $(2, \omega)$-divisible. Using [**49**, Theorem 8.9], it follows that every element is $(2n, \omega)$-divisible for every $n \in \mathbb{N}$. This proves that (i) implies (ii), and it is trivial that (ii) implies (iii).

To see that (iii) implies (i), assume for the sake of contradiction that A is not nowhere scattered. Then, we know from [**49**, Theorem 3.1] that there exist ideals $I \subseteq J \subseteq A$ such that J/I is elementary, that is, $\mathrm{Cu}(I/J) \cong \mathrm{Cu}(\mathbb{C}) \cong \overline{\mathbb{N}}$; see [**19**, Theorem 4.4.4].

Note that, since we are assuming that A satisfies (iii), both I and I/J satisfy (iii) as well. Let ϕ be an isomorphism from $\mathrm{Cu}(I/J)$ to $\overline{\mathbb{N}}$. Now, it follows from [**51**, Lemma 3.3] that for every positive element $b \in (I/J) \otimes \mathcal{K}$ there exists $a \in (I/J)_+$ such that $[b] \leqslant \infty[a]$ in $\mathrm{Cu}(I/J)$. Thus, let $a \in (I/J)_+$ be such that $\phi([a]) \neq 0$. We get $1 \ll 1 \leqslant \phi([a])$ and, using Rørdam's lemma (see e.g. [**46**, Theorem 2.30]), there exists $a' \in (I/J)_+$ such that $1 = \phi([a'])$.

Take $n \in \mathbb{N}$ such that $[a' \otimes 1_n]$ is weakly $(n+1, \omega)$-divisible. Since ϕ is an isomorphism, we must have that $\phi([a' \otimes 1_n]) = n$ is also weakly $(n+1, \omega)$-divisible. However, the only element $x \in \overline{\mathbb{N}}$ such that $(n+1)x \leqslant n$ is zero. This contradicts the weak $(n+1, \omega)$-divisibility of $[a' \otimes 1_n]$.

Thus, A has no nonzero elementary ideal-quotients, as desired. □

REMARK 5.5. In the proof of '(iii) $\Longrightarrow$ (i)' in lemma 5.4 above, we have used that a C*-algebra is nowhere scattered if and only if it has no elementary ideal-quotients. As shown in [**49**, Proposition 8.8], a Cu-semigroup S satisfying (O5)–(O8) has no elementary ideal-quotients if and only if every element in S is weakly $(2, \omega)$-divisible.

In light of this, one might expect '(iii) $\Longrightarrow$ (i)' to hold for every Cu-semigroup S satisfying (O5)–(O8), that is, that every element in S is weakly $(2, \omega)$-divisible whenever for every element $x \in S$ there exists n_x such that $n_x x$ is weakly $(n_x + 1, \omega)$-divisible.

However, this is not true: For example, $S = \{0, 1, \infty\}$ is a Cu-semigroup satisfying (O5)–(O8). S is not weakly $(2, \omega)$-divisible, but every element in S has a properly infinite multiple. Thus, it satisfies (iii), but not (i).

The reason for this disparity is due to the fact that, in the context of abstract Cuntz semigroups, $\overline{\mathbb{N}}$ is not the only elementary Cu-semigroup. In fact, $\{0, 1, \infty\}$ is elementary; see [**49**, Section 8].

Lemma 5.6 below is [**25**, Lemma 4.10], which uses some of the ideas from [**16**]. It was stated with the assumption of weak pure infiniteness, but an inspection of their proof shows that this is not actually needed. This was also stated, in a different way and with a different proof, in [**22**, Theorem 4.2].

Note that both of the strictly convergent sums in the right-hand side consist of pairwise orthogonal terms.

LEMMA 5.6. *Let A be a σ-unital* C*-*algebra, let $T \in \mathcal{M}(A)_+$ and let $\varepsilon > 0$. Then A has an increasing, countable, approximate unit $(e_n)_n$ of positive contractions with $e_{n+1}e_n = e_n$ such that*

$$T = a + \sum_{n=1}^{\infty} f_{2n-1}^{1/2} T f_{2n-1}^{1/2} + \sum_{n=1}^{\infty} f_{2n}^{1/2} T f_{2n}^{1/2}, \quad f_n := e_n - e_{n-1}$$

with $e_0 = 0$ and $a \in A$ satisfying $\|a\| \leqslant \varepsilon$.

In the proof of proposition 5.9 we will also need the following:

LEMMA 5.7. *Let A be a* C*-*algebra, and let $a, b \in A_+$ be elements such that $[a], [b] \in \mathrm{Cu}(A)$ have multiples of finite weak divisibility. Then, $[a+b]$ has a multiple of finite weak divisibility.*

In particular, there exists $n \in \mathbb{N}$ such that $(a+b)\otimes 1_n$ has a $(n+1,\,\omega)$-divisible class.

