3

Triangulated Categories

Contents

In this chapter we introduce triangulated categories. These provide the appropriate framework for studying derived functors and derived categories. A triangulated category is an additive category together with a suspension functor and a distinguished class of triangles. Important examples are stable categories of Frobenius categories. A basic tool is the localisation theory for triangulated categories. Another useful result is Brown's representability theorem for cohomological functors which requires the existence of generators satisfying certain finiteness conditions.

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3.1 Triangulated Categories

Triangulated categories are defined via a set of four axioms. Then we discuss some of the basic properties of triangulated categories.

The Axioms

A *suspended category* is a pair $(\mathcal{T}, \Sigma)$ consisting of an additive category $\mathcal{T}$ and an equivalence $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ which we call a *suspension* or *shift*. A *triangle* in $(\mathcal{T}, \Sigma)$ is a sequence (α, β, γ) of morphisms

$$X \xrightarrow{\ \alpha\ } Y \xrightarrow{\ \beta\ } Z \xrightarrow{\ \gamma\ } \Sigma X$$

and a morphism between triangles (α, β, γ) and $(\alpha', \beta', \gamma')$ is given by a triple (ϕ_1, ϕ_2, ϕ_3) of morphisms in $\mathcal{T}$ making the following diagram commutative.

$$\begin{array}{ccccccc} X & \xrightarrow{\ \alpha\ } & Y & \xrightarrow{\ \beta\ } & Z & \xrightarrow{\ \gamma\ } & \Sigma X \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \Sigma\phi_1} \\ X' & \xrightarrow{\ \alpha'\ } & Y' & \xrightarrow{\ \beta'\ } & Z' & \xrightarrow{\ \gamma'\ } & \Sigma X' \end{array}$$

A *triangulated category* is a triple $(\mathcal{T}, \Sigma, \mathcal{E})$ consisting of a suspended category $(\mathcal{T}, \Sigma)$ and a class $\mathcal{E}$ of distinguished triangles in $(\mathcal{T}, \Sigma)$ (called *exact triangles*) satisfying the following conditions.

(Tr1) A triangle isomorphic to an exact triangle is exact. For each object X, the triangle $0 \to X \xrightarrow{\text{id}} X \to \Sigma 0$ is exact. Each morphism α fits into an exact triangle (α, β, γ).

(Tr2) A triangle (α, β, γ) is exact if and only if $(\beta, \gamma, -\Sigma\alpha)$ is exact.

(Tr3) Given two exact triangles (α, β, γ) and $(\alpha', \beta', \gamma')$, each pair of morphisms ϕ_1 and ϕ_2 satisfying $\phi_2\alpha = \alpha'\phi_1$ can be completed to a morphism

$$\begin{array}{ccccccc} X & \xrightarrow{\ \alpha\ } & Y & \xrightarrow{\ \beta\ } & Z & \xrightarrow{\ \gamma\ } & \Sigma X \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \Sigma\phi_1} \\ X' & \xrightarrow{\ \alpha'\ } & Y' & \xrightarrow{\ \beta'\ } & Z' & \xrightarrow{\ \gamma'\ } & \Sigma X' \end{array}$$

of triangles.

(Tr4) Given exact triangles $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, and $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1 = \beta_1\alpha_1$, there exists an exact triangle $(\delta_1, \delta_2, \delta_3)$ making the following

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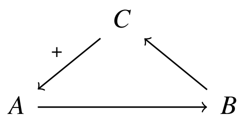
diagram commutative.

$$
\begin{array}{ccccccc}
X & \xrightarrow{\alpha_1} & Y & \xrightarrow{\alpha_2} & U & \xrightarrow{\alpha_3} & \Sigma X \\
\| & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \delta_1} & & \| \\
X & \xrightarrow{\gamma_1} & Z & \xrightarrow{\gamma_2} & V & \xrightarrow{\gamma_3} & \Sigma X \\
 & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \Sigma\alpha_1} \\
 & & W & = & W & \xrightarrow{\beta_3} & \Sigma Y \\
 & & \downarrow{\scriptstyle \beta_3} & & \downarrow{\scriptstyle \delta_3} & & \\
 & & \Sigma Y & \xrightarrow{\Sigma\alpha_2} & \Sigma U & &
\end{array}
$$

The axiom (Tr4) is also known as the *octahedral axiom*, because the objects and morphisms of the diagram can be arranged to produce the skeleton of an octahedron, four of whose faces are exact triangles, so of the form



corresponding to an exact triangle $A \to B \to C \to \Sigma A$.

Given a triangulated category $(\mathcal{T}, \Sigma, \mathcal{E})$, we simplify the notation and identify $\mathcal{T} = (\mathcal{T}, \Sigma, \mathcal{E})$.

Exact Functors

An *exact functor* (or *triangle functor*) $\mathcal{T} \to \mathcal{U}$ between triangulated categories is a pair (F, η) consisting of an additive functor $F \colon \mathcal{T} \to \mathcal{U}$ and a natural isomorphism $\eta \colon F \circ \Sigma_{\mathcal{T}} \xrightarrow{\sim} \Sigma_{\mathcal{U}} \circ F$ such that for every exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma_{\mathcal{T}} X$ in $\mathcal{T}$ the triangle

$$FX \xrightarrow{F\alpha} FY \xrightarrow{F\beta} FZ \xrightarrow{\eta_X \circ F\gamma} \Sigma_{\mathcal{U}}(FX)$$

is exact in $\mathcal{U}$. In the following we simplify the notation and identify $F = (F, \eta)$.

An exact functor $F \colon \mathcal{T} \to \mathcal{U}$ is called a *triangle equivalence* if F is an equivalence of categories. The terminology is justified by the following observation, because then a quasi-inverse is again exact.

Lemma 3.1.1. *Let (F, G) be an adjoint pair of functors between triangulated categories. Then F is exact if and only if G is exact.* □

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Cohomological Functors

Let $\mathcal{T}$ be a triangulated category. An additive functor $F\colon \mathcal{T} \to \mathcal{A}$ into an abelian category $\mathcal{A}$ is called *cohomological* if it sends each exact triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{T}$ to an exact sequence $FX \to FY \to FZ$ in $\mathcal{A}$.

Lemma 3.1.2. *For each object X in $\mathcal{T}$, the representable functors*

$$\operatorname{Hom}_{\mathcal{T}}(X,-)\colon \mathcal{T} \longrightarrow \mathrm{Ab} \quad \textit{and} \quad \operatorname{Hom}_{\mathcal{T}}(-,X)\colon \mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$$

into the category Ab *of abelian groups are cohomological functors.*

Proof We show that $\operatorname{Hom}_{\mathcal{T}}(X,-)$ is cohomological. For $\operatorname{Hom}_{\mathcal{T}}(-,X)$ the proof is dual.

Fix an exact triangle $U \xrightarrow{\alpha} V \xrightarrow{\beta} W \xrightarrow{\gamma} \Sigma U$. We need to show the exactness of the induced sequence

$$\operatorname{Hom}_{\mathcal{T}}(X,U) \longrightarrow \operatorname{Hom}_{\mathcal{T}}(X,V) \longrightarrow \operatorname{Hom}_{\mathcal{T}}(X,W).$$

To this end fix a morphism $\phi\colon X \to V$ and consider the following diagram.

$$\begin{array}{ccccccc} X & \xrightarrow{\mathrm{id}} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ & & \downarrow{\scriptstyle\phi} & & & & \\ U & \xrightarrow{\alpha} & V & \xrightarrow{\beta} & W & \xrightarrow{\gamma} & \Sigma U \end{array}$$

If ϕ factors through α, then (Tr3) implies the existence of a morphism $0 \to W$ making the diagram commutative. Thus $\beta \circ \phi = 0$. Now assume $\beta \circ \phi = 0$. Applying (Tr2) and (Tr3), we find a morphism $X \to U$ making the diagram commutative. Thus ϕ factors through α. □

We discuss some consequences. For example, we see that in any exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ the morphism α is a weak kernel of β. Also, the Yoneda functor $\mathcal{T} \to \operatorname{mod}\mathcal{T}$ is a universal cohomological functor.

Proposition 3.1.3. *The category* $\operatorname{mod}\mathcal{T}$ *is abelian and the Yoneda functor* $\mathcal{T} \to \operatorname{mod}\mathcal{T}$ *is cohomological. Any cohomological functor* $\mathcal{T} \to \mathcal{A}$ *factors uniquely (up to a unique isomorphism) through the Yoneda functor via an exact functor* $\operatorname{mod}\mathcal{T} \to \mathcal{A}$.

Proof Every morphism in $\mathcal{T}$ admits a weak kernel by Lemma 3.1.2. Therefore the category $\operatorname{mod}\mathcal{T}$ is abelian by Lemma 2.1.6. Moreover, Lemma 3.1.2 implies that the Yoneda functor is cohomological. Given a cohomological functor $F\colon \mathcal{T} \to \mathcal{A}$, the functor $\operatorname{mod}\mathcal{T} \to \mathcal{A}$ takes $\operatorname{Coker}\operatorname{Hom}_{\mathcal{T}}(-,\phi)$ (given by a morphism ϕ in $\mathcal{T}$) to $\operatorname{Coker} F(\phi)$. This functor is exact and essentially unique; see Lemma 2.1.8. □

Proposition 3.1.4. *A functor* $\mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ *is cohomological if and only if it is a filtered colimit of representable functors.*

Proof One direction is clear, since filtered colimits in Ab are exact and representable functors are cohomological. Now fix an additive functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$. Let $\mathcal{T}/F$ denote the category consisting of pairs (X, f) with $X \in \mathcal{T}$ and $f \in F(X)$. A morphism $(X, f) \to (X', f')$ is given by a morphism $\alpha\colon X \to X'$ in $\mathcal{T}$ such that $F(\alpha)(f') = f$. We write $\mathrm{Add}(\mathcal{T}^{\mathrm{op}}, \mathrm{Ab})$ for the category of additive functors $\mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$. Then F equals the colimit of the functor

$$\mathcal{T}/F \longrightarrow \mathrm{Add}(\mathcal{T}^{\mathrm{op}}, \mathrm{Ab}), \quad (X, f) \mapsto \mathrm{Hom}_{\mathcal{T}}(-, X)$$

(Lemma 11.1.8). It is easily checked that $\mathcal{T}/F$ is filtered when F is cohomological. □

Uniqueness of Exact Triangles

Let $\mathcal{T}$ be a triangulated category. Given a morphism $\alpha\colon X \to Y$ in $\mathcal{T}$ and two exact triangles $\Delta = (\alpha, \beta, \gamma)$ and $\Delta' = (\alpha, \beta', \gamma')$ which complete α, there exists a comparison morphism $(\mathrm{id}_X, \mathrm{id}_Y, \phi)$ between Δ and Δ', by (Tr3). The morphism ϕ is an isomorphism, by the following lemma, but it need not be unique.

Lemma 3.1.5. *Let* (ϕ_1, ϕ_2, ϕ_3) *be a morphism between exact triangles. If two of* ϕ_1, ϕ_2, ϕ_3 *are isomorphisms, then the third is also an isomorphism.*

Proof Use Lemma 3.1.2 and apply the five lemma. □

The third object Z in an exact triangle $X \xrightarrow{\alpha} Y \to Z \to \Sigma X$ is called the *cone* of α and is denoted by $\mathrm{Cone}\,\alpha$, despite the fact that it is not unique. Later on we will see specific constructions which justify this terminology.

