# METANILPOTENT FITTING CLASSES

**Dedicated to the memory of Hanna Neumann**

R. A. BRYCE and JOHN COSSEY

(Received 27 June 1972)

Communicated by M. F. Newman

Hawkes showed in [10] that classes of metanilpotent groups which are both formations and Fitting classes are saturated and subgroup closed; more, he characterized all such classes as those local formations with a local definition consisting of saturated formations (of nilpotent groups). In [3] we showed that those "Fitting formations" which are subgroup closed are also saturated, without restriction on nilpotent length; indeed such classes are, roughly speaking, recursively definable as local formations using a local definition consisting of such classes. It is natural to ask how these hypotheses may be weakened yet still produce the same classes of groups. Already in [10] Hawkes showed that Fitting formations need be neither subgroup closed nor saturated; and in [3] we showed that a saturated Fitting formation need not be subgroup closed (though a Fitting formation of groups of nilpotent length three is saturated if and only if it is subgroup closed).

In this paper we study Fitting classes of metanilpotent groups closed under some additional closure operation. The work began with the feeling that a Fitting class of metanilpotent groups is "almost" a formation in the sense that imposing another closure condition makes it one. (Of course once such a Fitting class is a formation it is covered by Hawkes' result above and all is known about it.) This naive feeling is easily shown to be false — see (C) below — but is true quite often. The results we get are these:

(A) A Fitting class of metanilpotent groups is a formation if it is either subgroup closed or saturated.

1. We are indebted to the referee for a number of helpful comments; for the information that Lemma 1.1 is in F. P. Lockett's Ph.D. Thesis at the University of Warwick and for the proof given there and for reminding us that Theorem 3.1 can be derived from [10] with minor changes and the use of our Lemma 3.5.

(B) The smallest quotient-group closed Fitting class containing a supersoluble group is a formation.

(C) There exists a subdirect product closed Fitting class of metanilpotent groups that is not a formation.

(D) There exists a Fitting class of metanilpotent groups that is not subgroup, quotient group or subdirect product, closed, nor is it saturated.

These examples—(C) and (D)—will be given in section 2. In section 1 we give some notation and prove a useful lemma; sections 3 and 4 deal with subgroup closure and saturation respectively; and sections 5, 6 and 7 with quotient group closure.

## 1. Preliminaries

Recall that a closure operation $\mathrm{A}$ on the set of (isomorphism closed) classes of finite soluble groups is a map with the properties:

$$\mathrm{A}\mathfrak{X} \subseteq \mathrm{A}\mathfrak{Y} \quad \text{if } \mathfrak{X} \subseteq \mathfrak{Y}$$

and

$$\mathfrak{X} \subseteq \mathrm{A}\mathfrak{X} = \mathrm{A}^2\mathfrak{X}.$$

A class $\mathfrak{X}$ is called $\mathrm{A}$-closed if $\mathrm{A}\mathfrak{X} \subseteq \mathfrak{X}$.

Perhaps the commonest closure operations are these; $\mathrm{S}, \mathrm{S}_n, \mathrm{Q}, \mathrm{N}_0, \mathrm{R}_0, \mathrm{E}_\phi$ where

$$\mathrm{S}\mathfrak{X} = \{H : H \leqq G \text{ for some } G \in \mathfrak{X}\};$$

$$\mathrm{S}_n\mathfrak{X} = \{H : H \text{ is a subnormal subgroup of some } G \in \mathfrak{X}\};$$

$$\mathrm{Q}\mathfrak{X} = \{G/N : G \in \mathfrak{X}, N \trianglelefteq G\};$$

$$\mathrm{N}_0\mathfrak{X} = \{G : G = N_1N_2\cdots N_r, N_i \text{ subnormal in } G, N_i \in \mathfrak{X}, \quad 1 \leqq i \leqq r\}$$

$$\mathrm{R}_0\mathfrak{X} = \{G : \exists N_i \trianglelefteq G\,(1 \leqq i \leqq r), \bigcap_{i=1}^{r} N_i = 1, G/N_i \in \mathfrak{X}\,(1 \leqq i \leqq r)\}$$

$$\mathrm{E}_\phi\mathfrak{X} = \{G : \exists N \trianglelefteq G, N \leqq \Phi(G), G/N \in \mathfrak{X}\}.$$

Of course if $\mathrm{A}$ is one of $\mathrm{S}, \mathrm{S}_n, \mathrm{Q}, \mathrm{N}_0, \mathrm{R}_0$ an $\mathrm{A}$-closed class is often termed subgroup closed, subnormal subgroup closed, quotient group closed, normal product closed and subdirect product closed respectively. An $\mathrm{E}_\phi$- closed class is termed saturated.

If $\mathrm{A}, \mathrm{B}$ are closure operations we define a closure operation $\{\mathrm{A}, \mathrm{B}\}$ by : for all classes $\mathfrak{X}$

$$\{\mathrm{A}, \mathrm{B}\}\mathfrak{X} = \cap\{\mathfrak{Y} : \mathfrak{X} \subseteq \mathfrak{Y}, \mathfrak{Y} \text{ is a } \mathrm{A}\text{-closed and } \mathrm{B}\text{-closed}\}.$$

In these terms a formation is a $\{\mathrm{Q}, \mathrm{R}_0\}$-closed class; a saturated formation is a $\{\mathrm{Q}, \mathrm{R}_0, \mathrm{E}_\phi\}$-closed class; a Fitting class is a $\{\mathrm{S}_n, \mathrm{N}_0\}$-closed class. A Fitting formation is a class which is both a formation and a Fitting class.

Denote by $\mathfrak{S}_p$ the class of all finite $p$-groups, and by $\mathfrak{N}$ the class of all finite nilpotent groups.

There are several notations that will be used frequently. Suppose that $A$ is a group and $B$ a group of operators on it. We denote by $AB$ the natural splitting extension of $A$ by $B$ using the implied homomorphism of $B$ as abstract group into aut$A$, the group of automorphisms of $A$. Thus also if $A$ is a $B$-module we write $AB$ for the natural split extension. In this context it should be noted that an abelian normal subgroup of a group may frequently be regarded as a module for the group and written additively if it seems to suit the occasion.

If $p$ is a prime, $\alpha$ a positive integer and $n$ is an integer prime to $p$ we denote by $C(p^\alpha, n)$ the (unique) group which is a splitting extension of a homocyclic group $P$ of exponent $p^\alpha$ by a cycle $C$ of order $n$ with $C$ acting faithfully and irreducibly on $P/\Phi(P)$.

Throughout all groups are finite and soluble.

There is one simple lemma that embodies a much used technique when dealing with Fitting classes which we state and prove here. If will be quoted often in the sequel.

LEMMA 1.1 *Let $G$ be a group with normal subgroups $N_1, N_2$ such that*

$$N_1 \cap N_2 = 1 \text{ and } G/N_1N_2 \text{ is nilpotent.}$$

*If $\mathfrak{F}$ is a Fitting class containing $G/N_2$ then $G \in \mathfrak{F}$ if and only if $G/N_1 \in \mathfrak{F}$.*

PROOF. Put $A = G/N_1$, $B = G/N_2$ and $N = N_1N_2$. Then $g \to (gN_1, gN_2)$ embeds $G$ in $A \times B$: write $G_0$ for its image. Now $G_0$ contains $N/N_1 \times N/N_2$ so $G_0$ is subnormal in $A \times B$ since $G/N$ is nilpotent. If $G/N_i \in \mathfrak{F} (i = 1, 2)$ then $G_0 \in \mathrm{S}_n\mathrm{N}_0\mathfrak{F}$ so $G_0 \in \mathfrak{F}$. On the other hand since $(gN_1, N_2) = (gN_1, gN_2)(N_1, gN_2)^{-1}$ for all $g \in G$ we have $A \times B = G_0B \in \mathrm{N}_0\mathfrak{F}$ if $G, B \in \mathfrak{F}$; and $A \times B \in \mathfrak{F}$ means $A \in \mathfrak{F}$ as required.