Proof. Let $n_a,\, n_b \in \mathbb{N}$ be such that $\mathrm{div}_{2n_a}(n_a[a]),\, \mathrm{div}_{2n_b}(n_b[b]) < \infty$.

Using that $n_a n_b[a+b] = [a\otimes 1_{n_a n_b} + b\otimes 1_{n_a n_b}]$, we can apply lemma 3.3 (iii) and get

$$\mathrm{div}_{2n_a n_b}(n_a n_b[a+b]) \leqslant \mathrm{div}_{2n_a n_b}(n_a n_b[a]) + \mathrm{div}_{2n_a n_b}(n_a n_b[b]).$$

Thus, applying lemma 3.4 to both summands, we obtain

$$\mathrm{div}_{2n_a n_b}(n_a n_b[a+b]) \leqslant \mathrm{div}_{2n_a}(n_a[a]) + \mathrm{div}_{2n_b}(n_b[b]) < \infty,$$

as desired. □

LEMMA 5.8. *Let A be a σ-unital C^*-algebra, and let $(a_i)_{i=1}^{\infty}$ be a bounded sequence in A of pairwise orthogonal elements such that $R=\sum_i a_i$ is strictly convergent and $a_i \perp e_{i-1}$ for every $i \geqslant 1$, where $(e_i)_i$ is an approximate unit as in lemma 5.6.*

Let $m \in \mathbb{N}$. Then for every $\varepsilon_0 > 0$, one has

$$\mathrm{div}_{2m}(m[R]) \leqslant \sup_i \sup_{0\leqslant\varepsilon\leqslant\varepsilon_0} \mathrm{div}_{2m}(m[(a_i-\varepsilon)_+])$$

in $\mathrm{Cu}(\mathcal{M}(A))$.

Proof. Let $\varepsilon_0 > 0$. Assume that $n := \sup_i \sup_{\varepsilon\leqslant\varepsilon_0} \mathrm{div}_{2m}(m[(a_i-\varepsilon)_+])$ is finite, since otherwise there is nothing to prove. Fix $\varepsilon \leqslant \varepsilon_0$ positive. By assumption, $(a_i - \frac{\varepsilon}{4})_+ \otimes 1_m$ is weakly $(2m,\, n)$-divisible for each i. Thus, we can find elements $b_{i,j} \in A\otimes\mathcal{K}$ for $j=1,\,\dots,\,n$ such that

$$m[(a_i-\varepsilon/3)_+] \leqslant [b_{i,1}] + \ldots + [b_{i,n}], \qquad \text{and} \qquad 2m[b_{i,j}] \leqslant m[(a_i-\varepsilon/4)_+]$$

for each i and j.

It follows from [**40**, Lemma 2.3(i)] that there exist elements $c_{i,j} \in M_m(A)_+$ with $c_{i,j} \precsim b_{i,j}$ such that

$$(a_i - 2\varepsilon/3)_+ \otimes 1_m = \sum_{j=1}^{n} c_{i,j}.$$

Set $d_{i,j} := ((a_i-2\varepsilon/3)_+ \otimes 1_m)c_{i,j}((a_i-2\varepsilon/3)_+\otimes 1_m) \in M_m(A)$. Note that every entry in $d_{i,j}$ is in $\overline{a_i A a_i}$, and is thus orthogonal with e_{i-1}. To see that the sums $R_j := \sum_i d_{i,j}$ are strictly convergent, we proceed as in [**22**, Proposition 4.4]. Using that $c_{i,j} \leqslant a_i \otimes 1_m$ at the third step, we have

$$\begin{aligned} d_{i,j}^2 &= ((a_i-2\varepsilon/3)_+\otimes 1_m)\left(c_{i,j}((a_i-2\varepsilon/3)_+\otimes 1_m)^2 c_{i,j}\right)((a_i-2\varepsilon/3)_+\otimes 1_m) \\ &\leqslant \|c_{i,j}\|^2 \|(a_i-2\varepsilon/3)_+\otimes 1_m\|^3 ((a_i-2\varepsilon/3)_+\otimes 1_m) \\ &\leqslant \left(\sup_i \|a_i\otimes 1_m\|^5\right)((a_i-2\varepsilon/3)_+\otimes 1_m), \end{aligned}$$

where note that the supremum over i is finite because the sequence $(a_i)_i$ is bounded.