Triangulated and Thick Subcategories

Let $\mathcal{T}$ be a triangulated category. A full subcategory $\mathcal{S}$ is a *triangulated subcategory* if $\mathcal{S}$ is non-empty and the following conditions hold.

(TS1) $\Sigma^n X \in \mathcal{S}$ for all $X \in \mathcal{S}$ and $n \in \mathbb{Z}$.

(TS2) Let $X \to Y \to Z \to \Sigma X$ be an exact triangle in $\mathcal{T}$. Then $X, Y \in \mathcal{S}$ implies $Z \in \mathcal{S}$.

A triangulated subcategory $\mathcal{S}$ is *thick* if in addition the following condition holds.

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(TS3) Every direct summand of an object in $\mathcal{S}$ belongs to $\mathcal{S}$, that is, a decomposition $X = X' \oplus X''$ for $X \in \mathcal{S}$ implies $X' \in \mathcal{S}$.

Note that a triangulated subcategory $\mathcal{S}$ inherits a canonical triangulated structure from $\mathcal{T}$.

Example 3.1.6. The kernel of an exact functor $\mathcal{T} \to \mathcal{U}$ between triangulated categories is a thick subcategory of $\mathcal{T}$.

Example 3.1.7. An object X in $\mathcal{T}$ is *homologically finite* if for every object Y in $\mathcal{T}$ we have $\operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n Y) = 0$ for almost all $n \in \mathbb{Z}$. The homologically finite objects form a thick subcategory of $\mathcal{T}$.

Dévissage

For a triangulated category $\mathcal{T}$ and a class of objects $\mathcal{C} \subseteq \mathcal{T}$ let $\operatorname{Thick}(\mathcal{C})$ denote the smallest thick subcategory of $\mathcal{T}$ that contains $\mathcal{C}$.

Lemma 3.1.8. *Let $F \colon \mathcal{T} \to \mathcal{U}$ be an exact functor between triangulated categories and let $\mathcal{C} \subseteq \mathcal{T}$ be a class of objects in $\mathcal{T}$. If the induced map*

$$\operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n Y) \to \operatorname{Hom}_{\mathcal{U}}(FX, \Sigma^n FY)$$

is bijective for all $X, Y \in \mathcal{C}$ and $n \in \mathbb{Z}$, then F restricted to $\operatorname{Thick}(\mathcal{C})$ *is fully faithful.*

Proof Use Lemma 3.1.2 and apply the five lemma. □

3.2 Localisation of Triangulated Categories

We introduce the localisation of a triangulated category with respect to a triangulated subcategory. Localising amounts to annihilating a class of objects, and the triangulated structure is preserved.

Verdier Localisation

Let $\mathcal{T}$ be a triangulated category and fix a triangulated subcategory $\mathcal{S}$. Set

$$S(\mathcal{S}) = \{\sigma \in \operatorname{Mor} \mathcal{T} \mid \operatorname{Cone} \sigma \in \mathcal{S}\}.$$

Also, we set

$$\mathcal{S}^{\perp} = \{Y \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(X, Y) = 0 \text{ for all } X \in \mathcal{S}\}$$

and

$${}^{\perp}\mathcal{S} = \{X \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(X,Y) = 0 \text{ for all } Y \in \mathcal{S}\}.$$

Lemma 3.2.1. *For a triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$ the following holds.*

(1) *$S(\mathcal{S})$ admits a calculus of left and right fractions.*
(2) *An object in $\mathcal{T}$ is $S(\mathcal{S})$-local if and only if it is in $\mathcal{S}^{\perp}$.*

Proof Set $S = S(\mathcal{S})$.

(1) We check for S the conditions (LF1)–(LF3) to admit a calculus of left fractions. The proof that S admits a calculus of right fractions is dual.

(LF1) The class S contains the identity morphisms by (Tr1) and the composite of two morphisms in S by (Tr4).

(LF2) Fix a pair of morphisms $X' \xleftarrow{\sigma} X \xrightarrow{\alpha} Y$ in $\mathcal{T}$ with $\sigma \in S$. Completing the composite $\Sigma^{-1}(\operatorname{Cone}\sigma) \to X \to Y$ to an exact triangle and applying (Tr3) yields a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\tau} \\ X' & \longrightarrow & Y' \end{array}$$

with $\operatorname{Cone}\sigma \cong \operatorname{Cone}\tau$. Thus $\tau \in S$.

(LF3) Let $\alpha, \beta \colon X \to Y$ be morphisms in $\mathcal{T}$ and suppose there is $\sigma \colon X' \to X$ in S such that $\alpha\sigma = \beta\sigma$. Complete σ to an exact triangle $X' \xrightarrow{\sigma} X \xrightarrow{\phi} \operatorname{Cone}\sigma \to \Sigma X'$. Then $\alpha-\beta$ factors through ϕ via a morphism $\psi \colon \operatorname{Cone}\sigma \to Y$. Now complete ψ to an exact triangle $\operatorname{Cone}\sigma \xrightarrow{\psi} Y \xrightarrow{\tau} Y' \to \Sigma(\operatorname{Cone}\sigma)$. Then $\tau\alpha = \tau\beta$ and $\tau \in S$.

(2) Fix $Y \in \mathcal{T}$ and suppose that $\operatorname{Hom}_{\mathcal{T}}(X,Y) = 0$ for all $X \in \mathcal{S}$. Then every $\sigma \in S$ induces a bijection $\operatorname{Hom}_{\mathcal{T}}(\sigma, Y)$ because $\operatorname{Hom}_{\mathcal{T}}(-,Y)$ is cohomological. Thus Y is S-local.

Now suppose that Y is S-local. If X belongs to $\mathcal{S}$, then the morphism $\sigma \colon X \to 0$ belongs to S and therefore induces a bijection $\operatorname{Hom}_{\mathcal{T}}(\sigma, Y)$. Thus Y belongs to $\mathcal{S}^{\perp}$. □

The *Verdier localisation* of $\mathcal{T}$ with respect to $\mathcal{S}$ is by definition the localisation

$$\mathcal{T}/\mathcal{S} = \mathcal{T}[S(\mathcal{S})^{-1}]$$

together with the canonical functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$.

Proposition 3.2.2. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a triangulated subcategory. Then the following holds.*

(1) *The category* $\mathcal{T}/\mathcal{S}$ *carries a unique triangulated structure such that the canonical functor* $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ *is exact and annihilates* $\mathcal{S}$.

(2) *If* $\mathcal{U}$ *is a triangulated category and* $F\colon \mathcal{T} \to \mathcal{U}$ *is an exact functor that annihilates* $\mathcal{S}$*, then there exists a unique exact functor* $\bar{F}\colon \mathcal{T}/\mathcal{S} \to \mathcal{U}$ *such that* $F = \bar{F} \circ Q$.

Proof (1) We apply Lemma 3.2.1. Thus $S(\mathcal{S})$ admits a calculus of left and right fractions. The category $\mathcal{T}/\mathcal{S}$ is additive by Lemma 2.2.1. The class $S(\mathcal{S})$ is invariant under the suspension Σ. Thus Σ induces an equivalence $\mathcal{T}/\mathcal{S} \xrightarrow{\sim} \mathcal{T}/\mathcal{S}$. A triangle in $\mathcal{T}/\mathcal{S}$ is by definition exact if it is isomorphic to the image under Q of an exact triangle in $\mathcal{T}$. It is straightforward to check the conditions (Tr1)–(Tr4), and the functor Q is exact by construction. Clearly, $Q|_{\mathcal{S}} = 0$.

(2) If $F\colon \mathcal{T} \to \mathcal{U}$ is an exact functor and $F|_{\mathcal{S}} = 0$, then F inverts all morphisms in $S(\mathcal{S})$. Thus F factors through $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ via a unique functor $\bar{F}\colon \mathcal{T}/\mathcal{S} \to \mathcal{U}$. The functor $\bar{F}$ is exact, because any exact triangle in $\mathcal{T}/\mathcal{S}$ is up to isomorphism the image under Q of an exact triangle in $\mathcal{T}$. □

Remark 3.2.3. (1) The properties (1)–(2) in Proposition 3.2.2 provide a universal property that determines the canonical functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ up to a unique isomorphism.

(2) The canonical functor $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ annihilates a morphism α in $\mathcal{T}$ if and only if α factors through an object in $\mathcal{S}$. In particular, $QX = 0$ for an object X in $\mathcal{T}$ if and only if X is a direct summand of an object in $\mathcal{S}$. Thus $\operatorname{Ker} Q = \operatorname{Thick}(\mathcal{S})$.

(3) A cohomological functor $H\colon \mathcal{T} \to \mathcal{A}$ factors through $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ via a unique cohomological functor $\mathcal{T}/\mathcal{S} \to \mathcal{A}$ if and only if $H|_{\mathcal{S}} = 0$.

(4) The canonical functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ preserves all coproducts in $\mathcal{T}$ if and only if $\mathcal{S}$ is closed under coproducts; see Lemma 1.1.8.

The following provides a useful fact about the morphisms in $\mathcal{T}/\mathcal{S}$.

Lemma 3.2.4. *Let* $\mathcal{S} \subseteq \mathcal{T}$ *be a triangulated subcategory and* $X \in \mathcal{T}$*. Then the canonical map*

$$\operatorname{Hom}_{\mathcal{T}}(X', X) \longrightarrow \operatorname{Hom}_{\mathcal{T}/\mathcal{S}}(X', X)$$

is a bijection for all $X' \in \mathcal{T}$ *if and only if* $X \in \mathcal{S}^{\perp}$*. Analogously,*

$$\operatorname{Hom}_{\mathcal{T}}(X, X') \longrightarrow \operatorname{Hom}_{\mathcal{T}/\mathcal{S}}(X, X')$$

is a bijection for all $X' \in \mathcal{T}$ *if and only if* $X \in {}^{\perp}\mathcal{S}$.

Proof This follows from Lemma 1.1.2 and Lemma 3.2.1. □

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Localisation of Subcategories

We consider a Verdier localisation and its triangulated subcategories. The following lemma provides a useful criterion.

Lemma 3.2.5. *Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{T}$ be triangulated subcategories of a triangulated category $\mathcal{T}$. Suppose that one of the following conditions holds.*

(1) *Every morphism $\mathcal{V} \ni V \to U \in \mathcal{U}$ factors through an object in $\mathcal{U} \cap \mathcal{V}$.*
(2) *Every morphism $\mathcal{U} \ni U \to V \in \mathcal{V}$ factors through an object in $\mathcal{U} \cap \mathcal{V}$.*

Then the induced functor $\mathcal{U}/(\mathcal{U} \cap \mathcal{V}) \to \mathcal{T}/\mathcal{V}$ is fully faithful.

We capture the situation in the following commutative diagram

$$\begin{array}{ccccc} \mathcal{U} \cap \mathcal{V} & \rightarrowtail & \mathcal{U} & \twoheadrightarrow & \mathcal{U}/(\mathcal{U} \cap \mathcal{V}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{V} & \rightarrowtail & \mathcal{T} & \twoheadrightarrow & \mathcal{T}/\mathcal{V} \end{array}$$

and provide a criterion for the functor on the right to be fully faithful.