One other fact, not unrelated to (1.1), is the following useful result which we shall use often: roughly that a Fitting class contains all relevant nilpotent groups (see Remark 1 on p. 204 in [9]).

LEMMA 1.2 *Let $\mathfrak{F}$ be a Fitting class, $G$ a group in $\mathfrak{F}$ and $p$ a prime dividing the order of $G$. Then $\mathfrak{S}_p \subseteq \mathfrak{F}$.*

## 2. Examples

These examples are easily obtained. It seems to be well known for example, that:

(2.1) *The class of those groups whose socle is central is a Fitting class.*

No reference for this fact is known to us: we first heard of it in conversation with Dr. Brian Hartley. In any case it is easy to prove and no proof is given here. Let

$G$ be the class described in (2.1) and note that $G$ is $\mathrm{R}_0$-closed. Hence $\mathfrak{G} \cap \mathfrak{N}^2$ is an $\mathrm{R}_0$-closed Fitting class of metanilpotent groups. However $\mathfrak{G} \cap \mathfrak{N}^2$ is not a formation, Q-closure fails: $SL(2,3)$ is in $\mathfrak{G} \cap \mathfrak{N}^2$, for example, but $PSL(2,3)$ is not. This takes care of (C).

In a recent paper Gaschütz and Blessenohl [6] have described some unusual Fitting classes as follows. Let $p$ be a prime and $S$ a subgroup of the multiplicative group of $GF(p)$, the field with $p$ elements. Let $G$ be a group and $g$ an element of it. In the usual way consider $G$ to be represented on its $p$-chief factors by conjugation, so that, to each $p$-chief factor $H/K$ in a given chief series of $G$, we have associated a matrix $M(g; H/K)$ unique up to similarity. Define the map $w_G : G \to GF(p)$ by

$$w_G : g \to \prod_{H/K} \det M(g; H/K).$$

Now let $\mathfrak{G}_p(S)$ denote the class of all groups $G$ such that $w_G(g) \in S$ for all $g \in G$. Then

THEOREM 2.2 (Gaschütz-Blessenohl). *$\mathfrak{G}_p(S)$ is a Fitting class.*

The case we want is $p = 3$ and $S = 1$. Then, of course, $\mathfrak{F}_0 = \mathfrak{G}_3(1) \cap \mathfrak{S}_3\mathfrak{S}_2$ is a Fitting class.

THEOREM 2.3 *$\mathfrak{F}_0$ is closed under none of* $\mathrm{S}, \mathrm{Q}, \mathrm{R}_0, \mathrm{E}_\Phi$.

PROOF. The fact we use here is that a group $G$ with $|G/0_3(G)| = 2$ is in $\mathfrak{F}_0$ if and only if it has in a chief series an even number of non-central 3-chief factors —this from Theorem 2.2.

Consider the group $G_1$ which has $O_3(G_1) \cong C_3 \times C_3, |G_1/O_3(G_1)| = 2$ and $Z(G_1) = 1$. Then $G_1 \in \mathfrak{F}_0$ but has a subgroup and a factor group isomorphic to $S_3$, which is not in $\mathfrak{F}_0$. Let $G_2$ be such that $O_3(G_2) \cong C_3 \times C_3 \times C_3, |G_2/O_3(G_2)| = 2$ and $Z(G_2) = 1$. Then $G_2$ is a subdirect product of two copies of $G_1$, but $G_2 \notin \mathfrak{F}_0$. Finally consider $G_3$ where $O_3(G_3) \cong C_9 \times C_3, |G_3/O_3(G_3)| = Z(G_3) = 1$. Then $G_3/\Phi(G_3) \cong G_1 \in \mathfrak{F}_0$ but $G_3 \notin \mathfrak{F}_0$.

## 3. Subgroup closure

THEOREM 3.1 *A subgroup closed Fitting class of metanilpotent groups is a formation.*

Before embarking on the proof of this theorem it is perhaps worth noting that Makan [13] has shown that a Fischer class of metanilpotent groups is subgroup closed. (Recall that a Fitting class $\mathfrak{F}$ is a Fischer class if, whenever $G \in \mathfrak{F}$ and $H$ is a subgroup of $G$ such that $H/\mathrm{core}_G H$ is nilpotent, then $H \in \mathfrak{F}$.) Hence we have an obvious corollary to Theorem 3.1.

COROLLARY 3.2 *A Fischer class of metanilpotent groups is a formation.*

https://doi.org/10.1017/S1446788700017067 Published online by Cambridge University Press

In the proof of Theorem 3.1 we use the next two lemmas, variations on some results in [2].

LEMMA 3.3 *Let $\mathfrak{F}$ be a subgroup closed Fitting class, $H$ a group in $\mathfrak{F}$ and $p$ a prime. Then the class of $GF(p)H$-modules*

$$\mathscr{M} = \{M : MH \in \mathfrak{F}\}$$

*is closed in the following sense:*

(i) *every submodule of a module in $\mathscr{M}$ is in $\mathscr{M}$,*

(ii) *the tensor product of modules in $\mathscr{M}$ is in $\mathscr{M}$,*

*and* (iii) *if $U$ is a $GF(p)H$-module with a submodule $V$ such that $V \in \mathscr{M}$ and $U/V$ is a trivial $GF(p)H$-module, then $U \in \mathscr{M}$.*

PROOF. The details are in the proof of Theorem 1 in section 4 of [2], and are not repeated here.

LEMMA 3.4 *Let $\mathscr{M}$ be a class of $GF(p)H$-modules for some group $H$, with the closure properties* (i), (ii), (iii) *of* (3.3). *Then, if $\mathscr{M}$ contains a module whose kernel in $H$ is precisely $O_p(H)$, it contains all $GF(p)H$-modules.*

PROOF. First note that $\mathscr{M}$ is direct-sum closed: if $M, N \in \mathscr{M}$ and $T$ is the one-dimensional trivial $GF(p)H$-module, $M \oplus T$ and $N \oplus T \in \mathscr{M}$ by (iii), then

$$(M \oplus T) \otimes (N \oplus T) \cong M \otimes N \oplus M \oplus N \oplus T$$

is in $\mathscr{M}$ by (ii) whence $M \oplus N \in \mathscr{M}$ by (i). Now use the proof of Lemma 2.1 in [2] except that where, in the proof of (2.2), the fact that every module is a homomorphic image of a free module is used, use that every module is a submodule of a free module.

One final result will be needed which tells us that S-closed Fitting classes have a measure of Q-closure.

LEMMA 3.5 *If $\mathfrak{F}$ is an* S*-closed Fittings class, $G$ a group in $\mathfrak{F}$ and $N$ a central subgroup of $G$, then $G/N \in \mathfrak{F}$.*

PROOF. Consider the diagonal subgroup $G_0 = \{(g, g) : g \in G\}$ in $G \times G$ and the (central) subgroup $N_0 = \{(n, n) : n \in N\}$. Plainly

$$G/N \cong G_0/N_0 \in \mathrm{S}\{G \times G)/N_0\} \subseteq \mathrm{SN}_0\{G\} \subseteq \mathfrak{F}.$$

PROOF OF 3.1. It is sufficient to show that an $\{\mathrm{S}, \mathrm{N}_0\}$-closed class of metanilpotent groups is Q-closed. Suppose to the contrary that there exists an $\{\mathrm{S}, \mathrm{N}_0\}$-closed class $\mathfrak{F}$ which is not Q-closed. Choose $G \in \mathfrak{F}$ of least order with the property that some quotient group of $G$ is not in $\mathfrak{F}$: let $N$ be normal in $G$ and of greatest order with $G/N \notin \mathfrak{F}$. Write $H = G/N$.