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Now, given any $t \in M_m(A)$ and any pair $n < k$, and using that the $d_{i,j}$'s are pairwise orthogonal on i (and that so are the a_i's), one has

$$\begin{aligned}\left\| t\sum_{i=n}^{k} d_{i,j}\right\|^2 &= \left\| t\left(\sum_{i=n}^{k} d_{i,j}^2\right)t^*\right\| \\ &\leqslant \left(\sup_i \|a_i\otimes 1_m\|^5\right)\left\| t\left(\sum_{i=n}^{k}(a_i-2\varepsilon/3)_+\otimes 1_m\right)t^*\right\| \\ &\leqslant \left(\sup_i \|a_i\otimes 1_m\|^5\right)\|t\|\left\| t\left(\sum_{i=n}^{k}(a_i-2\varepsilon/3)_+\otimes 1_m\right)\right\|\end{aligned}$$

and, similarly,

$$\left\|\left(\sum_{i=n}^{k} d_{i,j}\right)t\right\|^2 \leqslant \left(\sup_i \|a_i\otimes 1_m\|^5\right)\|t\|\left\|\left(\sum_{i=n}^{k}(a_i-2\varepsilon/3)_+\otimes 1_m\right)t\right\|$$

Since $\sum_i a_i$ is strictly convergent, so is $\sum_i (a_i - 2\varepsilon/3)_+ \otimes 1_m$. Thus, it follows that $R_j = \sum_i d_{i,j}$ is a strictly convergent sum. Further, note that we have

$$(R-2\varepsilon/3)_+^3 \otimes 1_m = \sum_{j=1}^{n} R_j$$

and, therefore, $m[(R-\varepsilon)_+] \ll \sum_{j=1}^{n}[R_j]$.

Let $\delta > 0$ be such that $m[(R-\varepsilon)_+] \leqslant \sum_{j=1}^{n}[(R_j-\delta)_+]$. Applying [**22**, Lemma 2.2] at

$$d_{i,j}\otimes 1_{2m} \precsim c_{i,j}\otimes 1_{2m} \precsim b_{i,j}\otimes 1_{2m} \precsim (a_i-\varepsilon/4)_+\otimes 1_m$$

we find elements $r_{i,j}$ such that

$$(d_{i,j}-\delta)_+\otimes 1_{2m} = r_{i,j}(a_i\otimes 1_m)r_{i,j}^*, \quad \text{ and } \quad \sup_i \|r_{i,j}\|^2 \leqslant \frac{4}{\varepsilon}\sup_i \|d_{i,j}\otimes 1_{2m}\|.$$

Set $x_{i,j} := (a_i\otimes 1_m)^{1/2} r_{i,j}^*(d_{i,j}\otimes 1_{2m}-\delta)_+$. We get

$$x_{i,j}^*x_{i,j} = (d_{i,j}-\delta)_+^3\otimes 1_{2m}, \quad \text{ and } \quad x_{i,j}x_{i,j}^* \in \overline{(a_i\otimes 1_m)M_m(A)(a_i\otimes 1_m)},$$

with both $((a_i\otimes 1_m)^{1/2}r_{i,j}^*)_i$ and $(r_{i,j}^*(d_{i,j}\otimes 1_{2m}-\delta)_+)_i$ being bounded sequences for each fixed j.

Let j be fixed. To see that $\sum_i x_{i,j}$ is strictly convergent, we argue as before. First, note that one has

$$\begin{aligned} x_{i,j}^*x_{i,j} &\leqslant \sup_i \|d_{i,j}\otimes 1_{2m}\|^2((d_{i,j}-\delta)_+\otimes 1_{2m}), \quad \text{ and } \\ x_{i,j}x_{i,j}^* &\leqslant \frac{4}{\varepsilon}\sup_i \|d_{i,j}\otimes 1_{2m}\|^3(a_i\otimes 1_m),\end{aligned}$$

where each suprema is bounded because $(d_{i,j})_i$ is a bounded sequence.

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For any given pair $n < k$ and any t, one has

$$\left\|\left(\sum_{i=n}^{k} x_{i,j}\right) t\right\|^2 \leqslant \left(\sup_i \|d_{i,j}\otimes 1_{2m}\|^2\right)\|t\| \left\|\left(\sum_{i=n}^{k}(d_{i,j}-\delta)_+ \otimes 1_{2m}\right) t\right\|, \text{ and}$$

$$\left\|t\left(\sum_{i=n}^{k} x_{i,j}\right)\right\|^2 \leqslant \frac{4}{\varepsilon}\sup_i \|d_{i,j}\otimes 1_{2m}\|^3 \left\|t\left(\sum_{i=n}^{k} a_i\otimes 1_m\right)\right\|.$$