Proof Suppose (1) holds; the other case is dual. We claim that $\mathcal{U}$ is left cofinal with respect to $S(\mathcal{V})$. Then the inclusion $\mathcal{U} \to \mathcal{T}$ induces a fully faithful functor $\mathcal{U}/(\mathcal{U} \cap \mathcal{V}) \to \mathcal{T}/\mathcal{V}$ by Lemma 1.2.5, since $S(\mathcal{U} \cap \mathcal{V}) = S(\mathcal{V}) \cap \mathcal{U}$.

To prove the claim choose a morphism $U \to Y$ in $S(\mathcal{V})$ with $U \in \mathcal{U}$. This yields an exact triangle $V \to U \to Y \to \Sigma V$. The first morphism factors through an object $X \in \mathcal{U} \cap \mathcal{V}$. Applying the octahedral axiom yields a commutative diagram

$$\begin{array}{ccccccc} & & \Sigma^{-1} Z & = & \Sigma^{-1} Z & & \\ & & \downarrow & & \downarrow & & \\ V & \longrightarrow & X & \longrightarrow & Y' & \longrightarrow & \Sigma V \\ \| & & \downarrow & & \downarrow & & \| \\ V & \longrightarrow & U & \longrightarrow & Y & \longrightarrow & \Sigma V \\ & & \downarrow & & \downarrow & & \\ & & Z & = & Z & & \end{array}$$

with exact rows and columns. Then $Y \to Z$ is the desired morphism with $Z \in \mathcal{U}$. □

Next we describe all triangulated subcategories of a Verdier localisation.

Proposition 3.2.6. *Let $\mathcal{V} \subseteq \mathcal{U} \subseteq \mathcal{T}$ be triangulated subcategories of a triangulated category $\mathcal{T}$. Then $\mathcal{U}/\mathcal{V}$ identifies with a triangulated subcategory of*

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$\mathcal{T}/\mathcal{V}$, and every triangulated subcategory of $\mathcal{T}/\mathcal{V}$ is of this form. Moreover, the canonical functor $\mathcal{T} \to \mathcal{T}/\mathcal{V}$ induces an isomorphism $\mathcal{T}/\mathcal{U} \xrightarrow{\sim} (\mathcal{T}/\mathcal{V})/(\mathcal{U}/\mathcal{V})$.

We capture the situation in the following commutative diagram.

$$\begin{array}{ccccc}
\mathcal{V} & \rightarrowtail & \mathcal{U} & \twoheadrightarrow & \mathcal{U}/\mathcal{V} \\
\| & & \downarrow & & \downarrow \\
\mathcal{V} & \rightarrowtail & \mathcal{T} & \twoheadrightarrow & \mathcal{T}/\mathcal{V} \\
 & & \downarrow & & \downarrow \\
 & & \mathcal{T}/\mathcal{U} & \xrightarrow{\sim} & (\mathcal{T}/\mathcal{V})/(\mathcal{U}/\mathcal{V})
\end{array}$$

Proof The inclusion $\mathcal{U} \to \mathcal{T}$ induces a fully faithful functor $\mathcal{U}/\mathcal{V} \to \mathcal{T}/\mathcal{V}$ by the above Lemma 3.2.5. It is easily checked that $\mathcal{U}/\mathcal{V}$ yields a triangulated subcategory of $\mathcal{T}/\mathcal{V}$. If $\mathcal{W} \subseteq \mathcal{T}/\mathcal{V}$ is a triangulated subcategory, set $\mathcal{U} := Q^{-1}(\mathcal{W})$. Then $\mathcal{U}/\mathcal{V} \xrightarrow{\sim} \mathcal{W}$. The final assertion is clear, since the kernel of the composite $\mathcal{T} \to \mathcal{T}/\mathcal{V} \to (\mathcal{T}/\mathcal{V})/(\mathcal{U}/\mathcal{V})$ equals $\mathcal{U}$. □

Localisation and Adjoints

Let $\mathcal{T}$ be a triangulated category and $\mathcal{S} \subseteq \mathcal{T}$ a triangulated subcategory. Suppose that the canonical functor $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ admits a right adjoint $Q_\rho\colon \mathcal{T}/\mathcal{S} \to \mathcal{T}$. Then Q_ρ is fully faithful and induces an equivalence

$$\mathcal{T}/\mathcal{S} \xrightarrow{\sim} \mathcal{S}^\perp \qquad \text{with quasi-inverse} \qquad \mathcal{S}^\perp \hookrightarrow \mathcal{T} \xrightarrow{Q} \mathcal{T}/\mathcal{S}.$$

This follows from Proposition 1.1.3 and Lemma 3.2.1. The unit of the adjunction yields for X in $\mathcal{T}$ an exact triangle

$$X' \longrightarrow X \xrightarrow{\eta} Q_\rho Q(X) \longrightarrow \Sigma X'$$

with X' a direct summand of an object in $\mathcal{S}$ since $Q(\eta)$ is invertible.

Lemma 3.2.7. *When $\mathcal{S} \subseteq \mathcal{T}$ is thick and Q admits a right adjoint, then the assignment $X \mapsto X'$ provides a right adjoint of the inclusion $\mathcal{S} \to \mathcal{T}$.*

Proof The map $\operatorname{Hom}_{\mathcal{T}}(-, X') \to \operatorname{Hom}_{\mathcal{T}}(-, X)$ is bijective when restricted to $\mathcal{S}$ since $Q_\rho Q(X)$ and $\Sigma^{-1} Q_\rho Q(X)$ are in $\mathcal{S}^\perp$. □

The following proposition expresses the symmetry which arises from localising a triangulated category with respect to a thick subcategory.

Proposition 3.2.8. *Let $\mathcal{S} \subseteq \mathcal{T}$ be a thick subcategory. Then the following are equivalent.*

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(1) *The inclusion* $\mathcal{S} \to \mathcal{T}$ *admits a right adjoint.*
(2) *For each* $X \in \mathcal{T}$ *there exists an exact triangle* $X' \to X \to X'' \to \Sigma X'$ *with* $X' \in \mathcal{S}$ *and* $X'' \in \mathcal{S}^{\perp}$.
(3) *The canonical functor* $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ *admits a right adjoint.*
(4) *The composite* $\mathcal{S}^{\perp} \hookrightarrow \mathcal{T} \twoheadrightarrow \mathcal{T}/\mathcal{S}$ *is a triangle equivalence.*

In that case the right adjoint $\mathcal{T} \to \mathcal{S}$ *induces a triangle equivalence*

$$\mathcal{T}/(\mathcal{S}^{\perp}) \xrightarrow{\sim} \mathcal{S} \qquad \text{and} \qquad {}^{\perp}(\mathcal{S}^{\perp}) = \mathcal{S}.$$

Proof (1) ⇒ (2): Suppose that the inclusion $I\colon \mathcal{S} \to \mathcal{T}$ admits a right adjoint $I_\rho\colon \mathcal{T} \to \mathcal{S}$, and consider for X in $\mathcal{T}$ the exact triangle

$$\Sigma^{-1}X'' \longrightarrow II_\rho(X) \longrightarrow X \longrightarrow X''$$

given by the counit of the adjunction. Then we have $II_\rho(X) \in \mathcal{S}$ and $X'' \in \mathcal{S}^{\perp}$.

(2) ⇒ (3): Suppose there is for X in $\mathcal{T}$ an exact triangle $X' \to X \to X'' \to \Sigma X'$ with $X' \in \mathcal{S}$ and $X'' \in \mathcal{S}^{\perp}$. The assignment $X \mapsto X''$ provides a left adjoint for the inclusion $\mathcal{S}^{\perp} \to \mathcal{T}$, say $F\colon \mathcal{T} \to \mathcal{S}^{\perp}$. The kernel of F equals ${}^{\perp}(\mathcal{S}^{\perp}) = \mathcal{S}$, and F induces an equivalence $\mathcal{T}/\mathcal{S} \xrightarrow{\sim} \mathcal{S}^{\perp}$. Composing this with the inclusion $\mathcal{S}^{\perp} \to \mathcal{T}$ provides the desired right adjoint of $\mathcal{T} \to \mathcal{T}/\mathcal{S}$.

(3) ⇒ (4): Combine Proposition 1.1.3 and Lemma 3.2.1.

(4) ⇒ (1): A quasi-inverse of $\mathcal{S}^{\perp} \xrightarrow{\sim} \mathcal{T}/\mathcal{S}$ composed with the inclusion $\mathcal{S}^{\perp} \to \mathcal{T}$ provides a right adjoint of $\mathcal{T} \to \mathcal{T}/\mathcal{S}$. Then the inclusion $\mathcal{S} \to \mathcal{T}$ admits a right adjoint, by Lemma 3.2.7.

This completes the first part of the proof. We have already seen that a right adjoint $I_\rho\colon \mathcal{T} \to \mathcal{S}$ of the inclusion $\mathcal{S} \to \mathcal{T}$ arises from a localisation, by Proposition 1.1.3, and its kernel equals $\mathcal{S}^{\perp}$. Thus I_ρ induces a triangle equivalence $\mathcal{T}/(\mathcal{S}^{\perp}) \xrightarrow{\sim} \mathcal{S}$. □

We capture the situation in the following diagram

$$\mathcal{S} \underset{I_\rho}{\overset{I}{\rightleftarrows}} \mathcal{T} \underset{Q_\rho}{\overset{Q}{\rightleftarrows}} \mathcal{T}/\mathcal{S}$$

which is a localisation sequence. The adjunctions yield for each object $X \in \mathcal{T}$ an exact triangle

$$II_\rho(X) \longrightarrow X \longrightarrow Q_\rho Q(X) \longrightarrow \Sigma II_\rho(X).$$

The following proposition complements Proposition 3.2.8.

Proposition 3.2.9. *Let* (F, G) *be an adjoint pair of functors*

$$\mathcal{T} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{U}$$

between triangulated categories such that F is exact and set $\mathcal{S} = \operatorname{Ker} F$. Then G is fully faithful if and only if F induces a triangle equivalence $\mathcal{T}/\mathcal{S} \xrightarrow{\sim} \mathcal{U}$.

Proof Let $S = \{\sigma \in \operatorname{Mor} \mathcal{T} \mid F\sigma \text{ is invertible}\}$. Then G is fully faithful if and only if F induces an equivalence $\mathcal{T}[S^{-1}] \xrightarrow{\sim} \mathcal{U}$, by Proposition 1.1.3. It remains to observe that $\mathcal{T}[S^{-1}] = \mathcal{T}/\mathcal{S}$, since $S = S(\mathcal{S})$. Here we use that F is exact. □

We note the symmetry for triangulated categories which differs from that for abelian categories. For an abelian category $\mathcal{A}$ and a Serre subcategory $\mathcal{C} \subseteq \mathcal{A}$, a right adjoint of $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ implies the existence of a right adjoint of $\mathcal{C} \hookrightarrow \mathcal{A}$ (Lemma 2.2.10), but the converse is not true without further assumptions. Also, a right adjoint of $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ need not be exact.

