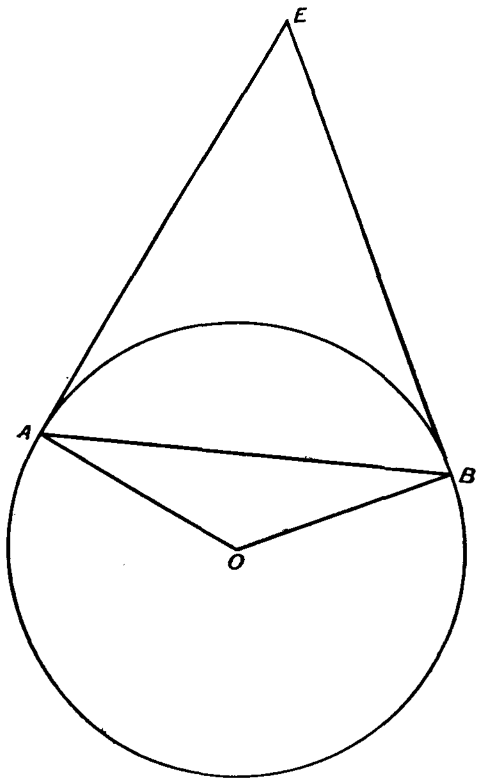
The Fallacy of the Figure in Geometry.—(1) Let EA, EB be tangents to a circle with centre O. To prove EA = EB, the following is often served up:—

Join OA, OB, AB.

$\angle$ OAB = $\angle$ OBA.



By subtraction from the equal angles EAO, EBO

$\angle$ EAB = $\angle$ EBA

$\therefore$ EA = EB.

Now we could prove on the same lines that the triangles in the ambiguous case (Euc. IA) are always congruent, for we could put the triangles on opposite sides of the same base with the equal angles

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facing that base. We ought in fact to prove above that AB does not pass through O.

(2) The ambiguity of the conclusion in IA is apt to be neglected when a partial figure is before one to deceive the eye.

Example. Find the locus of a point at which the equal sides of an isosceles $\triangle$ subtend equal angles.

The locus obviously consists of a circular arc and a straight line. It is astonishing how often one or other of these is omitted.

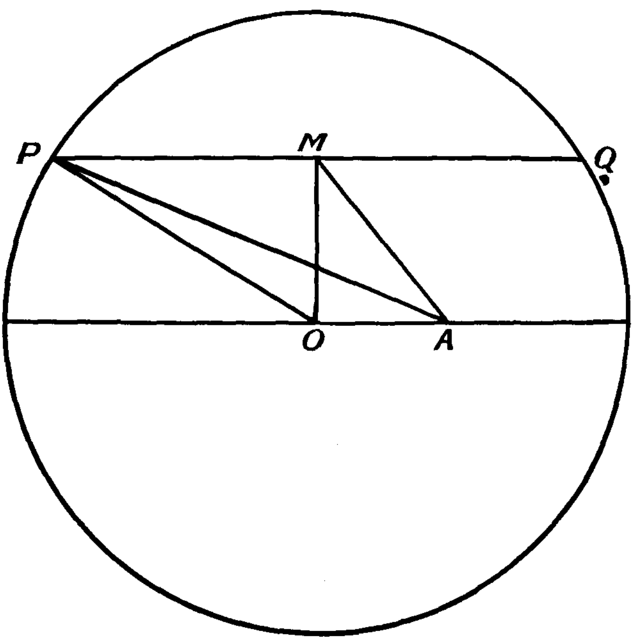
This is an instance of the risk of dealing with loci synthetically. The student sees that if a point be taken on the circle round the triangle, the condition is satisfied, and he proceeds with a synthetic proof.

(3) A more subtle source of error occurs when two points (or lines), which in a figure drawn at random are distinct, may coincide in certain cases or in a whole class of cases.

Example. In a circle with centre O, radius r, A is a fixed point, and PQ a variable chord such that

$$AP^2 + AQ^2 = 2AO^2 + 2r^2.$$

Prove that PQ is parallel to OA.



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Let M be the mid point of PQ.

$AM^2 + PM^2 = AO^2 + r^2$,

$\therefore$ $AM^2 + r^2 - OM^2 = AO^2 + r^2$,

$\therefore$ $\angle AOM = 90°$

But $\angle OMP = 90°$

$\therefore$ PQ is parallel to OA.

I venture to say that 99 per cent. of casual readers will see nothing wrong about this. But what if M is at the centre?

(The student of elementary geometrical conics will easily prove that if, more generally, $AP^2 + AQ^2 = c^2$, then PQ envelopes a parabola with focus at O. In the special case the parabolic envelope breaks down into a couple of points, one at infinity, the other the centre).

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The solution of "homogeneous" quadratics.

$$\left.\begin{array}{l} 2x^2 - 5xy + 4y^2 = 4 \\ 3y^2 - x^2 \quad\quad = 3 \end{array}\right\} \quad \ldots\ldots\ldots\ldots(1)$$

A common method is to put $y = mx$.

$$\left.\begin{array}{l} x^2(2 - 5m + 4m^2) = 4 \\ x^2(3m^2 - 1) \quad\quad = 3 \end{array}\right\} \quad \ldots\ldots\ldots\ldots(2)$$

By division $$\frac{x^2(2 - 5m + 4m^2)}{x^2(3m^2 - 1)} = \frac{4}{3} \quad \ldots\ldots\ldots\ldots(3)$$

or $$\frac{2 - 5m + 4m^2}{3m^2 - 1} = \frac{4}{3} \quad \ldots\ldots\ldots\ldots(4)$$

$$m = \frac{2}{3},$$

and from either of (1), $x = \pm 3$, $y = \pm 2$.

The point is that we have missed the obvious solutions $x = 0$, $y = \pm 1$. We dropped them at the passage from (3) to (4). In fact from (3) we can only infer (4), *if x is not zero*, so that we should say, *either* $x = 0$, *or*

$$\frac{2 - 5m + 4m^2}{3m^2 - 1} = \frac{4}{3},$$

and then try $x = 0$ in (1).