Let $x_{i,j}(s, r)$ denote the (s, r)-th entry of $x_{i,j}$. Then, the sums $x_j(s, r) := \sum_i x_{i,j}(s, r)$ are all strictly convergent. Set $x_j := (x_j(s, r))_{s,r} \in M_{m,2m^2}(\mathcal{M}(A))$. One gets

$$x_j^* x_j = (R_j-\delta)_+^3 \otimes 1_{2m} \sim (R_j-\delta)_+ \otimes 1_{2m},$$

and

$$x_j x_j^* \in \overline{(R\otimes 1_m)M_m(\mathcal{M}(A))(R\otimes 1_m)}.$$

It follows that, in $\mathrm{Cu}(\mathcal{M}(A))$, we have $2m[(R_j-\delta)_+] \leqslant m[R]$ for each $j\in\mathbb{N}$. Since $[R] = \sup_{0\leqslant\varepsilon\leqslant\varepsilon_0}[(R-\varepsilon)_+]$, we have that $m[R]$ is weakly $(2m, n)$-divisible, as desired. □

Let us say that a sequence $(a_i)_i$ of positive elements in a C*-algebra A is *orthogonal* if the elements in the sequence are pairwise orthogonal, that is, if $a_i a_j = 0$ whenever $i \neq j$.

Proposition 5.9. *Let A be a σ-unital* C*-*algebra. Assume that for every orthogonal sequence $(a_i)_i$ of positive elements in A there exist $m, n \in \mathbb{N}$ such that $\mathrm{div}_{2m}(m[a_i]) \leqslant n$ for each i. Then $\mathrm{div}_{2m}(m[T]) < \infty$ for every $T \in \mathcal{M}(A)/A$.*

In particular, $\mathcal{M}(A)/A$ is nowhere scattered.

Proof. Let $\pi\colon \mathcal{M}(A) \to \mathcal{M}(A)/A$ denote the quotient map. Using lemmas 5.6 and 5.8, we deduce that each positive element in the corona algebra can be written as $\pi(R_1)+\pi(R_2)$ with $[R_1]$, $[R_2]$ having multiples of finite weak divisibility.

By lemma 5.7, every element of the form $[\pi(T)]$ in $\mathrm{Cu}(\mathcal{M}(A)/A)$ with $T \in (\mathcal{M}(A)/A)_+$ has a multiple of finite weak divisibility. Lemma 5.4 shows that the corona algebra is nowhere scattered, as desired. □

Remark 5.10. Let A be the simple, nowhere scattered C*-algebra from example 5.3. It follows from proposition 5.9 above that there must exist an element $a \in (\oplus_{i=1}^{\infty} A)_+$ whose Cuntz class does not have a multiple of finite weak divisibility.

Using proposition 5.9, we can now prove the main result of the paper.

Theorem 5.11. *Let A be a σ-unital* C*-*algebra. Assume that for every orthogonal sequence $(a_i)_i$ in A_+ there exist $m, n \in \mathbb{N}$ such that $\mathrm{div}_{2m}(m[a_i]) \leqslant n$ for each i. Then $\mathcal{M}(A)$ is nowhere scattered.*

Proof. Using that nowhere scatteredness works well with extensions (see [**49**, Proposition 4.2]), the result is a direct consequence of proposition 5.9. □

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It follows from the results in [**37**] that the multiplier algebra of a simple, separable C*-algebra of real rank zero does not admit a character. Theorem 5.12 (i) below generalizes this result.

THEOREM 5.12. *Let A be a σ-unital* C*-*algebra. Assume that A has*

(i) *real rank zero, or*

(ii) *finite nuclear dimension, or*

(iii) *k-comparison and a surjective rank map.*

Then A is nowhere scattered if and only if $\mathcal{M}(A)$ is nowhere scattered.

Proof. Assume that $\mathcal{M}(A)$ is nowhere scattered. [**49**, Proposition 4.2] implies that A is nowhere scattered as well.

Conversely, assume that A is nowhere scattered. It follows from example 3.11, proposition 4.1 and proposition 4.3 respectively that (i), (ii) and (iii) imply that every sequence of positive elements (orthogonal or not) in A_+ has uniformly bounded multiple divisibility. Thus, we can use theorem 5.11 to get the desired result. □

QUESTION 5.13. Let A be a nowhere scattered C*-algebra such that $[a]$ has a multiple of finite weak divisibility for every $a \in A_+$. Does it follow that $\mathcal{M}(A)$ is nowhere scattered?

Note that this would imply that the simple C*-algebra from example 5.3 contains a $(2, \omega)$-divisible element of infinite weak divisibility.