3.3 Frobenius Categories

Stable categories of Frobenius categories provide important examples of triangulated categories. The exact structure of a Frobenius category induces a canonical triangulated structure of the stable category. In particular, there are canonical choices of exact triangles and morphisms between such triangles. With these choices the formation of cones becomes functorial.

Stable Categories of Frobenius Categories

An exact category $\mathcal{A}$ is a *Frobenius category* if there are enough projective and enough injective objects, and if projective and injective objects coincide. Let $\mathcal{P}$ denote the full subcategory of projective objects. The *stable category* $\operatorname{St}\mathcal{A}$ is by definition the additive quotient $\mathcal{A}/\mathcal{P}$. For objects X, Y we set

$$\underline{\operatorname{Hom}}_{\mathcal{A}}(X,Y) = \operatorname{Hom}_{\operatorname{St}\mathcal{A}}(X,Y).$$

Let $\mathcal{A}$ be a Frobenius category and fix for each object X an admissible monomorphism $X \to IX$ such that IX is an injective object. The *cone* of a morphism $\phi\colon X \to Y$ is obtained by forming the following pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & IX & \longrightarrow & \Sigma X & \longrightarrow & 0 \\ & & {\scriptstyle\phi}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{\phi'} & \operatorname{Cone}\phi & \xrightarrow{\phi''} & \Sigma X & \longrightarrow & 0 \end{array}$$

and we call (ϕ, ϕ', ϕ'') a *cone sequence* induced by ϕ. Note that this diagram depends on the choice of $X \to IX$, but it is unique up to an isomorphism when

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one passes to the stable category of $\mathcal{A}$. In particular, a morphism in $\mathcal{A}$ has a projective cone if and only if its image under $\mathcal{A} \to \operatorname{St}\mathcal{A}$ is invertible in $\operatorname{St}\mathcal{A}$.

Now let $\xi\colon 0 \to X \to Y \to Z \to 0$ be an admissible exact sequence in $\mathcal{A}$. We consider the induced commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & X & \longrightarrow & IX & \longrightarrow & \Sigma X & \longrightarrow & 0 \end{array}$$

and call (α, β, γ) a *standard triangle* induced by ξ. Again, the triangle is unique up to isomorphism in $\operatorname{St}\mathcal{A}$.

Let us compare cone sequences and standard triangles by taking them into the stable category $\operatorname{St}\mathcal{A}$.

Lemma 3.3.1. *In* $\operatorname{St}\mathcal{A}$, *a triangle* (α, β, γ) *is isomorphic to a cone sequence induced by a morphism in* $\mathcal{A}$ *if and only if* (α, β, γ) *is isomorphic to a standard triangle induced by an admissible exact sequence in* $\mathcal{A}$.

Proof Given a morphism $\phi\colon X \to Y$ in $\mathcal{A}$, the pushout defining the cone sequence (ϕ, ϕ', ϕ'') yields an admissible exact sequence $0 \to X \to Y \oplus IX \to \operatorname{Cone}\phi \to 0$. On the other hand, an admissible exact sequence $0 \to X \to Y \to Z \to 0$ yields the following pushout diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & X & \longrightarrow & IX & \longrightarrow & \Sigma X & \longrightarrow & 0 \\ & & {\scriptstyle\phi}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{\phi'} & \operatorname{Cone}\phi & \xrightarrow{\phi''} & \Sigma X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & Z & = & Z & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

and it is clear that $\operatorname{Cone}\phi \to Z$ is an isomorphism in $\operatorname{St}\mathcal{A}$. □

Proposition 3.3.2. *Let* $\mathcal{A}$ *be a Frobenius category. Then the assignment* $X \mapsto \Sigma X$ *induces an equivalence* $\operatorname{St}\mathcal{A} \xrightarrow{\sim} \operatorname{St}\mathcal{A}$, *and the category* $\operatorname{St}\mathcal{A}$ *together with all triangles isomorphic to the image of a standard triangle in* $\mathcal{A}$ *is a triangulated category.*

A triangulated category that is triangle equivalent to the stable category

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of a Frobenius category is called *algebraic*. In fact, all specific triangulated categories arising in this book are algebraic. Further descriptions are provided in Proposition 9.1.5 and Proposition 9.1.15.

The proof of Proposition 3.3.2 requires some preparation. For each $X \in \mathcal{A}$ fix an exact sequence

$$\omega_X\colon \quad 0 \longrightarrow X \xrightarrow{x} IX \xrightarrow{\bar{x}} \Sigma X \longrightarrow 0.$$

Lemma 3.3.3. *Multiplication by ω_X induces a natural isomorphism*

$$\underline{\mathrm{Hom}}(-, \Sigma X) \xrightarrow{\sim} \mathrm{Ext}^1(-, X).$$

A standard triangle (α, β, γ) corresponding to an exact sequence $\xi\colon 0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ in $\mathcal{A}$ induces an exact sequence of functors

$$\mathrm{Hom}(-, X) \xrightarrow{(-,\alpha)} \mathrm{Hom}(-, Y) \xrightarrow{(-,\beta)} \mathrm{Hom}(-, Z) \xrightarrow{(-,\gamma)} \underline{\mathrm{Hom}}(-, \Sigma X)$$

which is functorial in X and Z. Moreover, we have $\omega_X \cdot \gamma = \xi$.

Proof The cokernel of $\mathrm{Hom}(-, IX) \to \mathrm{Hom}(-, \Sigma X)$ equals $\underline{\mathrm{Hom}}(-, \Sigma X)$ which is therefore isomorphic to $\mathrm{Ext}^1(-, X)$. Thus for $Z \in \mathcal{A}$ the isomorphism

$$\underline{\mathrm{Hom}}(Z, \Sigma X) \xrightarrow{\sim} \mathrm{Ext}^1(Z, X)$$

maps ϕ to $\omega_X \cdot \phi$. The identity $\omega_X \cdot \gamma = \xi$ follows from the definition of a standard triangle, and then the exact sequence of functors is clear. □

Proof of Proposition 3.3.2 The first assertion is easily checked. For the verification of the axioms of a triangulated category we use Lemma 3.3.1 and standard properties of exact categories.

(Tr1) The class of exact triangles is closed under isomorphisms by definition. The standard triangle given by the exact sequence $0 \to 0 \to X \xrightarrow{\mathrm{id}} X \to 0$ equals $0 \to X \xrightarrow{\mathrm{id}} X \to 0$. From the definition of a cone sequence (ϕ, ϕ', ϕ'') it is clear that each morphism ϕ fits into an exact triangle.

(Tr2) Fix a standard triangle (α, β, γ) given by the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & X & \xrightarrow{x} & IX & \xrightarrow{\bar{x}} & \Sigma X & \longrightarrow & 0
\end{array}$$

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Then consider the following diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\left[\begin{smallmatrix}\beta\\ \alpha'\end{smallmatrix}\right]} & Z\oplus IX & \xrightarrow{\left[\begin{smallmatrix}\gamma & -\bar{x}\end{smallmatrix}\right]} & \Sigma X & \longrightarrow & 0\\
 & & \Vert & & \big\downarrow{\scriptstyle \left[\begin{smallmatrix}\beta' & I\alpha\end{smallmatrix}\right]} & & \big\downarrow{\scriptstyle -\Sigma\alpha} & & \\
0 & \longrightarrow & Y & \xrightarrow{y} & IY & \xrightarrow{\bar{y}} & \Sigma Y & \longrightarrow & 0
\end{array}$$

From the identity

$$(y - I\alpha\alpha')\alpha = y\alpha - I\alpha x = 0$$

we obtain $\beta' \colon Z \to IY$ satisfying $y - I\alpha\alpha' = \beta'\beta$; so the left hand square commutes. For the commutativity of the other square we compute

$$\bar{y}\beta'\beta = \bar{y}y - \bar{y}I\alpha\alpha' = -\Sigma\alpha\bar{x}\alpha' = -\Sigma\alpha\gamma\beta.$$

Thus $\bar{y}\beta' = -\Sigma\alpha\gamma$ since β is an epimorphism. Now the diagram yields a standard triangle which is isomorphic to $(\beta, \gamma, -\Sigma\alpha)$.

(Tr3) Fix exact triangles (α, β, γ) and $(\alpha', \beta', \gamma')$ with a pair of morphisms ϕ_1 and ϕ_2 satisfying $\phi_2\alpha = \alpha'\phi_1$. We may assume them to be standard triangles and that the equality $\phi_2\alpha = \alpha'\phi_1$ holds in $\mathcal{A}$, by adding to α an injective summand if necessary. This yields the following commutative diagram with exact rows.

$$\begin{array}{cccccccccc}
\xi\colon & 0 & \longrightarrow & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \longrightarrow & 0\\
 & & & \big\downarrow{\scriptstyle \phi_1} & & \big\downarrow{\scriptstyle \phi_2} & & \big\downarrow{\scriptstyle \phi_3} & & \\
\xi'\colon & 0 & \longrightarrow & X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \longrightarrow & 0
\end{array}$$

We need to show that $\Sigma\phi_1\gamma = \gamma'\phi_3$. Clearly, this follows from a commutative diagram of the following form.

$$\begin{array}{ccccccc}
\operatorname{Hom}(-,X) & \longrightarrow & \operatorname{Hom}(-,Y) & \longrightarrow & \operatorname{Hom}(-,Z) & \longrightarrow & \underline{\operatorname{Hom}}(-,\Sigma X)\\
\big\downarrow{\scriptstyle (-,\phi_1)} & & \big\downarrow{\scriptstyle (-,\phi_2)} & & \big\downarrow{\scriptstyle (-,\phi_3)} & & \big\downarrow{\scriptstyle (-,\Sigma\phi_1)}\\
\operatorname{Hom}(-,X') & \longrightarrow & \operatorname{Hom}(-,Y') & \longrightarrow & \operatorname{Hom}(-,Z') & \longrightarrow & \underline{\operatorname{Hom}}(-,\Sigma X')
\end{array}$$

We obtain this from Lemma 3.3.3, since the horizontal sequence is functorial, and using that $\phi_1\xi = \xi'\phi_3$.