First note that $H$ is not nilpotent as $\mathfrak{F}$, being a Fitting class, contains all relevant nilpotent groups (Lemma 1.2). Also $H$ is monolithic since, if not, $G$ has normal subgroups $N_1, N_2$ properly containing $N$ so that

$$N = N_1 \cap N_2.$$

But since $N$ is maximally chosen, $G/N_1$ and $G/N_2$ both belong to $\mathfrak{F}$ whence $H$, being isomorphic to a subgroup of $G/N_1 \times G/N_2$, is in $\mathfrak{F}$, a contradiction.

Next we see that $H$ is co-monolithic (i.e. it has a unique maximal normal subgroup). For, if not, $H = (N_1/N)(N_2/N)$ where $N_1$ and $N_2$ are proper normal subgroups of $G$. However $N_1, N_2 \in \mathfrak{F}$ and so, by the minimality of $G, N_1/N, N_2/N \in \mathfrak{F}$, whence $H \in \mathfrak{F}$, again a contradiction.

The Frattini subgroup $\Phi(G)$ contains $N$, since if $G$ has a maximal subgroup $U$ not containing $N$ then

$$G/N = UN/N \cong U/U \cap N \in \mathfrak{F},$$

$U$ being in $\mathfrak{F}$ and $G$ being minimal, and this is a contradiction. In fact

$$N = \Phi(G). \tag{3.6}$$

For, if not, $N < \Phi(G), \Phi(H) = \Phi(G)/N$ and $H/\Phi(H) \in \mathfrak{F}$. Since $H$ is monolithic $F(H)$ is a $p$-group for some prime $p$, and $M_0 = F(H)/\Phi(H)$ as $GF(p)H$-module has kernel precisely $O_p(H)$. Also if $\sigma H$ is the monolith of $H$, then $M_0(H/\sigma H) \in \mathfrak{F}$ by Lemma 1.1, so it follows from Lemmas 3.3 and 3.4 that

$$\sigma H wr\, H/\sigma H \in F.$$

The Krasner-Kaloujnine embedding (22.21 in [14]) then yields $H \in \mathrm{s}\mathfrak{F} = \mathfrak{F}$, so we must conclude that $N = \Phi(G)$ and (3.6) is proved.

Since $H$ is co-monolithic, $F(H)$ is supplemented in $H$ by a cyclic subgroup of order a power of $q$, a prime different from $p$. Since $F(H) = F(G)/N, F(G)$ has a supplement $C$ whose order is a power of $q$. Then, writing $P$ for the Sylow $p$-subgroup of $F(G)$, $PCN = G$ and (3.6) implies that

$$G = PC.$$

The facts that $\Phi(H) = 1$ and $H$ is monolithic mean that $F(H)$ is the unique minimal normal subgroup of $H$, and it is not central in $H$. Hence a central $p$-subgroup $A$ of $G$ must be in $N$; but then by Lemma 3.5, $G/A \in \mathfrak{F}$ and $(G/A)/(N/A) \notin \mathfrak{F}$, contradicting the minimality of $G$, unless $A = 1$. It follows that $G$ has a non-central minimal normal $p$-subgroup, $M$ say. Then $MC \in \mathrm{s}\mathfrak{F} = \mathfrak{F}$, and $C$ does not centralize $M$. By Lemma 3.5, $MC/C_C(M) \in \mathfrak{F}$, say

$$MD \in \mathfrak{F}$$

where $1 \neq D = C/C_C(M)$ and $M$ faithfully and irreducibly represents $D$. If $D_0$ is the subgroup of order $q$ in $D$ then $MD_0 \in \mathfrak{F}$; and, as $D_0$-module, $M = M_0 \oplus M_1$ where $M_0$ faithfully and irreducibly represents $D_0$. Hence since $M_0 D_0 \in \mathfrak{F}$ and $M_0 D_0 \cong C(p,q)$ we conclude

$$C(p,q) \in \mathfrak{F}. \tag{3.7}$$

We now show

(3.8) *$\mathfrak{F}$ contains all extensions of an elementary abelian p-group by a q-group.*

For, suppose that $UT$ is such a group, $U$ being an elementary abelian $p$-group, normalized by the $q$-group $T$. If $|T| = q$ then, using Maschke's Theorem, $UT = G_1 \times G_2$ where $G_1$ is a subdirect product of copies of $C(p,q)$ and hence is in $\mathfrak{F}$ by (3.7), and $G_2$ is an elementary abelian $p$-group: in any event $UT \in \mathfrak{F}$. If $|T| > q$ but the exponent of $T$ is $q$ we write $T$ as a normal product $T_1 T_2$, where $T_1, T_2$ have orders smaller than $|T|$. By induction, therefore, $UT_1, UT_2 \in \mathfrak{F}$ hence

$$UT \in \mathrm{N}_0\{UT_1, UT_2\} \subseteq \mathfrak{F}.$$

A similar induction shows that $UT \in \mathfrak{F}$ when the exponent of $T$ is $q^\alpha$ provided it is true whenever $T$ is cyclic of order $q^\alpha$; and this much will follow if $C(p,q^\alpha) \in \mathfrak{F}$. To prove this let $W = C_q wr C_{q_{\alpha-1}}$ and let $U_0$ be a faithful irreducible module for it over $GF(p)$. Now $W$ is a normal product of groups of exponent $q^{\alpha-1}$ so $U_0 W \in \mathfrak{F}$ by induction on $\alpha$. But $W$ has a subgroup $W_0 \cong C_{q_\alpha}$ so $U_0 W_0 \in \mathfrak{F}$; and as $W_0$-module $U_0 = U_1 \oplus U_2$ where $U_1$ is faithful and irreducible for $W_0$. Hence $U_1 W_0 \in \mathfrak{F}$; or, to put it another way, $C(p,q^\alpha) \in \mathfrak{F}$ as required.

However (3.8) gives $H \in \mathfrak{F}$ which is a contradiction to the existence of an s-closed Fitting class that is not Q-closed. The proof of Theorem 3.1 is therefore complete.

## 4. Saturation

THEOREM 4.1 *A saturated Fitting class of metanilpotent groups is a formation.*

The following lemma will be crucial.

LEMMA 4.2 *Let $H = AB$ be a splitting extension of the nilpotent group $A$ by the nilpotent group $B$; and suppose that $A_0$ is a normal subgroup of $H$ in $A$ with $A/A_0$ an elementary abelian q-group. Then, if $\mathfrak{F}$ is an $\mathrm{E}_\phi$-closed Fitting class containing $H$, $H/A_0 \in \mathfrak{F}$.*

PROOF. Consider the second nilpotent product of $S$ with a cyclic group $C$ of order $p$:

$$X = A *_{N_2} C.$$

Define the homomorphism $\phi : B \to \operatorname{aut} X$ by

$$a(b\phi) = a^b, \ a \in A, \ b \in B,$$

$$c(b\phi) = c \ , \ c \in C, \ b \in B.$$

Form the splitting extension $XB$ according to $\phi$ and observe that $[A_0, C]$ is normal in it. Put $X_0 = X/[A_0, C]$ and consider $X_0B$. It is clear that as $B$-modules

$$[A, C]/[A_0, C] \cong A/A_0.$$

Also $XB/[A, C] \cong AB \times C \in \mathfrak{F}$; and since $[A_0, C] \leqq [A, C] \leqq \Phi(X) \leqq \Phi(XB)$ we have

$$X_0B \in \mathrm{E}_\phi\mathfrak{F} = \mathfrak{F}.$$

However $(A \times A/A_0)B$ is isomorphic to a subnormal subgroup of $X_0B$ and therefore $(A \times A/A_0)B \in \mathfrak{F}$. However $AB \in \mathfrak{F}$ so Lemma 1.1 yields $(A/A_0)B \in \mathfrak{F}$. In other words $H/A_0 \in \mathfrak{F}$ as required.