REMARK 5.14. The assumption from theorem 5.11 above is not equivalent to nowhere scatteredness, not even in the simple or σ-unital case:

(1) The example from example 5.2 is σ-unital. Thus, there exist σ-unital, nowhere scattered C*-algebras that do not satisfy the condition from theorem 5.11.

(2) By lemma 3.12, the set of classes with finite weak divisibility is sup-dense in the Cuntz semigroup of a simple, non-elementary C*-algebra A. However, and as made explicit in example 5.3, this is not enough for the multiplier algebra $\mathcal{M}(A)$ to be nowhere scattered.

Further, note that if the converse of theorem 5.11 holds for some family of C*-algebras, one must have that any unital, nowhere scattered C*-algebra A in the family must satisfy $\sup_{a\in A_+} \operatorname{div}_{2m}(m[a_i]) < \infty$ for some m.

As another application of lemma 5.8, we can study when multiplier algebras have a character:

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Proposition 5.15. *Let A be a σ-unital* C*-*algebra, and let $(e_i)_i$ be an approximate unit as in lemma 5.6. Assume that*

$$\sup_i \sup_{0\leqslant\varepsilon\leqslant\varepsilon_0} \mathrm{div}_{2m}(m[(e_i-e_{i-1}-\varepsilon)_+])<\infty$$

for some $\varepsilon_0>0$.

Then, $\mathcal{M}(A)$ does not admit a character.

Proof. Using lemma 5.8, we see that the unit $1=\sum_i(e_i-e_{i-1})$ in $\mathcal{M}(A)$ satisfies $\mathrm{div}_{2m}(m[1])<\infty$.

The result now follows from [**40**, Corollary 5.6]. □

In particular, proposition 5.15 recovers [**40**, Corollary 8.5]:

Corollary 5.16. *The product of unital* C*-*algebras A_i does not admit a character whenever the supremum of $\mathrm{div}_2([1_i])$ is finite.*

5.1. Stable C*-algebras

The multiplier algebra of a σ-unital, stable C*-algebra A has been studied extensively. For example, [**42**] studies when the corona algebra of A is simple, while in [**34**] it is shown, using its multiplier algebra, that a nowhere scattered C*-algebra of finite nuclear dimension has the corona factorization property.

In what follows, we show that $[a]$ must have a multiple of finite weak divisibility for every $a\in A_+$ if $\mathcal{M}(A)$ is to be nowhere scattered; see theorem 5.18. Thus, if question 5.13 has an affirmative answer, this would imply that the multiplier algebra of a σ-unital, stable C*-algebra is nowhere scattered if and only if every element in the algebra has a multiple of finite weak divisibility.

Lemma 5.17 below was stated for (m,ω)-divisibility in [**51**]. We now provide the analogous statement for weak (m,ω)-divisibility, which only needs to assume the element (and not the whole Cuntz semigroup) to be divisible.

Lemma 5.17. *Let A be a* C*-*algebra, and let $a\in(A\otimes\mathcal{K})_+$. Then, if $[a]$ is weakly (m,ω)-divisible, there exists a sequence $(y_n)_n\subseteq\mathrm{Cu}(A)$ such that $my_j\leqslant[a]\leqslant\sum_{j=1}^{\infty}y_j$ for every $j\in\mathbb{N}$.*

Proof. Let ε_n be a strictly decreasing sequence converging to 0, and let $x_n:=[(a-\varepsilon_n)_+]$. For each n, apply (m,ω)-divisibility to $x_n\ll[a]$ to find elements $y_{n,j}$ such that

$$x_n\leqslant\sum_j y_{n,j},\qquad\text{and}\qquad my_{n,j}\leqslant[a].$$

for every j. In particular, note that one gets

$$x_n\leqslant\sum_{k=1}^{n}\sum_j y_{k,j}\leqslant\sum_{k=1}^{\infty}\sum_j y_{k,j}$$

By taking supremum on n, one sees that the elements $(y_{n,j})_{n,j}$ satisfy the desired conditions. □

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Note that, in general, one cannot adapt the previous proof to show that there exist $y_1, \ldots, y_m$ with $2y_j \leqslant [a] \leqslant y_1 + \ldots + y_m$ whenever $[a]$ is $(2, m)$-divisible.

However, this is the case for σ-unital C*-algebras whose multiplier algebra is nowhere scattered:

THEOREM 5.18. *Let A be a σ-unital, stable* C*-*algebra. Assume that $\mathcal{M}(A)$ is nowhere scattered. Then, for every $a \in A_+$ and $m \in \mathbb{N}$ there exist finitely many elements $y_1, \ldots, y_n \in \mathrm{Cu}(A)$ such that*

$$my_j \leqslant [a] \leqslant y_1 + \ldots + y_n$$

for every $j \leqslant n$.