(Tr4) Fix exact triangles $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$, and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1 = \beta_1\alpha_1$. We may assume them to be standard and that the equality $\gamma_1 = \beta_1\alpha_1$ holds in $\mathcal{A}$. Then we obtain in $\mathcal{A}$ a commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\alpha_1} & Y & \xrightarrow{\alpha_2} & U & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \delta_1} & & \\
0 & \longrightarrow & X & \xrightarrow{\gamma_1} & Z & \xrightarrow{\gamma_2} & V & \longrightarrow & 0
\end{array}$$

From this we obtain a standard triangle $(\delta_1, \delta_2, \delta_3)$ making the following diagram commutative.

$$\begin{array}{ccccccc}
X & \xrightarrow{\alpha_1} & Y & \xrightarrow{\alpha_2} & U & \xrightarrow{\alpha_3} & \Sigma X \\
\| & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \delta_1} & & \| \\
X & \xrightarrow{\gamma_1} & Z & \xrightarrow{\gamma_2} & V & \xrightarrow{\gamma_3} & \Sigma X \\
 & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \delta_2} & & \\
 & & W & = & W & & \\
 & & \downarrow{\scriptstyle \beta_3} & & \downarrow{\scriptstyle \delta_3} & & \\
 & & \Sigma Y & \xrightarrow{\Sigma \alpha_2} & \Sigma U & &
\end{array}$$

It remains to check the identity $\beta_3 \delta_2 = \Sigma \alpha_1 \gamma_3$ in $\underline{\mathrm{Hom}}(V, \Sigma Y)$. This follows from the following commutative diagram

$$\begin{array}{ccccc}
\underline{\mathrm{Hom}}(V, \Sigma X) & \xrightarrow{(V, \Sigma \alpha_1)} & \underline{\mathrm{Hom}}(V, \Sigma Y) & \xleftarrow{(\delta_2, \Sigma Y)} & \underline{\mathrm{Hom}}(W, \Sigma Y) \\
\downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle \wr} \\
\mathrm{Ext}^1(V, X) & \longrightarrow & \mathrm{Ext}^1(V, Y) & \longleftarrow & \mathrm{Ext}^1(W, Y)
\end{array}$$

and the description of the third morphism in a standard triangle given in Lemma 3.3.1, since the maps in the bottom row send the extensions corresponding to β and γ to the same extension

$$0 \longrightarrow Y \xrightarrow{\left[\begin{smallmatrix} \alpha_2 \\ \beta_1 \end{smallmatrix}\right]} U \oplus Z \xrightarrow{\left[\begin{smallmatrix} -\delta_1 & \gamma_2 \end{smallmatrix}\right]} V \longrightarrow 0. \qquad \square$$

Example 3.3.4. For a ring Λ the following conditions are equivalent (Theorem 13.2.13).

(1) Projective and injective Λ-modules coincide.
(2) The category $\operatorname{Mod} \Lambda$ of Λ-modules is a Frobenius category.
(3) The ring Λ is right artinian and $\operatorname{mod} \Lambda$ is a Frobenius category.
(4) The ring Λ is right noetherian and the module Λ_Λ is injective.

A ring satisfying these equivalent conditions is called *quasi-Frobenius*. This notion is symmetric, so Λ is quasi-Frobenius if and only if Λ^{op} is quasi-Frobenius. A ring Λ is called *right self-injective* if the module Λ_Λ is injective, and Λ is *self-injective* if it is both right and left self-injective. Thus for noetherian rings the concepts 'quasi-Frobenius' and 'self-injective' coincide. For example, the group algebra kG of a finite group G over a field k is quasi-Frobenius and self-injective.

We write $\operatorname{StMod}\Lambda = \operatorname{St}(\operatorname{Mod}\Lambda)$ when Λ is quasi-Frobenius.

Frobenius Pairs

A *Frobenius pair* $(\mathcal{A}, \mathcal{A}_0)$ is a Frobenius category $\mathcal{A}$ together with a full additive subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $\mathcal{A}_0$ contains all projective objects of $\mathcal{A}$ and the *two out of three property* holds: for an admissible exact sequence in $\mathcal{A}$ with two terms in $\mathcal{A}_0$, the third term is also in $\mathcal{A}_0$.

We observe that for a fixed Frobenius category $\mathcal{A}$ the Frobenius pairs $(\mathcal{A}, \mathcal{A}_0)$ correspond bijectively to triangulated subcategories of $\operatorname{St}\mathcal{A}$. The assignment sends $\mathcal{A}_0$ to its stable category $\operatorname{St}\mathcal{A}_0$, where $\mathcal{A}_0$ is viewed as a Frobenius category having the same projective and injective objects as $\mathcal{A}$.

Let $(\mathcal{A}, \mathcal{A}_0)$ be a Frobenius pair and set

$$S = \{\phi \in \operatorname{Mor}\mathcal{A} \mid \operatorname{Cone}\phi \in \mathcal{A}_0\}.$$

The *derived category* $\mathbf{D}(\mathcal{A}, \mathcal{A}_0)$ of $(\mathcal{A}, \mathcal{A}_0)$ is obtained by formally inverting all morphisms in S. Thus one defines

$$\mathbf{D}(\mathcal{A}, \mathcal{A}_0) = \mathcal{A}[S^{-1}].$$

For a morphism ϕ in $\mathcal{A}$ we write $\bar{\phi}$ for the corresponding morphism in $\operatorname{St}\mathcal{A}$.

Proposition 3.3.5. *For a Frobenius pair $(\mathcal{A}, \mathcal{A}_0)$ the following holds.*

(1) *The class $\bar{S} = \{\bar{\phi} \mid \phi \in S\} \subseteq \operatorname{Mor}\operatorname{St}\mathcal{A}$ admits a calculus of left and right fractions, and the canonical functor $\mathcal{A} \to \mathbf{D}(\mathcal{A}, \mathcal{A}_0)$ induces an equivalence*

$$(\operatorname{St}\mathcal{A})[\bar{S}^{-1}] = \operatorname{St}\mathcal{A}/\operatorname{St}\mathcal{A}_0 \xrightarrow{\sim} \mathbf{D}(\mathcal{A}, \mathcal{A}_0).$$

(2) *The assignment $X \mapsto \Sigma X$ induces an equivalence $\mathbf{D}(\mathcal{A}, \mathcal{A}_0) \xrightarrow{\sim} \mathbf{D}(\mathcal{A}, \mathcal{A}_0)$, and the category $\mathbf{D}(\mathcal{A}, \mathcal{A}_0)$ together with all triangles isomorphic to the localisation of a cone sequence (ϕ, ϕ', ϕ'') in $\mathcal{A}$ is a triangulated category.*

Proof The stable category $\operatorname{St}\mathcal{A}$ is the localisation of $\mathcal{A}$ with respect to the class of morphisms ϕ in $\mathcal{A}$ such that $\operatorname{Cone}\phi$ is projective, by Lemma 2.2.2. Thus $(\operatorname{St}\mathcal{A})[\bar{S}^{-1}]$ identifies with $\mathcal{A}[S^{-1}]$. Next observe that $\operatorname{St}\mathcal{A}_0$ is a triangulated subcategory of $\operatorname{St}\mathcal{A}$. It follows from Lemma 3.2.1 that $\bar{S}$ admits a calculus of left and right fractions, and the localisation $(\operatorname{St}\mathcal{A})[\bar{S}^{-1}]$ equals the Verdier

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localisation $\operatorname{St}\mathcal{A}/\operatorname{St}\mathcal{A}_0$. Now the triangulated structure of $\mathbf{D}(\mathcal{A},\mathcal{A}_0)$ is induced by that of $\operatorname{St}\mathcal{A}$, using Proposition 3.2.2 and Proposition 3.3.2. □

The class $S \subseteq \operatorname{Mor}\mathcal{A}$ admits a calculus of left fractions if and only if $\mathcal{A}_0 = \mathcal{A}$. For instance, (LF3) fails for a pair $\alpha, \beta\colon X \to Y$ where $\alpha = 0$ and β is an epimorphism with projective X and $Y \in \mathcal{A} \setminus \mathcal{A}_0$.

The construction of the derived category $\mathbf{D}(\mathcal{A},\mathcal{A}_0)$ yields the following universal property.

Corollary 3.3.6. *Let $(\mathcal{A},\mathcal{A}_0)$ be a Frobenius pair. If $\mathcal{T}$ is a triangulated category and $F\colon \operatorname{St}\mathcal{A} \to \mathcal{T}$ is an exact functor such that $F|_{\mathcal{A}_0} = 0$, then there exists a unique exact functor $\bar{F}\colon \mathbf{D}(\mathcal{A},\mathcal{A}_0) \to \mathcal{T}$ making the following diagram commutative:*

$$\begin{array}{ccc} \mathcal{A} & \twoheadrightarrow & \mathbf{D}(\mathcal{A},\mathcal{A}_0) \\ \downarrow & & \downarrow \bar{F} \\ \operatorname{St}\mathcal{A} & \xrightarrow{F} & \mathcal{T} \end{array}$$

Proof Combine Proposition 3.2.2 and Proposition 3.3.5. □

3.4 Brown Representability

In this section we study triangulated categories that admit arbitrary coproducts. An important aspect in this context is the representability of cohomological functors. We discuss two versions of Brown's representability theorem. In each case the category needs to be generated by objects satisfying certain finiteness conditions. The most natural condition is 'compactness', which means that the functor $\operatorname{Hom}(X,-)$ preserves all coproducts. The construction of representing objects is fairly explicit and involves homotopy colimits.

Homotopy (Co)limits

Let $\mathcal{T}$ be a triangulated category and suppose that countable coproducts exist in $\mathcal{T}$. Let

$$X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \cdots$$

be a sequence of morphisms in $\mathcal{T}$. A *homotopy colimit* of this sequence is by definition an object X that occurs in an exact triangle

$$\Sigma^{-1}X \longrightarrow \textstyle\coprod_{n\geq 0} X_n \xrightarrow{\operatorname{id}-\phi} \coprod_{n\geq 0} X_n \xrightarrow{\mu} X.$$

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Here, the nth component of the morphism $\mathrm{id}-\phi$ is the composite

$$X_n \xrightarrow{\left[\begin{smallmatrix}\mathrm{id}\\ -\phi_n\end{smallmatrix}\right]} X_n \oplus X_{n+1} \xrightarrow{\mathrm{inc}} \textstyle\coprod_{n\ge 0} X_n.$$

We write $\mathrm{hocolim}_n X_n$ for X; this comes with canonical morphisms

$$\mu_i \colon X_i \longrightarrow \operatorname*{hocolim}_n X_n \qquad (i \ge 0).$$

Note that a homotopy colimit is unique up to a non-unique isomorphism. In some cases the obstruction for uniqueness is controlled by phantom morphisms; see Lemma 5.2.5.

Lemma 3.4.1. *Let $(\alpha_n \colon X_n \to Y)_{n\ge 0}$ be a sequence of morphisms in $\mathcal{T}$ such that $\alpha_n = \alpha_{n+1}\phi_n$ for all n. Then there exists a (usually non-unique) morphism $\bar{\alpha}\colon \mathrm{hocolim}_n X_n \to Y$ such that $\alpha_n = \bar{\alpha}\mu_n$ for all n.*

Proof The α_n yield a morphism $\alpha\colon \coprod_{n\ge 0} X_n \to Y$ satisfying $\alpha(\mathrm{id}-\phi) = 0$. Thus α factors through $\mathrm{Cone}(\mathrm{id}-\phi) = \mathrm{hocolim}_n X_n$. □

The dual construction requires the existence of countable products in $\mathcal{T}$ and yields the *homotopy limit* of a sequence

$$\cdots \xrightarrow{\phi_2} X_2 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_0} X_0$$

which is by definition an object X occurring in an exact triangle

$$X \longrightarrow \textstyle\prod_{n\ge 0} X_n \xrightarrow{\mathrm{id}-\phi} \prod_{n\ge 0} X_n \longrightarrow \Sigma X.$$

Again, this is unique up to a non-unique isomorphism and we write $\mathrm{holim}_n X_n$.