Proof of 4.1. Suppose that, on the contrary, there is an $\mathrm{E}_\phi$-closed Fitting class $\mathfrak{F}$ of metanilpotent groups which is not a formation. By Theorem 3.1 therefore, $\mathfrak{F}$ is not s-closed. Choose $G \in \mathfrak{F}$ minimal with respect to having a subgroup, $H$ say, not in $\mathfrak{F}$: let $H$ be chosen minimally among such subgroups of $G$. Note that $H$ is not nilpotent by Lemma 1.2. Also $H$ is co-monolithic since otherwise $H = N_1N_2$ for proper normal subgroups $N_1, N_2$ of $H$; but $N_1, N_2 \in \mathfrak{F}$, $H$ being minimal, and therefore $H \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$, a contradiction. It follows that $H/H \cap F(G)$ is a $q$-power cycle for some prime $q$, and hence that

$$H = (H \cap F(G))C \tag{4.3}$$

where $C$ is cyclic of $q$-qower order and $C \neq 1$.

Note that $G = F(G)H$, since otherwise the minimality of $G$ and the fact that $F(G)H \in \mathrm{s}_n\mathfrak{F} = \mathfrak{F}$ would mean $H \in \mathfrak{F}$; and (4.3) then yields

$$G = F(G)C.$$

In this set up we prove

(4.4) *$F(G)$ is a $p$-group for some prime $p \neq q$.*

For, suppose that $U \neq 1$ is the Sylow $q$-subgroup of $F(G)$, and write $F(G) = U \times V$ where $V$ is a $q'$-group. By Lemma 1.1 $G/U \in \mathfrak{F}$ and then the minimality of $G$ implies that $HU/U \in \mathfrak{F}$; that is $H/H \cap U \in \mathfrak{F}$. However $H \cap F(G) = (H \cap U) \times (H \cap V)$ and Lemma 1.2 again yields $H \in \mathfrak{F}$, a contradiction. We conclude that $U = 1$. Now suppose that $P$ is the Sylow $p$-subgroup of $F(G)$ for some prime $p$ other than $q$ (one exists since $G$ is not nilpotent): write $F(G) = P \times R$. Then, by Lemma 4.2, $G/\Phi(P) \times R \in \mathfrak{F}$, or, in other words,

$$(P/\Phi(P))C \in \mathfrak{F}. \tag{4.5}$$

https://doi.org/10.1017/S1446788700017067 Published online by Cambridge University Press

Since $\Phi(P) \leqq \Phi(PC)$ we see that $PC \in \mathrm{E}_{\Phi}\mathfrak{F} = \mathfrak{F}$. If $R \neq 1$ then $PC$ has smaller order than that of $G$, so, by minimality of $G$, $H \cap PC = (H \cap P) \in \mathfrak{F}$. But Lemma 1.2 gives from this that $H \in \mathfrak{F}$. It follows that $R = 1$ and (4.4) is proved.

Next we prove

(4.6) *Every extension of an elementary abelian p-group by C is in* $\mathfrak{F}$.

Put $M = F(G)/\Phi(G)$. Then (4.5) tells us that $MC \in \mathfrak{F}$. Note that $M$ faithfully represents $C$. Let $C_1$ be a subgroup of $C$, and observe that, as $C_1$-module,

$$M = M_1 \oplus M_2$$

where $M_1$ is faithful and irreducible for $C_1$. By Lemma 4.2, $M_1C_1 \in \mathfrak{F}$. If $\alpha : C \to C_1$ is the natural homomorphism of $C$ onto $C_1$ we may use $\alpha$ to form the split extension $M_1C$. But

$$\ker \alpha \leqq \Phi(M_1C) \text{ and } M_1C/\ker \alpha \cong M_1C_1 \in \mathfrak{F}$$

so $M_1C \in \mathfrak{F}$. Note that every irreducible $GF(p)C$-module can be obtained up to linear isomorphism as $M_1$ for suitable choice of $C_1$. Hence if $W$ is an arbitrary $GF(p)C$-module and

$$0 = W_0 < W_1 < \cdots < W_n = W$$

is a composition chain of it, $W_1C \in \mathfrak{F}$ (just proved);

$$W_iC \in \mathfrak{F} \Rightarrow W_{i+1}C \in \mathfrak{F}, \quad 0 \leqq i < n$$

(using Lemma 1.1 since $(W_{i+1}/W_i)V \in \mathfrak{F}$); and so, by induction, $WC \in \mathfrak{F}$. Hence (4.6) is proved.

Finally (4.6) implies $H/\Phi(H \cap F(G)) \in \mathfrak{F}$ and so since $\Phi(H \cap F(G) \leqq \Phi(H)$, $H \in \mathrm{E}_{\Phi}\mathfrak{F} = \mathfrak{F}$. This final contradiction completes the proof of Theorem 3.1.

## 5. A module lemma

This section and the next two deal with Q-closed Fitting classes, though we cannot give as complete an answer here as we have done for S-and $\mathrm{E}_{\Phi}$-closure: at two crucial points in the proof we have been unable to eliminate the assumption that the groups under discussion are supersoluble. However the result of this section is a quite general one about Q-closed Fitting classes of metanilpotent groups. The analogy with (3.3) and (3.4) is obvious: indeed the proof we give is a special case of our original proof of (3.4) since L. G. Kovács' proof of that no longer works.

LEMMA 5.1 *Let $H$ be a metanilpotent group in which $0_{p'}(H) = 1$ for some prime p. Then if $\mathfrak{F}$ is a* Q*-closed Fitting class containing $H$, the class of $GF(p)H$-modules*

$$\mathscr{M} = \{U : UH \in \mathfrak{F}\}$$

*contains all $GF(p)H$-modules.*

The proof will be by induction on the length of the lower Frattini series of a $FG(p)H$-module $U$. Note that, under the conditions imposed on $H/0_p(H)$ is a $p'$-group and therefore a $GF(p)H$-module is completely reducible if and only if its kernel contains $0_p(H)$. It follows that the terms of the lower Frattini series of a $GF(p)H$-module $U$ are the submodules $U(i) = [U, i0_p(H)]\,(i \geqq 0)$ where: $U(0) = U$, and $U(i)$ is the smallest submodule of $U(i-1)$ such that $U(i-1)/U(i)$ is a completely reducible $H$-module. Denote by $l(U) = l$ the smallest integer for which $U(l) = 0$ : $l(U)$ is the Frattini length of $U$.

Consider the case $l(U) = 1$. Write $U_0 = 0_p(H)/\Phi(H)$ as $GF(p)H$-module. It follows from Lemmas 1.1 and 1.2 that $U_0 \in \mathscr{M}$: note that $U_0$ is faithful for $H/0_p(H)$. It follows as in the proof of Theorem 1 in section 4 of [2] that all tensor powers $U_0^r$ of $U_0$ are also in $\mathscr{M}$ (since $s_n$-closure and not s-closure is all that is needed here). Hence using Steinberg's theorem (Satz 7.19 in Chapter VI of [12]) we conclude that $U$ is a direct summand of $U_0^r$ for some $r$, and so is in $\mathscr{M}$, $\mathscr{M}$ being quotient module closed.