In particular, $[a]$ has a multiple of finite weak divisibility for every $a \in A_+$.

Proof. Let $a \in A_+$ and $m \in \mathbb{N}$. By [**2**, Proposition 2.8] and its proof there exists a projection $p_a \in \mathcal{M}(A)$ such that, for every $x \in \mathrm{Cu}(A)$, one has $x \leqslant [a]$ if and only if $x \leqslant [p_a]$ in $\mathrm{Cu}(\mathcal{M}(A))$.

Assume that $\mathcal{M}(A)$ is nowhere scattered. Thus, $\mathrm{Cu}(\mathcal{M}(A))$ is weakly (m, ω)-divisible by [**49**, Theorem 8.9]. It follows from remark 3.2 that $\mathrm{div}_m([p]) < \infty$ for every projection $p \in \mathcal{M}(A)$.

Then, since we have $[a] \leqslant [p_a] \ll [p_a]$, there exist finitely many elements $z_1, \ldots, z_n$ in $\mathrm{Cu}(\mathcal{M}(A))$ such that

$$[a] \leqslant [p_a] \leqslant z_1 + \ldots + z_n, \qquad \text{and} \qquad mz_j \leqslant [p_a]$$

for each $j \leqslant n$.

Since $\mathcal{M}(A)$ is a C*-algebra, we know from [**1**, Remark 2.6] that the infimum $y_j := z_j \wedge \infty[a]$ exists for each j. Any representative of this element is contained in the ideal generated by a and, therefore, must be a positive element in A. Thus, we have $[a] \leqslant y_1 + \ldots + y_n$ in $\mathrm{Cu}(\mathcal{M}(A))$ with $y_j \in \mathrm{Cu}(A)$ for each j.

Further, we also get $my_j = m(z_j \wedge \infty[a]) \leqslant [p_a]$. Thus, $my_j \leqslant [a]$ in $\mathrm{Cu}(\mathcal{M}(A))$.

Finally, note that, since A is an ideal of $\mathcal{M}(A)$, a pair of elements in A are Cuntz subequivalent in $\mathcal{M}(A)$ if and only if they are Cuntz subequivalent in A; see, for example, [**46**, Proposition 2.18].

Consequently, we have

$$[a] \leqslant y_1 + \ldots + y_n, \qquad \text{and} \qquad my_j \leqslant [a]$$

in $\mathrm{Cu}(A)$, as desired. □

REMARK 5.19. Note that there exist stable, nowhere scattered C*-algebras with a multiplier algebra that is not nowhere scattered. Indeed, simply take A as in example 3.6. Then, $A \otimes \mathcal{K}$ is a nowhere scattered C*-algebra by [**49**, Proposition 4.12] that contains a $(2, \omega)$-divisible element of infinite weak divisibility.

It follows from theorem 5.18 that $\mathcal{M}(A \otimes \mathcal{K})$ cannot be nowhere scattered.

In this paper, we have studied conditions under which a multiplier algebra $\mathcal{M}(A)$ is nowhere scattered. A problem that seems to be much more involved is to determine when $\mathcal{M}(A)$ has the Global Glimm Property.

QUESTION 5.20. Assume that A has the Global Glimm Property. When does $\mathcal{M}(A)$ also have the Global Glimm Property?

Note that this has a positive solution whenever $\sup_{a\in A_+} \mathrm{Div}_{2m}(m[a]) < \infty$ and $\mathcal{M}(A)/A$ satisfies a condition that ensures that the Global Glimm Problem can be answered affirmatively. For example, Lin and Ng show in [**33**] that $\mathcal{M}(\mathcal{Z}\otimes\mathcal{K})/\mathcal{Z}\otimes\mathcal{K}$ has real rank zero. Thus, since $\mathcal{Z}\otimes\mathcal{K}$ is covered by theorem 5.11, we get that $\mathcal{M}(\mathcal{Z}\otimes\mathcal{K})/\mathcal{Z}\otimes\mathcal{K}$ is nowhere scattered. By [**18**] (see also [**51**, Proposition 7.4]), $\mathcal{M}(\mathcal{Z}\otimes\mathcal{K})/\mathcal{Z}\otimes\mathcal{K}$ has the Global Glimm Property. Consequently, $\mathcal{M}(\mathcal{Z}\otimes\mathcal{K})$ also has the Global Glimm Property by [**51**, Theorem 3.10].

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References

1 R. Antoine, F. Perera, L. Robert and H. Thiel. Edwards' condition for quasitraces on C*-algebras. *Proc. Roy. Soc. Edinburgh Sect. A* **151** (2021), 525–547.