Remark 3.4.2. Given sequences $X_0 \to X_1 \to X_2 \to \cdots$ and $Y_0 \to Y_1 \to Y_2 \to \cdots$ of morphisms in $\mathcal{T}$, we have

$$(\operatorname*{hocolim}_n X_n) \oplus (\operatorname*{hocolim}_n Y_n) \cong \operatorname*{hocolim}_n (X_n \oplus Y_n).$$

Let us compute the functor $\mathrm{Hom}_{\mathcal{T}}(-, \mathrm{hocolim}_n X_n)$. To this end observe that a sequence

$$A_0 \xrightarrow{\phi_0} A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \cdots$$

of maps between abelian groups induces an exact sequence

$$0 \longrightarrow \textstyle\coprod_{n\ge 0} A_n \xrightarrow{\mathrm{id}-\phi} \coprod_{n\ge 0} A_n \longrightarrow \mathrm{colim}_n A_n \longrightarrow 0$$

because it identifies with the colimit of the exact sequences

$$0 \longrightarrow \textstyle\coprod_{i=0}^{n-1} A_i \xrightarrow{\mathrm{id}-\phi} \coprod_{i=0}^{n} A_i \longrightarrow A_n \longrightarrow 0.$$

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Lemma 3.4.3. *Let C be an object in $\mathcal{T}$ such that* $\operatorname{Hom}_{\mathcal{T}}(C,-)$ *preserves all coproducts. Then any sequence $X_0 \to X_1 \to X_2 \to \cdots$ in $\mathcal{T}$ induces an isomorphism*

$$\operatorname*{colim}_n \operatorname{Hom}_{\mathcal{T}}(C, X_n) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{T}}(C, \operatorname*{hocolim}_n X_n).$$

Proof The above observation gives an exact sequence

$$0 \to \textstyle\coprod_n \operatorname{Hom}_{\mathcal{T}}(C, X_n) \to \coprod_n \operatorname{Hom}_{\mathcal{T}}(C, X_n) \to \operatorname{colim}_n \operatorname{Hom}_{\mathcal{T}}(C, X_n) \to 0.$$

Now apply $\operatorname{Hom}_{\mathcal{T}}(C,-)$ to the defining triangle for $\operatorname{hocolim}_n X_n$. Comparing both sequences yields the assertion, since

$$\coprod_n \operatorname{Hom}_{\mathcal{T}}(C, X_n) \cong \operatorname{Hom}_{\mathcal{T}}(C, \coprod_n X_n). \qquad \square$$

Example 3.4.4. Let $\phi \colon X \to X$ be an idempotent morphism in $\mathcal{T}$. Consider the following sequences:

$$X \xrightarrow{\phi} X \xrightarrow{\phi} X \xrightarrow{\phi} \cdots$$

$$X \xrightarrow{\mathrm{id}-\phi} X \xrightarrow{\mathrm{id}-\phi} X \xrightarrow{\mathrm{id}-\phi} \cdots$$

Write X' for a homotopy colimit of the first sequence and X'' for a homotopy colimit of the second sequence. Then we have $X \cong X' \oplus X''$ with $X' = \operatorname{Ker}(\mathrm{id}-\phi)$ and $X'' = \operatorname{Ker}\phi$. In particular, a triangulated category with countable coproducts is idempotent complete.

Proof The object $X' \oplus X''$ is isomorphic to the homotopy colimit of the sequence

$$X \oplus X \xrightarrow{\left[\begin{smallmatrix}\phi & 0\\ 0 & 1-\phi\end{smallmatrix}\right]} X \oplus X \xrightarrow{\left[\begin{smallmatrix}\phi & 0\\ 0 & 1-\phi\end{smallmatrix}\right]} X \oplus X \xrightarrow{\left[\begin{smallmatrix}\phi & 0\\ 0 & 1-\phi\end{smallmatrix}\right]} \cdots$$

by Remark 3.4.2. Now consider the following commutative diagram

$$\begin{array}{ccccccc}
X \oplus X & \xrightarrow{\left[\begin{smallmatrix}\phi & 0\\ 0 & 1-\phi\end{smallmatrix}\right]} & X \oplus X & \xrightarrow{\left[\begin{smallmatrix}\phi & 0\\ 0 & 1-\phi\end{smallmatrix}\right]} & X \oplus X & \xrightarrow{\left[\begin{smallmatrix}\phi & 0\\ 0 & 1-\phi\end{smallmatrix}\right]} & \cdots \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \\
X \oplus X & \xrightarrow{\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]} & X \oplus X & \xrightarrow{\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]} & X \oplus X & \xrightarrow{\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]} & \cdots
\end{array}$$

with α given by $\left[\begin{smallmatrix}\phi & 1-\phi\\ 1-\phi & \phi\end{smallmatrix}\right]$. Observe that α is an isomorphism, since $\alpha^2 = \mathrm{id}$. The homotopy colimit of the bottom row is X, again using Remark 3.4.2, and therefore $X \cong X' \oplus X''$. $\square$

A Brown Representability Theorem

Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. A triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$ is called *localising* if it is closed under all coproducts. Given a class $\mathcal{X} \subseteq \mathcal{T}$ of objects we denote by $\operatorname{Loc}(\mathcal{X})$ the smallest localising subcategory of $\mathcal{T}$ that contains $\mathcal{X}$.

A set $\mathcal{S}$ of objects in $\mathcal{T}$ is called *perfectly generating* if $\operatorname{Loc}(\mathcal{S}) = \mathcal{T}$ and the following holds

(PG) Given a countable family of morphisms $X_i \to Y_i$ in $\mathcal{T}$ such that the map $\operatorname{Hom}_{\mathcal{T}}(S, X_i) \to \operatorname{Hom}_{\mathcal{T}}(S, Y_i)$ is surjective for all i and $S \in \mathcal{S}$, the induced map

$$\operatorname{Hom}_{\mathcal{T}}\Big(S, \coprod_i X_i\Big) \longrightarrow \operatorname{Hom}_{\mathcal{T}}\Big(S, \coprod_i Y_i\Big)$$

is surjective.

The condition $\operatorname{Loc}(\mathcal{S}) = \mathcal{T}$ can be reformulated, saying that $\operatorname{Hom}_{\mathcal{T}}(\Sigma^n S, X) = 0$ for all $S \in \mathcal{S}$ and $n \in \mathbb{Z}$ implies $X = 0$; see Corollary 3.4.8. The triangulated category $\mathcal{T}$ is called *perfectly generated* if $\mathcal{T}$ admits a perfectly generating set.

We have the following *Brown representability theorem* for a perfectly generated triangulated category.

Theorem 3.4.5 (Brown)**.** *Let $\mathcal{T}$ be a perfectly generated triangulated category. Then a functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ is cohomological and sends all coproducts in $\mathcal{T}$ to products if and only if $F \cong \operatorname{Hom}_{\mathcal{T}}(-, X)$ for some object X in $\mathcal{T}$.*

The proof employs the category $\operatorname{mod}\mathcal{T}$ of finitely presented functors on $\mathcal{T}$. The following lemma explains the basic facts which are needed; it is independent of the triangulated structure of $\mathcal{T}$. In particular, the crucial condition (PG) is explained.

Lemma 3.4.6. *Let $\mathcal{T}$ be an additive category with arbitrary coproducts and weak kernels. Let $\mathcal{S}_0$ be a set of objects in $\mathcal{T}$, and denote by $\mathcal{S}$ the full subcategory of all coproducts of objects in $\mathcal{S}$.*

(1) *The category* $\operatorname{mod}\mathcal{T}$ *is abelian and has arbitrary coproducts. Moreover, the Yoneda functor* $\mathcal{T} \to \operatorname{mod}\mathcal{T}$ *preserves all coproducts.*
(2) *The category $\mathcal{S}$ has weak kernels and* $\operatorname{mod}\mathcal{S}$ *is an abelian category.*
(3) *The assignment $F \mapsto F|_{\mathcal{S}}$ induces an exact functor* $\operatorname{mod}\mathcal{T} \to \operatorname{mod}\mathcal{S}$.
(4) *The functor* $\mathcal{T} \to \operatorname{mod}\mathcal{S}$ *sending X to* $\operatorname{Hom}_{\mathcal{T}}(-, X)|_{\mathcal{S}}$ *preserves countable coproducts if and only if condition* (PG) *holds.*

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Proof First observe that for every X in $\mathcal{T}$, there exists an *approximation* $X' \to X$ such that $X' \in \mathcal{S}$ and $\operatorname{Hom}_{\mathcal{T}}(T, X') \to \operatorname{Hom}_{\mathcal{T}}(T, X)$ is surjective for all $T \in \mathcal{S}$. Take $X' = \coprod_{S \in \mathcal{S}_0} \coprod_{\alpha \in \operatorname{Hom}_{\mathcal{T}}(S,X)} S$ and the canonical morphism $X' \to X$.

(1) The category $\operatorname{mod} \mathcal{T}$ is abelian since every morphism in $\mathcal{T}$ has a weak kernel; see Lemma 2.1.6.

Let $(F_i)_{i \in I}$ be a family of functors in $\operatorname{mod} \mathcal{T}$ with presentations

$$\operatorname{Hom}_{\mathcal{T}}(-, X_i) \longrightarrow \operatorname{Hom}_{\mathcal{T}}(-, Y_i) \longrightarrow F_i \longrightarrow 0.$$

Then the coproduct $\coprod_i F$ is given by the presentation

$$\operatorname{Hom}_{\mathcal{T}}\Big(-, \coprod_i X_i\Big) \longrightarrow \operatorname{Hom}_{\mathcal{T}}\Big(-, \coprod_i Y_i\Big) \longrightarrow \coprod_i F_i \longrightarrow 0.$$

To see this we need to check that

$$\operatorname{Hom}\Big(\coprod_i F_i, G\Big) \cong \prod_i \operatorname{Hom}(F_i, G)$$

for each $G \in \operatorname{mod} \mathcal{T}$. This reduces to the case that $G = \operatorname{Hom}_{\mathcal{T}}(-, Z)$ is representable, and then it follows from Yoneda's lemma. In particular, the coproduct is *not* computed pointwise in Ab.

(2) To prove that $\operatorname{mod} \mathcal{S}$ is abelian, it is sufficient to show that every morphism in $\mathcal{S}$ has a weak kernel. In order to obtain a weak kernel of a morphism $Y \to Z$ in $\mathcal{S}$, take the composite of a weak kernel $X \to Y$ in $\mathcal{T}$ and an approximation $X' \to X$.

(3) It follows from Proposition 2.2.20 that restriction to $\mathcal{S}$ yields a functor $\operatorname{mod} \mathcal{T} \to \operatorname{mod} \mathcal{S}$. Clearly, restriction is exact.

(4) We denote by $i \colon \mathcal{S} \to \mathcal{T}$ the inclusion and write $i^* \colon \operatorname{mod} \mathcal{T} \to \operatorname{mod} \mathcal{S}$ for the restriction functor. Then i^* induces an equivalence $(\operatorname{mod} \mathcal{T})/(\operatorname{Ker} i^*) \xrightarrow{\sim} \operatorname{mod} \mathcal{S}$; see again Proposition 2.2.20.

Thus the functor $\mathcal{T} \to \operatorname{mod} \mathcal{S}$ preserves countable coproducts if and only if i^* preserves countable coproducts, and this happens if and only if if $\operatorname{Ker} i^*$ is closed under countable coproducts; see Remark 2.2.7.