Suppose therefore that $U$ is a $GF(p)H$-module with $l(U) > 1$. Since $U$ is finite dimensional we may write

$$U = \sum_{i=1}^{s} U_i$$

where each $U_i$ is co-monolithic. Let $P_i$ be the projective cover of $U_i$ $(1 \leqq i \leqq s)$ and write $N_i = U_i/\rho U_i$, the unique minimal factor module of $U_i$. Recall that there exists a homomorphism from $P_i$ onto $U_i$. It is well known (see for example Exercise 2 on p. 426 of Curtis and Reiner [5]) that, if $P$ is the projective cover of the one dimensional trivial $GF(p)H$-module, $P_i$ is a direct summand of $P \otimes N_i$. Write

$$N = \bigoplus_{i=1}^{s} N_i$$

so that $P \otimes N \cong \oplus_{i=1}^{s} P \otimes N_i$. It follows that $\oplus_{i=1}^{s} P_i$ is isomorphic to a direct summand of $P \otimes N$. But there is a homomorphism from $\oplus_{i=1}^{s} P_i$ onto $\Sigma_{U_i} = M$ so we have shown:

(5.2) *There exists a completely reducible module $N$ and an onto homomorphism*

$$\theta : P \otimes N \to U$$

*where $P$ is the projective cover of the one-dimensional trivial $GF(p)H$-module.*

Next we prove

(5.3) $$[P \otimes N, i0_p(H)] = [P, i0_p(H)] \otimes N, i \geqq 0.$$

The proof is by induction on $i$, the case $i = 0$ being trivial. We do the case $i = 1$ only, the general step being the same. Note that

$$P \otimes N/[P, 0_p(H)] \otimes N \cong (P/[P, 0_p(H)]) \otimes N$$

has $0_p(H)$ in its kernel and therefore

$$[P, 0_p(H)] \otimes N \geqq [P \otimes N, 0_p(H)].$$

Now observe that $[P, 0_p(H)]$ is spanned by elements of the form $u(1-x)$ $(u \in P, x \in 0_p(H))$ and hence $[P, 0_p(H)] \otimes N$ is spanned by elements of the form $u(1-x) \otimes n\, (u \in P, x \in 0_p(H), n \in N)$. But since $0_p(H) \leqq \ker N$,

$$\begin{aligned} u(1-x) \otimes n &= (u \otimes n)(1-x) \\ &\in [P \otimes N, 0_p(H)], \end{aligned}$$

and the other inclusion follows.

(5.4) $$[P \otimes N, i0_p(H)]\theta = [U, i0_p(H)],\ i \geqq 0.$$

Again the proof is by induction and we give the case $i = 1$ only. First

$$\begin{aligned} U/[P \otimes N, 0_p(H)]\theta &\cong ((P \otimes N)/\ker \theta)/([P \otimes N, 0_p(H)] + \ker \theta)/\ker \theta \\ &\cong P \otimes N/([P \otimes N, 0_p(H)] + \ker \theta) \end{aligned}$$

which has $0_p(H)$ in its kernel, so $[U, 0_p(H)] \leqq [P \otimes N, 0_p(H)]\theta$. Conversely

$$\begin{aligned} P \otimes N/[U, 0_p(H)]\theta^{-1} &\cong (P \otimes N/\ker \theta)/([U, 0_p(H)]\theta^{-1}/\ker \theta) \\ &\cong U/[U, 0_p(H)] \end{aligned}$$

which has $0_p(H)$ in its kernel. Hence $[P \otimes N, 0_p(H)] \leqq [U, 0_p(H)]\theta^{-1}$ which gives the other inclusion.

Put $L = P/[P, l(U)0_p(H)]$. Then, since (5.3) and (5.4) imply that

$$[P, l(U)0_p(H)] \otimes N \leqq \ker \theta,$$

there is a homomorphism from $L \otimes N$ onto $U$ induced by $\theta$. Note that

(5.5) $$l(L) = l(U),\ L/[L, 0_p(H)] \text{ is trivial and } l[L, 0_p(H)] = l(U) - 1.$$

Finally consider the outer tensor product $K = L \not\# N$ as a module for $H \times H/0_p(H)$. Now $K_H$ is a direct sum of copies of $L$ so

$$l[K_H, 0_p(H)] = l(K) - 1 = l(U) - 1$$

and therefore, by induction,

$$[K, 0_p(H)]H \in \mathfrak{F}.$$

But $K_H/[K_H, 0_p(H)]$ is a direct sum of trivial $H$-modules and so $KH$ is a normal product

$$KH = [K, 0_p(H)]H \cdot K \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}.$$

Also $K_{H/0_p(H)}$ is completely reducible and so, by the second paragraph of this proof and Lemma 1.1, $K(H)0_p(H) \in \mathfrak{F}$. But if we write

$$H_0 = \{(x, x0_p(H)) : x \in H\}$$

then $KH_0 \in \mathrm{S}_n\mathrm{N}_0\{KH, K(H/0_p(H))\} \subseteq \mathfrak{F}$. However $KH_0 \cong (L \otimes N)H$ and so $L \otimes N \in \mathscr{M}$. Since $U$ is a homomorphic image of $L \otimes N$ it is also in $\mathscr{M}$, as required. This concludes the induction and with it the proof of Lemma 5.1.

## 6. Lifting automorphisms

The results in this section will be used in the next in our attack on Q-closed Fitting classes. The somewhat elaborate set up which follows Lemma 6.1 below in particular is important there. A reference for results and terminology about varieties of groups is Neumann [13], and for projective groups in varieties Bryant [1].

A common situation is to have an onto homomorphism $\nu : A \to B$ and to want to infer a homomorphism from a subgroup of the automorphism group of $A$ onto a subgroup of the automorphism group of $B$ which "commutes" with $\nu$. For example see Theorem 4.2.2 in [4] and section 2.7 in Higman [11] (where the existence of the groups $C(p^\alpha, n)$ is proved or inferred by this means); section 3 in [3]; and [16].

LEMMA 6.1 *Let $\mathfrak{V}$ be a locally finite variety, $B$ a finite group in $\mathfrak{V}$ and $A$ the projective cover of $B$ in $\mathfrak{V}$: let $\nu : A \to B$ be the natural homomorphism. Then if $T$ is a subgroup of aut$B$ there exists a subgroup $S$ of aut$A$ and an onto homomorphism $\mu : S \to T$ such that*

$$\sigma\nu = \nu(\sigma\mu), \quad \sigma \in S.$$

PROOF. Recall from [1] that ker $\nu \leqq \Phi(A)$. Now for each $\tau \in T$ there exists, by the definition of projective group, an endomorphism $\sigma(\tau)$ of $A$ such that

$$\nu\tau = \sigma(\tau)\nu. \tag{6.2}$$

Since $A\sigma(\tau)$ ker $\nu = A$ it follows that $\sigma(\tau)$ is onto and therefore is an automorphism. Let $S$ be the subgroup of aut$A$ generated by $\{\sigma(\tau) : \tau \in T\}$. We claim that there is a homomorphism $\mu : S \to T$ such that

$$\sigma(\tau)\mu = \tau, \quad \tau \in T.$$

For, if

$$\sigma(\tau_1)^{\varepsilon_1} \cdots \sigma(\tau_r)^{\varepsilon_r} = 1$$

https://doi.org/10.1017/S1446788700017067 Published online by Cambridge University Press

is a relation among the $\sigma(\tau)$ we conclude from repeated applications of (6.2) that

$$\nu\tau_1^{\varepsilon_1}\cdots\tau_r^{\varepsilon_r} = \nu,$$

and hence that

$$\tau_1^{\varepsilon_1}\cdots\tau_r^{\varepsilon_r} = 1,$$

being onto. The existence of $\mu$ now follows from van Dyck's theorem.

Continuing the same notation we have

LEMMA 6.3 *If $B$ is a $p$-group then*

(a) *the kernel of $\mu$ is a $p$-group*

*and*

(b) *if $T$ is a $p'$-group then $S$ contains a $p'$-subgroup $S_0$ isomorphic to $T$, and $S_0\mu = T$.*

PROOF. That ker $\mu$ is a $p$-group follows from a well known result of P. Hall (Theorem 12.2.2 in M. Hall [7]); and (b) then follows from the Schur-Zassenhaus theorem (Theorem 15.2.2 in [7]).