2 R. Antoine, F. Perera, L. Robert and H. Thiel. C*-algebras of stable rank one and their Cuntz semigroups. *Duke Math. J.* **171** (2022), 33–99.

3 R. Antoine, F. Perera and H. Thiel. Tensor products and regularity properties of Cuntz semigroups. *Mem. Amer. Math. Soc.* **251** (2018), viii+191.

4 R. Antoine, F. Perera and H. Thiel. Cuntz semigroups of ultraproduct C*-algebras. *J. Lond. Math. Soc. (2)* **102** (2020), 994–1029.

5 P. Ara, F. Perera and A. S. Toms, K-theory for operator algebras. Classification of C*-algebras, in *Aspects of operator algebras and applications*, *Contemp. Math.* Vol. 534 (Amer. Math. Soc., Providence, RI, 2011), pp. 1–71.

6 A. Asadi-Vasfi, H. Thiel and E. Vilalta, Ranks of soft operators in nowhere scattered C*-algebras, preprint arXiv:2310.00663 [math.OA], 2023.

7 J. Bosa. Stable elements and property (S). *J. Math. Anal. Appl.* **512** (2022), 126171. 16.

8 L. G. Brown and G. K. Pedersen. C*-algebras of real rank zero. *J. Funct. Anal.* **99** (1991), 131–149.

9 L. G. Brown and G. K. Pedersen. Limits and C*-algebras of low rank or dimension. *J. Operator Theory* **61** (2009), 381–417.

10 N. P. Brown and A. Ciuperca. Isomorphism of Hilbert modules over stably finite C*-algebras. *J. Funct. Anal.* **257** (2009), 332–339.

11 A. Chand and L. Robert. Simplicity bounded normal generation, and automatic continuity of groups of unitaries. *Adv. Math.* **415** (2023), 108894. 52 p.

12 M. S. Christensen, *Regularity of C*-algebras and central sequence algebras*, 2017, Thesis (Ph.D.). University of Copenhagen (Denmark).

13 A. Ciuperca, L. Robert and L. Santiago. The Cuntz semigroup of ideals and quotients and a generalized Kasparov stabilization theorem. *J. Operator Theory* **64** (2010), 155–169.

14 K. T. Coward, G. A. Elliott and C. Ivanescu. The Cuntz semigroup as an invariant for C*-algebras. *J. Reine Angew. Math.* **623** (2008), 161–193.

15 J. Cuntz. Dimension functions on simple C*-algebras. *Math. Ann.* **233** (1978), 145–153.

16 G. A. Elliott. Derivations of matroid C^*-algebras. II. *Ann. Math. (2)* **100** (1974), 407–422.

17 G. A. Elliott, L. Robert and L. Santiago. The cone of lower semicontinuous traces on a C*-algebra. *Amer. J. Math.* **133** (2011), 969–1005.

18 G. A. Elliott and M. Rørdam, Perturbation of Hausdorff moment sequences, and an application to the theory of C*-algebras of real rank zero, in *Operator Algebras: The Abel Symposium 2004*, *Abel Symp.*, Vol. 1 (Springer, Berlin, 2006), pp. 97–115.

19 M. Engbers, Decomposition of simple Cuntz semigroups, Ph.D. Thesis, WWU Münster, 2014.

20 I. Farah, B. Hart, M. Lupini, L. Robert, A. Tikuisis, A. Vignati and W. Winter. Model theory of C*-algebras. *Mem. Amer. Math. Soc.* **271** (2021), viii+127.

21 E. Gardella and F. Perera, The modern theory of Cuntz semigroups of C*-algebras, preprint arXiv:2212.02290 [math.OA], 2022.

22 V. Kaftal, P. W. Ng and S. Zhang. Strict comparison of positive elements in multiplier algebras. *Canad. J. Math.* **69** (2017), 373–407.

23 V. Kaftal, P. W. Ng and S. Zhang. Purely infinite corona algebras. *J. Operator Theory* **82** (2019), 307–355.

24 E. Kirchberg, Central sequences in C*-algebras and strongly purely infinite algebras, in *Operator Algebras: The Abel Symposium 2004*, *Abel Symp.* Vol. 1 (Springer, Berlin, 2006), pp. 175–231.

25 E. Kirchberg and M. Rørdam. Infinite non-simple C*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$. *Adv. Math.* **167** (2002), 195–264.

26 E. Kirchberg and M. Rørdam. When central sequence C*-algebras have characters. *Internat. J. Math.* **26** (2015), 1550049. 32.