Now observe that $\operatorname{Ker} i^*$ being closed under countable coproducts is a reformulation of the condition (PG). □

Proof of Theorem 3.4.5 Fix a perfectly generating set $\mathcal{S}_0$ and denote by S the coproduct of all suspensions of objects in $\mathcal{S}_0$. It is easily checked that $\{S\}$ is perfectly generating. Taking coproducts and suspensions does not affect the condition (PG). Also, $\operatorname{Loc}(S) = \operatorname{Loc}(\mathcal{S}_0)$ because a triangulated subcategory closed under countable coproducts is closed under direct summands; see Example 3.4.4.

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We construct inductively a sequence

$$X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \cdots$$

of morphisms in $\mathcal{T}$ and elements π_i in FX_i as follows. Set $X_0 = 0$ and $\pi_0 = 0$. Let $X_1 = S^{[FS]}$ be the coproduct of copies of S indexed by the elements in FS, and let π_1 be the element corresponding to id_{FS} in $FX_1 \cong (FS)^{FS}$. Suppose we have already constructed ϕ_{i-1} and π_i for some $i > 0$. Let

$$K_i = \{\alpha \in \mathrm{Hom}_{\mathcal{T}}(S, X_i) \mid (F\alpha)\pi_i = 0\}$$

and complete the canonical morphism $\chi_i \colon S^{[K_i]} \to X_i$ to an exact triangle

$$S^{[K_i]} \xrightarrow{\chi_i} X_i \xrightarrow{\phi_i} X_{i+1} \longrightarrow \Sigma S^{[K_i]}.$$

Now choose an element π_{i+1} in FX_{i+1} such that $(F\phi_i)\pi_{i+1} = \pi_i$. This is possible since $(F\chi_i)\pi_i = 0$ and F is cohomological.

Let $\mathcal{S}$ denotes the full subcategory of all coproducts of copies of S in $\mathcal{T}$. We identify each π_i via Yoneda's lemma with a morphism $\mathrm{Hom}_{\mathcal{T}}(-, X_i) \to F$ and obtain in $\mathrm{mod}\,\mathcal{S}$ the following commutative diagram with split exact rows, where $\psi_i = \mathrm{Hom}_{\mathcal{T}}(-, \phi_i)|_{\mathcal{S}}$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}\,\pi_i|_{\mathcal{S}} & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(-, X_i)|_{\mathcal{S}} & \xrightarrow{\pi_i} & F|_{\mathcal{S}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \psi_i} & & \| & & \\
0 & \longrightarrow & \mathrm{Ker}\,\pi_{i+1}|_{\mathcal{S}} & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(-, X_{i+1})|_{\mathcal{S}} & \xrightarrow{\pi_{i+1}} & F|_{\mathcal{S}} & \longrightarrow & 0
\end{array}$$

We wish to compute the colimit of the sequence $(\psi_i)_{i\geq 0}$. Taking coproducts yields the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \coprod_{i\geq 0} \mathrm{Ker}\,\pi_i|_{\mathcal{S}} & \longrightarrow & \coprod_{i\geq 0} \mathrm{Hom}_{\mathcal{T}}(-, X_i)|_{\mathcal{S}} & \longrightarrow & \coprod_{i\geq 0} F|_{\mathcal{S}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}-0} & & \downarrow{\scriptstyle \mathrm{id}-\psi} & & \downarrow{\scriptstyle \mathrm{id}-\mathrm{id}} & & \\
0 & \longrightarrow & \coprod_{i\geq 0} \mathrm{Ker}\,\pi_i|_{\mathcal{S}} & \longrightarrow & \coprod_{i\geq 0} \mathrm{Hom}_{\mathcal{T}}(-, X_i)|_{\mathcal{S}} & \longrightarrow & \coprod_{i\geq 0} F|_{\mathcal{S}} & \longrightarrow & 0
\end{array}$$

and then the snake lemma yields the following exact sequence.

$$0 \longrightarrow \coprod_{i\geq 0} \mathrm{Hom}_{\mathcal{T}}(-, X_i)|_{\mathcal{S}} \xrightarrow{\mathrm{id}-\psi} \coprod_{i\geq 0} \mathrm{Hom}_{\mathcal{T}}(-, X_i)|_{\mathcal{S}} \longrightarrow F|_{\mathcal{S}} \longrightarrow 0. \tag{3.4.7}$$

Next consider the exact triangle

$$\Sigma^{-1}X \longrightarrow \coprod_{i\geq 0} X_i \xrightarrow{\mathrm{id}-\phi} \coprod_{i\geq 0} X_i \longrightarrow X$$

and observe that

$$(\pi_i) \in \prod_{i\geq 0} FX_i \cong F\Big(\coprod_{i\geq 0} X_i\Big)$$

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induces a morphism

$$\pi\colon \operatorname{Hom}_{\mathcal{T}}(-,X) \longrightarrow F$$

by Yoneda's lemma. We have an isomorphism

$$\coprod_{i\geq 0} \operatorname{Hom}_{\mathcal{T}}(-,X_i)|_{\mathcal{S}} \cong \operatorname{Hom}_{\mathcal{T}}\Big(-,\coprod_{i\geq 0} X_i\Big)\Big|_{\mathcal{S}}$$

because of the reformulation of condition (PG) in Lemma 3.4.6, and we obtain in $\operatorname{mod}\mathcal{S}$ the following exact sequence:

$$\coprod_{i\geq 0} \operatorname{Hom}_{\mathcal{T}}(-,X_i)|_{\mathcal{S}} \xrightarrow{\mathrm{id}-\psi} \coprod_{i\geq 0} \operatorname{Hom}_{\mathcal{T}}(-,X_i)|_{\mathcal{S}} \longrightarrow \operatorname{Hom}_{\mathcal{T}}(-,X)|_{\mathcal{S}}$$
$$\longrightarrow \coprod_{i\geq 0} \operatorname{Hom}_{\mathcal{T}}(-,\Sigma X_i)|_{\mathcal{S}} \xrightarrow{\mathrm{id}-\Sigma\psi} \coprod_{i\geq 0} \operatorname{Hom}_{\mathcal{T}}(-,\Sigma X_i)|_{\mathcal{S}}.$$

A comparison with the exact sequence (3.4.7) shows that

$$\pi|_{\mathcal{S}}\colon \operatorname{Hom}_{\mathcal{T}}(-,X)|_{\mathcal{S}} \longrightarrow F|_{\mathcal{S}}$$

is an isomorphism since $\mathrm{id}-\Sigma\psi$ is a monomorphism. Here one uses that $\Sigma S \cong S$.

Finally, observe that the objects Y in $\mathcal{T}$ such that π_Y is an isomorphism form a localising subcategory of $\mathcal{T}$. We conclude that π is an isomorphism, since $\operatorname{Loc}(\mathcal{S}_0) = \mathcal{T}$. □

We collect several consequences of the Brown representability theorem. For instance, the following provides a useful reformulation of the definition of a perfectly generating set.

Corollary 3.4.8. *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts and let $\mathcal{S}_0$ be a set of objects satisfying* (PG). *Then* $\operatorname{Loc}(\mathcal{S}_0) = \mathcal{T}$ *if and only if* $\operatorname{Hom}_{\mathcal{T}}(\Sigma^n S, X) = 0$ *for all $S \in \mathcal{S}_0$ and $n \in \mathbb{Z}$ implies $X = 0$ for each $X \in \mathcal{T}$.*

Proof Suppose that $\operatorname{Loc}(\mathcal{S}_0) = \mathcal{T}$ holds. Let $X \in \mathcal{T}$ satisfy $\operatorname{Hom}_{\mathcal{T}}(\Sigma^n S, X) = 0$ for all $S \in \mathcal{S}_0$ and $n \in \mathbb{Z}$. The objects $U \in \mathcal{T}$ satisfying $\operatorname{Hom}_{\mathcal{T}}(\Sigma^n U, X) = 0$ for all $n \in \mathbb{Z}$ form a localising subcategory of $\mathcal{T}$ containing $\mathcal{S}_0$. Thus $X = 0$.

For the other implication fix an object $X \in \mathcal{T}$. The above proof yields for $F = \operatorname{Hom}_{\mathcal{T}}(-,X)$ an object $X' \in \operatorname{Loc}(\mathcal{S}_0)$ and a morphism $\pi\colon X' \to X$ which restricts to an isomorphism

$$\operatorname{Hom}_{\mathcal{T}}(-,X')|_{\mathcal{S}} \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{T}}(-,X)|_{\mathcal{S}}.$$

The condition on $\mathcal{S}_0$ implies $\operatorname{Cone}\pi = 0$. Thus $X' \cong X$, so $\operatorname{Loc}(\mathcal{S}_0) = \mathcal{T}$. □

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Corollary 3.4.9. *Let $\mathcal{S}$ be a perfectly generating set for $\mathcal{T}$. Then every object in $\mathcal{T}$ can be written as a homotopy colimit of a sequence*

$$X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \cdots$$

of morphisms in $\mathcal{T}$ such that $X_0 = 0$ and the cone of each ϕ_i is a coproduct of suspensions of objects in $\mathcal{S}$.

Proof Let $X \in \mathcal{T}$ and consider the functor $F = \operatorname{Hom}_{\mathcal{T}}(-, X)$. Then the construction of the representing object in the above proof yields X as a homotopy colimit of a sequence having the desired properties. □

Corollary 3.4.10. *A perfectly generated triangulated category has arbitrary products.*

Proof Given a family of objects X_i, the product $\prod_i X_i$ is the object representing the functor $\prod_i \operatorname{Hom}(-, X_i)$. □

Corollary 3.4.11. *Let $\mathcal{T}$ be a perfectly generated triangulated category. Then an exact functor $\mathcal{T} \to \mathcal{U}$ between triangulated categories preserves all coproducts if and only if it has a right adjoint.*

Proof Let $F : \mathcal{T} \to \mathcal{U}$ be an exact functor. If F preserves all coproducts, then one defines the right adjoint G by sending an object X in $\mathcal{U}$ to the object in $\mathcal{T}$ representing $\operatorname{Hom}_{\mathcal{U}}(F-, X)$. Thus

$$\operatorname{Hom}_{\mathcal{U}}(F-, X) \cong \operatorname{Hom}_{\mathcal{T}}(-, GX).$$

Conversely, given a right adjoint of F, it is automatic that F preserves all coproducts. □

Remark 3.4.12. There is the dual concept of a *perfectly cogenerating* set for a triangulated category. The dual Brown representability theorem for a perfectly cogenerated triangulated category $\mathcal{T}$ characterises the representable functors $\operatorname{Hom}_{\mathcal{T}}(X, -)$ as the cohomological and product preserving functors $\mathcal{T} \to \mathrm{Ab}$.

Compact Objects

Let $\mathcal{T}$ be a triangulated category. An object X in $\mathcal{T}$ is called *compact* (or *small*) if for any morphism $\phi : X \to \coprod_{i \in I} Y_i$ in $\mathcal{T}$ there is a finite set $J \subseteq I$ such that ϕ factors through $\coprod_{i \in J} Y_i$. It is easily checked that X is compact if and only if the canonical map

$$\coprod_{i \in I} \operatorname{Hom}_{\mathcal{T}}(X, Y_i) \longrightarrow \operatorname{Hom}_{\mathcal{T}}\Big(X, \coprod_{i \in I} Y_i\Big)$$

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is bijective for all coproducts $\coprod_{i \in I} Y_i$ in $\mathcal{T}$. It follows that the compact objects form a thick subcategory of $\mathcal{T}$.