We carry on this development, assuming still that $B$ is a $p$-group, and $T$ a $p'$-group. Suppose that, as $GF(p)T$-modules, we have the direct decomposition

$$B/\Phi(B) \cong A/\Phi(A) \cong \bigoplus_{i=1}^{r} A_i/\Phi(A),$$

so that each $A_i/\Phi(A)$ is a $GF(p)S_0$-modules as well. Now by P. Hall [8], $A$ is a relatively free group and we may choose a generating set $X$ for it with a partition

$$X = \bigcup_{i=1}^{r} X_i$$

where

$$X_i \subseteq A_i\backslash\Phi(A),\ 1 \leqq i \leqq r.$$

Note that $F_i$, the subgroup of $A$ generated by $X_i$, is a free group of rank $|X_i|$ in var $A$ and hence that

$$\Phi(F_i) = F_i \cap \Phi(A),\ 1 \leqq i \leqq r$$

by 12.63 in [14] since the Frattini subgroup of a finite $p$-group is the $\mathfrak{A}_p$-subgroup. Hence

$$F_i/\Phi(F_i) \cong A_i/\Phi(A),\ 1 \leqq i \leqq r.$$

Now $T$ acts on each $A_i/\Phi(A)$ as a group $T_i$ of operators. It follows from Lemma 6.3, since $F_i$ is the projective cover of $A_i/\Phi(A)$ in var $F_i$, that there exists a group of operators $S_i$ on $F_i$ and an isomorphism $\mu_i : S_i \cong T_i$ such that

$$\sigma\nu_i = \nu_i(\sigma\mu_i),\ \sigma \in S_i,\ 1 \leqq i \leqq r$$

where $\nu_i | F_i$ $(1 \leqq i \leqq r)$. Extend the action of each $S_i$ to all of $A$ by

$$x\sigma_i = x,\ x \in X \setminus X_i,\ \sigma_i \in S_i.$$

Since the elements of different $S_i, S_j$ commute in their action on $A$ we see that

(6.4) *there exists a homomorphism* $\xi : T^r \to \mathrm{aut}A$ (where $T^r$ is the $r$-fold direct power of $T$) *such that*

$$(\tau, \tau, \cdots, \tau)\xi\nu = \nu\tau,\ \tau \in T.$$

Finally in this development put $T_0$ for the diagonal subgroup

$$\{(\tau, \tau, \cdots, \tau) : \tau \in T\}$$

of $T^r$ and form the splitting extension $AT_0$ from $\xi$. Then (6.4) yields that

(6.5) $\nu$ *extends to a homomorphism of* $AT_0$ *onto* $BT$.

We find it convenient to finish this section with a result of a quite different nature. With each finite $p$-group $P$ associate the ordered pair $\xi(P) = (c, \alpha)$ where $c$ is the class of $P$ and $p_\alpha$ is the exponent of $\gamma_c(P)$, the last non-trivial term of the lower central series of $P$.

LEMMA 6.6 *Let $P$ be a finite non-abelian $p$-group and $M$ a subgroup of $P$ such that the index of $M \cap \Phi(P)$ in $M$ is at most $p$. Then in the lexicographic ordering of ordered pairs,*

$$\xi(M) < \xi(P).$$

PROOF. If $M$ has smaller class than does $P$ we are done, so suppose that $M$ and $P$ have the same class $c$ greater than 1. Now $M\Phi(P)$ is generated by $\Phi(P)$ and at most one element $x \in M \backslash \Phi(P)$. Every commutator of weight $c$ in elements of $M\Phi(P)$ can be written as a product of commutators of weight $c$ in the generating set consisting of $x$, commutators in $P$ and $p$th powers of elements of $P$. But $c$ is at least 2 so one of the entries in each such non-trivial commutator is not $x$, nor is it a commutator and so must be a $p$th power of an element of $P$. It follows that $\gamma_c(M\Phi(P))$ is generated by $p$th powers of elements of $\gamma_c(P)$ and hence has exponent strictly less than the exponent of $\gamma_c(P)$. The same is therefore true of $M$.

## 7. Q-closure

We prove here the result (B) of the Introduction and get it as a corollary of this theorem.

THEOREM 7.1. *Let $p,q$ be primes such that $q$ divides $p-1$. Then a* Q*-closed Fitting class containing $C(p,q)$ contains $\mathfrak{S}_p\mathfrak{S}_q$.*

COROLLARY 7.2. *The smallest* Q*-closed Fitting class containing a supersoluble group is a formation.*

https://doi.org/10.1017/S1446788700017067 Published online by Cambridge University Press

The single most difficult step in the proof of Theorem 7.1 has proved to be the next lemma. Accordingly its proof is deferred until later.

LEMMA 7.3. *A Q-closed Fitting class which contains $C(p^{\alpha-1}, q)$ for some $\alpha \geqq 1$ contains $C(p^{\alpha}, q)$ where $p$ is prime and $q \mid p-1$.*

Modulo this result the proof divides naturally into the two parts of an induction. We bend the usual varietal notation and write $\mathfrak{B}_n$ for the class of (finite) groups of exponent dividing $n$. Throughout $\mathfrak{F}$ denotes a Q-closed Fitting class.

(I) *If $p, q$ are primes such that $q$ divides $p-1$ and $C(p, q) \in \mathfrak{F}$ then $\mathfrak{S}_p\mathfrak{B}_q \subseteq \mathfrak{F}$.*

(II) *If $\mathfrak{S}_p\mathfrak{B}_{q^{\alpha-1}} \subseteq \mathfrak{F}$ then $\mathfrak{S}_p\mathfrak{B}_{q^{\alpha}} \subseteq \mathfrak{F}$, $\alpha > 1$.*

Clearly it suffices to prove (I) and (II) in order to prove Theorem 7.1.

PROOF OF (I). Consider first the case of a group $G = UT$ where $U$ is a normal $p$-subgroup of $G$ and $|T| = q$. We show by induction on $\xi(U)$ that $G \in \mathfrak{F}$. We begin the induction by considering the case when $U$ is abelian. In this case it follows from a result of Taunt [15] that $U$ is an unrefinable direct product of normal homocyclic subgroups of $G$, say

$$U = U_1 \times \cdots \times U_s.$$

Since $q$ divides $p-1$ each $U_i$ is cyclic and so $U_iT \cong C(p^{\beta}, q)$ for some positive integer $\beta$, or $U_iT \cong U_i \times T$. In any case, using Lemma 7.3, $U_iT \in \mathfrak{F}$ $(1 \leqq i \leqq s)$. Repeated application of Lemma 1.1 then gives $G \in \mathfrak{F}$.

Suppose therefore that the class of $U$ is at least 2. Let $V$ be the projective cover of $U$ in var $U$: note that then $\xi(V) = \xi(U)$ since $U, V$ generate the same variety. Since $q$ divides $p-1$, $U/\Phi(U)$ as $GF(p)T$-module is a direct sum of one dimensional submodules. It follows from (6.4) that a free generating set $\{x_1, x_2, \cdots, x_r\}$ for $V$ and groups of operators $S_1, \cdots, S_r$ of $V$, each of order $q$, may be chosen such that

$$V_i = \langle \Phi(V), x_i \rangle \text{ admits } S_i,\ 1 \leqq i \leqq r.$$

Now Lemma 6.6 ensures that $\xi(V_i) < \xi(U)$ and hence, by induction,

$$V_iS_i \in \mathfrak{F},\ i \leqq i \leqq r.$$

But by construction $VS_i$ is a normal product of $V_iS_i$ and $V$ each of which is in $\mathfrak{F}$, so $VS_i \in \mathfrak{F}$. Hence from (6.4) and (6.5) (using the notation there)

$$VT_0 \in \mathrm{S}_n\mathrm{N}_0\ \{VS_i : 1 \leqq i \leqq r\} \subseteq \mathfrak{F}.$$

https://doi.org/10.1017/S1446788700017067 Published online by Cambridge University Press

However (6.5) then yields that $G \in Q\mathfrak{F} = \mathfrak{F}$ and so the inductive step is complete and with it the proof that $UT \in \mathfrak{F}$ whenever $|T| = q$.