27 D. Kucerovsky, P. W. Ng and F. Perera. Purely infinite corona algebras of simple C^*-algebras. *Math. Ann.* **346** (2010), 23–40.

28 D. Kucerovsky and F. Perera. Purely infinite corona algebras of simple C^*-algebras with real rank zero. *J. Operator Theory* **65** (2011), 131–144.

29 N. S. Larsen and H. Osaka. Extremal richness of multiplier algebras and corona algebras of simple C^*-algebras. *J. Operator Theory* **38** (1997), 131–149.

30 H. X. Lin. Simple C^*-algebras with continuous scales and simple corona algebras. *Proc. Amer. Math. Soc.* **112** (1991), 871–880.

31 H. Lin. Exponential rank of C*-algebras with real rank zero and the Brown-Pedersen conjectures. *J. Funct. Anal.* **114** (1993), 1–11.

32 H. Lin. Simple corona C^*-algebras. *Proc. Amer. Math. Soc.* **132** (2004), 3215–3224.

33 H. Lin and P. W. Ng. The corona algebra of the stabilized Jiang-Su algebra. *J. Funct. Anal.* **270** (2016), 1220–1267.

34 P. W. Ng and W. Winter. Nuclear dimension and the corona factorization property. *Int. Math. Res. Not.* **2010** (2010), 261–278.

35 G. K. Pedersen. Applications of weak* semicontinuity in C*-algebra theory. *Duke Math. J.* **39** (1972), 431–450.

36 F. Perera. Extremal richness of multiplier and corona algebras of simple C^*-algebras with real rank zero. *J. Operator Theory* **44** (2000), 413–431.

37 F. Perera and M. Rørdam. AF-embeddings into C*-algebras of real rank zero. *J. Funct. Anal.* **217** (2004), 142–170.

38 L. Robert. The cone of functionals on the Cuntz semigroup. *Math. Scand.* **113** (2013), 161–186.

39 L. Robert. On the Lie ideals of C^*-algebras. *J. Operator Theory* **75** (2016), 387–408.

40 L. Robert and M. Rørdam. Divisibility properties for C*-algebras. *Proc. Lond. Math. Soc. (3)* **106** (2013), 1330–1370.

41 L. Robert and A. Tikuisis. Nuclear dimension and $\mathcal{Z}$-stability of non-simple C*-algebras. *Trans. Amer. Math. Soc.* **369** (2017), 4631–4670.

42 M. Rørdam. Ideals in the multiplier algebra of a stable C*-algebra. *J. Operator Theory* **25** (1991), 283–298.

43 M. Rørdam. On the structure of simple C*-algebras tensored with a UHF-algebra. II. *J. Funct. Anal.* **107** (1992), 255–269.

44 M. Rørdam and W. Winter. The Jiang-Su algebra revisited. *J. Reine Angew. Math.* **642** (2010), 129–155.

45 S. Sakai. Derivations of simple C*-algebras. III. *Tohoku Math. J. (2)* **23** (1971), 559–564.

46 H. Thiel, The Cuntz semigroup, lecture notes available at www.math.uni-muenster.de/u/hannes.thiel/, 2017.

47 H. Thiel. Ranks of operators in simple C*-algebras with stable rank one. *Comm. Math. Phys.* **377** (2020), 37–76.

48 H. Thiel and E. Vilalta. Covering dimension of Cuntz semigroups II. *Internat. J. Math.* **32** (2021), 2150100. 27 p.

49 H. Thiel and E. Vilalta, Nowhere scattered C*-algebras, J. Noncommut. Geom. (to appear), preprint arXiv:2112.09877 [math.OA], 2021.

50 H. Thiel and E. Vilalta. Covering dimension of Cuntz semigroups. *Adv. Math.* **394** (2022), 108016. 44 p.

51 H. Thiel and E. Vilalta. The Global Glimm Property. *Trans. Amer. Math. Soc.* **376** (2023), 4713–4744. DOI: 10.1090/tran/8880. 2022.

52 H. Thiel and E. Vilalta, Soft operators in C*-algebras, preprint arXiv:2304.11644 [math.OA], 2023.

53 E. Vilalta. A local characterization for the Cuntz semigroup of AI-algebras. *J. Math. Anal. Appl.* **506** (2022), 125612. 47.

54 W. Winter. Nuclear dimension and $\mathcal{Z}$-stability of pure C*-algebras. *Invent. Math.* **187** (2012), 259–342.

55 W. Winter and J. Zacharias. The nuclear dimension of C*-algebras. *Adv. Math.* **224** (2010), 461–498.

56 S. Zhang. A Riesz decomposition property and ideal structure of multiplier algebras. *J. Operator Theory* **24** (1990), 209–225.