We wish to describe all compact objects of a triangulated category. To this end we make the following definition.

For classes $\mathcal{U}$ and $\mathcal{V}$ of objects in a triangulated category $\mathcal{T}$ we denote by $\mathcal{U} * \mathcal{V}$ the class of objects $X \in \mathcal{T}$ that fit into an exact triangle $U \to X \to V \to \Sigma U$ such that $U \in \mathcal{U}$ and $V \in \mathcal{V}$. The octahedral axiom implies that the operation $*$ is associative. For a class $\mathcal{X}$ the objects of $\mathcal{X} * \mathcal{X} * \cdots * \mathcal{X}$ (n factors) are called *extensions of length n* of objects in $\mathcal{X}$.

Let $\mathcal{C} \subseteq \mathcal{T}$ be a class of objects and suppose that $\mathcal{C}$ is closed under all suspensions. We write $\coprod \mathcal{C}$ for the class of all coproducts of objects in $\mathcal{C}$.

Proposition 3.4.13. *Let $X \in \mathcal{T}$ be an object that is a direct summand of an extension of objects in $\coprod \mathcal{C}$. If X and all objects in $\mathcal{C}$ are compact, then X is a direct summand of an extension of objects in $\mathcal{C}$.*

Proof Let $X \to Y$ be a split monomorphism such that Y is an extension of objects in $\coprod \mathcal{C}$. Then the assertion follows from the lemma below by choosing $Y' = 0$. More precisely, complete the morphism $X' \to X$ in this lemma to an exact triangle $X' \to X \to X'' \to \Sigma X'$. The choice for Y' implies that the morphism $X \to Y$ factors through $X \to X''$. In particular, $X \to X''$ is a split monomorphism, so X is a direct summand of an extension of objects in $\mathcal{C}$. □

Lemma 3.4.14. *Let X and all objects in $\mathcal{C}$ be compact. Also, let $Y' \to Y$ be a morphism such that its cone is an extension of objects in $\coprod \mathcal{C}$. Then each morphism $X \to Y$ fits into a commutative square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

such that the cone of $X' \to X$ is an extension of objects in $\mathcal{C}$.

Proof Complete $\psi \colon Y' \to Y$ to an exact triangle $Y' \to Y \to Y'' \to \Sigma Y'$. We use induction on the length l of Y''. If $l = 1$, then $Y'' \in \coprod \mathcal{C}$ and the composite $X \to Y \to Y''$ factors through a summand X'' of Y'' that lies in $\mathcal{C}$ since X is compact. We complete $X \to X''$ to an exact triangle $X' \to X \to X'' \to \Sigma X'$ and $X' \to X$ factors through $Y' \to Y$ by construction. Now let $l > 1$ and write Y'' as an extension $Y''_0 \to Y'' \to Y''_1 \to \Sigma Y''_0$ of objects having smaller length than l. Using the octahedral axiom we obtain the following morphism of exact

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triangles

$$\begin{array}{ccccccc} Y' & \xrightarrow{\psi_0} & Y_0 & \longrightarrow & Y_0'' & \longrightarrow & \Sigma Y' \\ \| & & \downarrow{\scriptstyle \psi_1} & & \downarrow & & \| \\ Y' & \xrightarrow{\psi} & Y & \longrightarrow & Y'' & \longrightarrow & \Sigma Y' \end{array}$$

where ψ admits a factorisation $\psi = \psi_1\psi_0$ with $\operatorname{Cone}\psi_i = Y_i''$. By induction we have a pair of commutative squares

$$\begin{array}{ccccc} X' & \xrightarrow{\phi_0} & X_0 & \xrightarrow{\phi_1} & X \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \xrightarrow{\psi_0} & Y_0 & \xrightarrow{\psi_1} & Y \end{array}$$

such that the cone of each ϕ_i is an extension of objects in $\mathcal{C}$. Then the same holds for the cone of $\phi_1\phi_0$ by the octahedral axiom. □

Compact Generators

Let $\mathcal{T}$ be a triangulated category that admits arbitrary coproducts. A set $\mathcal{C}$ of compact objects is called *compactly generating* if $\mathcal{T}$ has no proper localising subcategory containing $\mathcal{C}$. In this case $\mathcal{T}$ is called *compactly generated*.

Proposition 3.4.15. *Let $\mathcal{T}$ be a compactly generated triangulated category and $\mathcal{C}$ a generating set of compact objects. Then $\mathcal{C}$ is a perfectly generating set for $\mathcal{T}$ and the full subcategory of compact objects equals* $\operatorname{Thick}(\mathcal{C})$.

Proof The first assertion follows easily from the fact that for any family of maps $\phi_i : A_i \to B_i$ between abelian groups we have

$$\prod_i \phi_i \text{ is an epimorphism} \iff \text{each } \phi_i \text{ is an epimorphism}$$
$$\iff \coprod_i \phi_i \text{ is an epimorphism.}$$

Clearly, the compact objects form a thick subcategory of $\mathcal{T}$. It follows from Corollary 3.4.9 that each object $X \in \mathcal{T}$ can be written as the homotopy colimit $\operatorname{hocolim} X_n$ of objects that are extensions of coproducts of suspension of objects in $\mathcal{C}$. If X is compact, then Lemma 3.4.3 implies that id_X factors through the canonical morphism $X_n \to X$ for some n. We conclude from Proposition 3.4.13 that X belongs to $\operatorname{Thick}(\mathcal{C})$. □

The following *Brown representability theorem* is an immediate consequence of Theorem 3.4.5. In fact, all corollaries of Theorem 3.4.5 apply to compactly

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generated triangulated categories as well. In particular, the definition of 'compactly generated' may be reformulated: a set $\mathcal{C}$ of compact objects generates if $\operatorname{Hom}(\Sigma^n C, X) = 0$ for all $C \in \mathcal{C}$ and $n \in \mathbb{Z}$ implies $X = 0$; see Corollary 3.4.8.

Theorem 3.4.16 (Brown). *Let $\mathcal{T}$ be a compactly generated triangulated category. Then a functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ is cohomological and sends all coproducts in $\mathcal{T}$ to products if and only if $F \cong \operatorname{Hom}_{\mathcal{T}}(-, X)$ for some object X in $\mathcal{T}$.* □

There is also a version of Brown representability for functors preserving products, keeping in mind that arbitrary products exist in a compactly generated triangulated category, by Corollary 3.4.10.

Theorem 3.4.17. *Let $\mathcal{T}$ be a compactly generated triangulated category. Then a functor $F\colon \mathcal{T} \to \mathrm{Ab}$ is cohomological and preserves all products in $\mathcal{T}$ if and only if $F \cong \operatorname{Hom}_{\mathcal{T}}(X, -)$ for some object X in $\mathcal{T}$.*

Proof Let $\mathcal{C}$ be a set of compact generators for $\mathcal{T}$. We claim that $\mathcal{T}^{\mathrm{op}}$ is also perfectly generated. Then the assertion follows from Theorem 3.4.5. For $C \in \mathcal{C}$ let C^* denote the object in $\mathcal{T}$ that represents $\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Hom}_{\mathcal{T}}(C, -), \mathbb{Q}/\mathbb{Z})$. Then it is straightforward to check that $\{C^* \mid C \in \mathcal{C}\}$ perfectly generates $\mathcal{T}^{\mathrm{op}}$, using the equivalent description from Corollary 3.4.8. □

We end our discussion of compact objects with a lemma that addresses the question when a right adjoint functor preserves coproducts.

Lemma 3.4.18. *Let $F\colon \mathcal{T} \to \mathcal{U}$ be an exact functor between triangulated categories that admit arbitrary coproducts, and suppose there exists a right adjoint G. If G preserves all coproducts, then F preserves compactness. The converse holds when $\mathcal{T}$ is compactly generated.*

Proof Fix objects $X \in \mathcal{T}$ and $\coprod_{i\in I} Y_i \in \mathcal{U}$, and suppose that X is compact. We consider the following commutative diagram.

$$\begin{array}{ccccc}
\coprod_i \operatorname{Hom}_{\mathcal{U}}(FX, Y_i) & & \xrightarrow{\quad\alpha\quad} & & \operatorname{Hom}_{\mathcal{U}}(FX, \coprod_i Y_i) \\
\downarrow \wr & & & & \downarrow \wr \\
\coprod_i \operatorname{Hom}_{\mathcal{T}}(X, GY_i) & \xrightarrow{\sim} & \operatorname{Hom}_{\mathcal{T}}(X, \coprod_i GY_i) & \xrightarrow{\beta} & \operatorname{Hom}_{\mathcal{T}}(X, G(\coprod_i Y_i))
\end{array}$$

Suppose that G preserves coproducts. Then β is an isomorphism, and therefore α is an isomorphism. Thus FX is compact. The converse requires that the compact objects of $\mathcal{T}$ are generating. □

An application of Brown representability provides a description of the localisation with respect to a localising subcategory generated by compact objects.

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Example 3.4.19. Let $\mathcal{T}$ be a triangulated category that admits arbitrary coproducts. Then a localising subcategory $\mathcal{S} \subseteq \mathcal{T}$ generated by a set of compact objects in $\mathcal{T}$ fits into a localisation sequence

$$\mathcal{S} \begin{smallmatrix}\rightarrowtail\\ \twoheadleftarrow\end{smallmatrix} \mathcal{T} \begin{smallmatrix}\twoheadrightarrow\\ \leftarrowtail\end{smallmatrix} \mathcal{T}/\mathcal{S}$$

because the inclusion $\mathcal{S} \to \mathcal{T}$ admits a right adjoint; see Corollary 3.4.11 and Proposition 3.2.8. In fact, the right adjoint $\mathcal{T} \to \mathcal{S}$ preserves all coproducts by Lemma 3.4.18. Applying Brown representability once more (assuming that $\mathcal{T}$ is perfectly generated) we obtain the following recollement.

$$\mathcal{S} \begin{smallmatrix}\rightarrowtail\\ \twoheadleftarrow\\ \rightarrowtail\end{smallmatrix} \mathcal{T} \begin{smallmatrix}\twoheadrightarrow\\ \leftarrowtail\\ \twoheadrightarrow\end{smallmatrix} \mathcal{T}/\mathcal{S}$$

Notes

Triangulated categories and derived categories were introduced simultaneously in 1963 by Verdier in his thesis, and most of the basic properties can be found in his work [199]. For a modern exposition we refer to Neeman's book [150]. A similar notion of a 'stable category' was defined by Puppe, but without the octahedral axiom [164]. There is no example known of a 'pre-triangulated category' (so all axioms except (Tr4) are required), which is not triangulated.

The study of Frobenius categories and their stable categories was initiated by Heller [108]; for Frobenius pairs see [181]. The terminology reflects the properties of modules for quasi-Frobenius and self-injective rings [40, 73].

In algebraic topology the Brown representability theorem for cohomology theories is due to Brown [42]. An analogue for compactly generated triangulated categories was established by Keller [121] and Neeman [148]. The method of describing the compact objects in such categories as the direct summands of extensions of compact generators goes back to Ravenel [167]. More general representability theorems for cohomological functors are due to Franke [74] and Neeman [150]; for the dual version see [149]. The formulation in terms of perfect generators, which is presented here, uses categories of finitely presented functors and is taken from [127].