The proof of (I) is completed by noting that if $G = UT$ where $U$ is a normal $p$-subgroup of $G$ and $T$ has exponent $q$, we may assume $T$ is not cyclic; hence that $T$ is a normal product, $T_1T_2$ say, and hence that $G$ is a normal product $(UT_1)(UT_2)$ so, by induction on $|T|$, $G \in \mathfrak{F}$.

PROOF OF (II). For exactly the same reason as in the last paragraph it is sufficient to show that $G \in \mathfrak{F}$ whenever $G = UT$ with $U$ a normal $p$-subgroup and $T$ a cyclic group of order $q^\alpha$.

Let $W = C_q wr C_{q^{\alpha-1}}$ and identify $T$ with a cyclic subgroup of order $q^\alpha$ in $W$ Since the regular $GF(p)W$-module restricted to $T$ gives a direct sum of regular $GF(p)T$-modules, it follows that, for some direct power $M$ of the regular $GF(p)$-module of $W$.

$$U/\Phi(U) \text{ is a direct summand of } M_T.$$

Now write $A$ for the projective cover of $M$ (regarded as an elementary abelian $p$-group) in var $U$. By Lemma 6.3 there is an isomorphism of $W$ into aut$A$: form the splitting extension $AW$ and note that, since $W$ is a normal product of groups of exponent dividing $q^{\alpha-1}$, $AW \in \mathfrak{F}$. It follows that

$$AT \in S_n\mathfrak{F} \subseteq \mathfrak{F}.$$

But $A/\Phi(A)$ as $T$-module decomposes

$$A/\Phi(A) = A_1/\Phi(A) \oplus A_2/\Phi(A)$$

where $A_1/\Phi(A) \cong U/\Phi(U)$. If $T_0$ is defined as in section 6 then (6.5) demands that $AT_0$ and $AT$ being of the same order are in fact isomorphic. However the normal closure of $F_2$ in $A$ admits $T_0$ and intersects $F_1$ trivially, so

$$F_1T_0 \in Q\{AT_0\} \subseteq \mathfrak{F}.$$

Finally, $F_1$ is the projective cover of $U$ in var $U$ so another application of (6.5) yields that $UT \in Q\{F_1T_0\} \subseteq \mathfrak{F}$ as required.

PROOF OF LEMMA 7.3. The crux of the proof lies in the construction of a certain $p$-group. Let $A, B$ be cyclic of order $p^\alpha, p^{\alpha-1}$ respectively, generated by $a, b$ say. Then the wreath product $W = A$ wr $B$ has an automorphism $\sigma$ of order $q$ such that

$$a\sigma = a^s,\ b\sigma = b^s$$

where $s$ has order $q$ modulo $p^\alpha$: note that $s \not\equiv 1 (\bmod\ p^{\alpha-1})$; and $C = B\langle\sigma\rangle \cong C(p^{\alpha-1}, q)$. Let $M_1$ be the base group of $W$ regarded as $Z_{p^\alpha}C$-module.

Consider the submodule of $M_1$ defined by

$$M_2 = p[M_1, B].$$

https://doi.org/10.1017/S1446788700017067 Published online by Cambridge University Press

Modulo $M_2$ the submodules $p^{\alpha-1}M_1$ and $[M_1,(p^{\alpha-1}-1)B]$ are isomorphic; for, the first is simply $\langle p^{\alpha-1}a\rangle$ and the second is $\langle [a,(p^{\alpha-1}-1)b]\rangle$; $B$ is in the kernel of each; and

$$\begin{aligned}(p^{\alpha-1}a)\sigma &= s(p^{\alpha-1}a),\\ [a,(p^{\alpha-1}-1)b]\sigma &= s\cdot s^{p^{\alpha-1}-1}[a,(p^{\alpha-1}-1)b]\\ &= s[a,(p^{\alpha-1}-1)b]\end{aligned}$$

since $s\not\equiv 0 \pmod p$ and $s^{p-1}\equiv 1 \pmod p$. It follows that if

$$U = \langle p^{\alpha-1}a + [a,(p^{\alpha-1}-1)b]\rangle$$

then $U + M_2/M_2$ is a submodule of $M_1/M_2$, call it $M_3/M_2$ say. Put $M = M_1/M_3$; for convenience we use a instead of its coset $a + M_3$ in $M$.

(7.5) *Every element of* $MB\backslash MB^p$ *has order exactly* $p^{\alpha-1}$.

For, if $x\in MB\backslash MB^p$, $x = b^t m$ for some $m\in M$ and $t$ such that $p\nmid t$. Now

$$\begin{aligned}x^{p^{\alpha-1}} &= (b^t)^{p^{\alpha-1}} m^{p^{\alpha-1}}[m,(p^{\alpha-1}-1)b^t]\\ &= m^{p^{\alpha-1}}[m,(p^{\alpha-1}-1)b]\\ &= m^{p^{\alpha-1}}[m,(p^{\alpha-1}-1)b]\end{aligned}$$

since $p\nmid t$. But $m = a^u d$ for some integer $u$ and some $d\in[M,B]$, and so

$$\begin{aligned}x^{p^{\alpha-1}} &= (a^{p^{\alpha-1}})^u[a,(p^{\alpha-1}-1)b]^u\\ &= 1.\end{aligned}$$

Plainly $x$ can have no smaller order.

(7.6) *There exists* $b_0\in MB\backslash MB^p$ *with the properties*

(i) $$b_0\sigma = b_0^s$$

(ii) $$\langle b_0, b\rangle = MB.$$

In fact we prove by induction on $i\in\{1,\cdots,p^\alpha\}$ that if $N_i = [M,iB]$ there exists $y_i\in MB\backslash MB^p$ and satisfying (i) and (ii) but *modulo* $N_i$.

If $i = 1$, $y_1 = ba$ will do. For $1\leqq i\leqq p^\alpha$ consider $\langle y_i, N_i\rangle$ modulo $N_{i+1}$: it is an abelian $p$-group of type $(\alpha-1,1)$ on which $\sigma$ acts. It follows from Taunt [15] again that there exists $y_{i+1}\in\langle y_i,N_i\rangle$ of order $p^{\alpha-1}$ with

$$y_{i+1}\sigma = y_{i+1}^s$$

modulo $N_{i+1}$ and with $\langle y_i,N_i\rangle = \langle y_{i+1},N_i\rangle$ modulo $N_{i+1}$. This means in particular that $y_{i+1}\notin MB^p$; and also that

$$\langle y_{i+1}, b, N_i\rangle N_{i+1} = \langle y_i, b, N_i\rangle N_{i+1}$$

whence $\langle y_{i+1}, b\rangle N_i = \langle y_i, b\rangle N_i = MB$ by induction. Since $N_i \leqq \Phi(MB)$, $\langle y_{i+1}, b\rangle = MB$. The induction is therefore complete, and $b_0 = y_{p^\alpha}$ is the element we want to satisfy (7.6).

(7.7) $$MC \in \mathfrak{F}.$$

But $B_0 = \langle b_0\rangle$, and let $\mathfrak{B}$ be the variety generated by $MB$. Write

$$R = B_0 *_{\mathfrak{B}} B$$

the $\mathfrak{B}$-product of $B_0$ and $B$ (see 18.31 in [14]). We need the facts that $B_0 \cap [B_0, B] = B \cap [B_0, B] = 1$ and that $[B_0, B]$ is an elementary abelian $p$-group.

Define automorphisms $\sigma_0, \sigma_1$ of R by:

$$b_0\sigma_0 = b_0^s, \qquad b\sigma_0 = b,$$

$$b_0\sigma_1 = b_0, \qquad b\sigma_1 = b^s.$$

Then $B_0\langle\sigma_0\rangle \cong C(p^{\alpha-1}, q) \in \mathfrak{F}$ and hence, by Lemma 5.1,

$$[B_0, B]B_0\langle\sigma_0\rangle \in F.$$

Similarly $[B_0, B]B\langle\sigma_1\rangle \in \mathfrak{F}$. But $R\langle\sigma_0\rangle$ is a normal product of $[B_0, B]B\langle\sigma_0\rangle$ and $R$, both of which are in $\mathfrak{F}$, so $R\langle\sigma_0\rangle \in \mathfrak{F}$, and of course $R\langle\sigma_1\rangle \in \mathfrak{F}$ similarly. If we put $\sigma' = \sigma_0\sigma_1$, therefore, we conclude that

$$R\langle\sigma'\rangle \in \mathrm{S}_n\mathrm{N}_0\{R\langle\sigma_0\rangle, R\langle\sigma_1\rangle\} \subseteq \mathfrak{F}.$$

Let $\nu : R \to MB$ the natural homomorphism given by

$$b_0\nu = b_0, \qquad b\nu = b.$$

It is a simple matter to check that $\nu\sigma = \sigma'\nu$, so ker $\nu$ admits $\sigma'$, and hence $\sigma' \mapsto \sigma$ extends $\nu$ to a homomorphism of $R\langle\sigma'\rangle$ onto $MC$. Therefore $MC \in \mathfrak{F}$ as required in (7.7).

Finally we observe that $W = A \text{ wr } B$ has an automorphism $\tau$ given by

$$a\tau = a, \qquad b\tau = b^s.$$

Since

$$(p^{\alpha-1}a)\tau = p^{\alpha-1}a$$

and

$$[a, (p^{\alpha-1} - k)b]\tau = s^{p^{\alpha-1}-1}[a, (p^{\alpha-1} - 1)b]$$
$$= [a, (p^{\alpha-1} - 1)b],$$

$M_3$ admits $\tau$ so $\tau$ acts on $MB$. Put $D = B\langle\tau\rangle \cong C(p^{\alpha-1}, q)$.

(7.8) $$MD \in \mathfrak{F}.$$

https://doi.org/10.1017/S1446788700017067 Published online by Cambridge University Press

For $i \in \{0, \cdots, \alpha\}$ define $D$-submodules $U_i$ of $M$ by

$$U_0 = [x, B],$$

and

$$U_i = \langle p^{\alpha-1}a\rangle + U_{i-1}, \qquad 1 \leqq i \leqq \alpha.$$

Now $U_0 D \in \mathfrak{F}$ by Lemma 5.1; and since for $i \geqq 1$ $U_i D$ is a normal product of $U$ and $U_{i-1}D$ it follows by induction that $U_i D \in \mathfrak{F}$ for all $i \in \{0, \cdots, \alpha\}$. But $U_\alpha = M$ so (7.8) is proved.

It is clear that, as automorphisms of $MB$, $\sigma\tau = \tau\sigma$ and hence if $\chi = \sigma\tau^{-1}$,

$$MB\langle\chi\rangle \in \mathrm{S}_n\mathrm{N}_0\{MC, MD\} \subseteq \mathfrak{F}.$$

However $b\chi = b$ so $M\langle\chi\rangle \in \mathrm{S}_n\mathfrak{F} = \mathfrak{F}$; and as $\langle\gamma\rangle$-module over $Z_{p^\alpha}$,

$$M = \langle a\rangle \oplus [M, B].$$

Since $a\chi = a\sigma\tau^{-1} = a^s$ we conclude that $C(p^\alpha, q) \cong \langle a\rangle\langle\chi\rangle \in \mathrm{Q}\mathfrak{F} = \mathfrak{F}$. The proof of Lemma 7.3 is now complete.

Proof of Corollary 7.2. Let $A$ be a supersoluble group: it is then certainly metanilpotent, even nilpotent by abelian. The proof relies on finding a suitable Q-closed Fitting class containing $A$. This is accomplished as follows. Let $p, q$ be different primes and $\mathfrak{F}(p, q)$ the class of all groups $B$ with the property that every $q$-element of $B$ centralizes every $p$-chief factor of $B$. It is easy to check that $\mathfrak{F}(p, q)$ is a Q-closed Fitting formation.

Suppose that $\mathfrak{F}$ is the smallest Q-closed Fitting class containing $A$ and suppose that it is not S-closed: choose a group $G \in \mathfrak{F}$ of least order with respect to having a subgroup $H$ not in $\mathfrak{F}$. Suppose that $H$ is chosen minimally. Much as in the proof of Theorem 4.1 we conclude that $F(G)$ is a $p$-group for some prime $p$ complemented by a $q$-power cycle for a prime $q \neq q$. It follows that $G \notin \mathfrak{F}(p, q)$. In other words $A$ has a $p$-chief factor with a $q$-element acting non-trivially on it. The supersolubility of $A$ then means that $q$ divides $p - 1$. But $C(p, q)$ is a factor group of a normal subgroup of $G$ and therefore lies in $\mathfrak{F}$, whence $H \in \mathfrak{F}$ by Theorem 7.1, a contradiction, so $\mathfrak{F}$ is S-closed. Finally $\mathfrak{F}$ is a formation by Theorem 3.1, and the proof of Corollary 7.2 is complete.

## References

[1] Roger M. Bryant, 'Finite splitting groups in varieties of groups', *Quart. J. Math.* (2) 22 (1971), 169–172.

[2] R. A. Bryce, and John Cossey, 'Fitting formations of finite soluble groups', *Math. Z.* 127 (1972), 217–223.

[3] R. A. Bryce, and John Cossey, 'Some product varieties of groups', *Bull. Austral. Math. Soc.* 3 (1970), 231–264.

[4] P. J. Cossey, *On varieties of A-groups*, (Ph. D. Thesis, Australian National University, 1966).
[5] Charles W. Curtis, and Irving Reiner, *Representation Theory of Finite Groups and Associative Algebras*, (Interscience New York and London, 1962.)
[6] Wolfgang Gaschütz, and Dieter Blessenohl, 'Über normale Schunk-und Fittingklassen', *Math. Z.* 118 (1970), 1–8.
[7] Marshall Hall, Jr. *The Theory of Groups*, (Macmillan, New York, 1959.)
[8] P. Hall, 'The splitting properties of relatively free groups', *Proc. London. Math. Soc.* (3) 4 (1954), 343–356.
[9] B. Hartley, 'On Fischer's dualization of formation theory', *Proc. London Math. Soc.* (3) 19 (1969), 193–207.
[10] Trevor O. Hawkes, 'On Fitting formations', *Math. Z.* 117 (1970), 177–182.
[11] Graham. Higman, *The orders of relatively free groups*, (Proc. Internat. Conf. Theory of Groups, Austral. Nat. Univ. Canberra, August 1965, Gordon and Breach, New York, London, Paris, 1967. pp. 153–165.)
[12] B. Huppert, *Endliche Gruppen I*, (Die Grundlehren der Mathematischen Wissenschaften, Bd 134, Springer-Verlag, Berlin, Heidelberg. New York, 1967).
[13] Amirali Makan. *On some aspects of finite soluble groups*, (Ph.D. Thesis, Australian National University, 1971.)
[14] Hanna Neumann, *Varieties of Groups*, (Ergebnisse der Mathematik und ihrer Grenzegbiete. Springer-Verlag. Berlin, Heidelberg, New York, 1967.)
[15] D. R. Taunt, 'On *A*-groups', *Proc. Cambridge Philos. Soc.* 45 (1959), 24–42.
[16] D. R. Taunt, 'Remarks on the isomorphism problem in theories of construction of finite groups', *Proc. Cambridge Philos. Soc.* 51 (1955), 16–24.

School of General Studies
Australian National University
P. O. Box 4
Canberra, A.C.T., 2600
Australia