AUTOMORPHY FACTORS FOR A HILBERT MODULAR GROUP

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Dedicated to Professor I. Satake on the occasion of his 60th birthday

Let K be a totally real algebraic number field of degree $n>1$, and let O_K be the maximal order. We denote by Γ_K, the Hilbert modular group $\mathrm{SL}_2(O_K)$ associated with K. On the extent of the weight of an automorphy factor for Γ_K, some restrictions are imposed, not as in the elliptic modular case. Maass [5] showed that the weight is integral for $K=\mathbb{Q}(\sqrt{5})$. It was shown by Christian [1] that for any Hilbert modular group it is a rational number with the bounded denominator depending on the group.

$\mathrm{SL}_2(K)$ acts on the product H^n of n copies of the upper half plane $H=\{z_1\in\mathbb{C}\mid \mathrm{Im}\, z_1>0\}$ by the usual modular substitution;

$$z=(z_1,\ldots,z_n)\rightarrow Mz=\left(\frac{\alpha^{(1)}z_1+\beta^{(1)}}{\gamma^{(1)}z_1+\delta^{(1)}},\ldots,\frac{\alpha^{(n)}z_n+\beta^{(n)}}{\gamma^{(n)}z_n+\delta^{(n)}}\right)$$

for $M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in \mathrm{SL}_2(K)$, $\alpha^{(1)},\ldots,\alpha^{(n)}$ being the conjugates of $\alpha\in K$. We consider an automorphy factor J for Γ_K which is of the following general form;

$$J(M,z)=v(M)\prod_{i=1}^{n}(\gamma^{(i)}z_i+\delta^{(i)})^{k_i},\quad M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma_K$$

where $(k_1,\ldots,k_n)\in\mathbb{Q}^n$, and $v(M)$ is a complex number. $(k_1,\ldots,k_n)$ is called the *weight* of J. v is called a *multiplier* and it has only roots of unity as the value. By Freitag [2], it is known that possible automorphy factors are of the above form, up to trivial automorphy factors, provided that $n\geqq 3$. Gundlach has obtained the bound of the denominator of $\sum k_i\in\mathbb{Q}$ for a possible automorphy factor J in his paper [3] (see also Tsuyumine [8]), where $2\sum k_i\in\mathbb{Z}$ is proved for n even. However his bound for n odd, is very large in general. The aim of the present paper is to give the proof valid regardless of n odd or even as well as a better bound in the case that an ideal (2) satisfies a ramification condition. Our result is as follows; $2\sum k_i\in\mathbb{Z}$, *and if the ideal* (2) *in* O_K *is unramified at any prime of degree one, then* $\frac{1}{2}\sum k_i\in\mathbb{Z}$. This result will be applied to study the structure of a Hilbert modular variety in a later paper [9].

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1. Let $\mathfrak{a}$ be a non-zero integral ideal of O_K. $\alpha\in O_K$ is said to be an $\mathfrak{a}$-*unit* if the image of α in $O_K/\mathfrak{a}$ is invertible. Let $\mathfrak{q}_2$ (resp. $\mathfrak{q}_2'$) denote the ideal which is the product with multiplicity one, of all primes of degree one at which the ideal (2) in O_K is unramified (resp. ramified), where it denotes O_K if there are no such primes. Similarly let $\mathfrak{q}_3$ denote the ideal which is the product with multiplicity one, of all primes of degree one in the ideal (3) of O_K.

LEMMA 1. *Let $\mathfrak{a}$ be any non-zero integral ideal. The integers α^2-1, α running over the set of $\mathfrak{a}$-units, generate the ideal containing $\mathfrak{q}_2^3\mathfrak{q}_2'^2\mathfrak{q}_3$.*

Proof. The set of $\mathfrak{a}$-units is not enlarged if $\mathfrak{a}$ is replaced by $\mathfrak{a}\mathfrak{q}_2\mathfrak{q}_2'\mathfrak{q}_3$. So we assume that $\mathfrak{a}\subset\mathfrak{q}_2\mathfrak{q}_2'\mathfrak{q}_3$. Let $\mathfrak{p}$ be a prime ideal. The reduction map mod $\mathfrak{p}$ gives rise to a surjective map of the set of $\mathfrak{a}$-units onto $O_K/\mathfrak{p}$ or $O_K/\mathfrak{p}-\{0\}$ according as $\mathfrak{a}\not\subset\mathfrak{p}$ or $\mathfrak{a}\subset\mathfrak{p}$, by the Chinese remainder theorem. If $\mathfrak{p}$ contains none of $\mathfrak{q}_2$, $\mathfrak{q}_2'$, $\mathfrak{q}_3$, then there is obviously an $\mathfrak{a}$-unit α for which the image of α^2-1 in $O_K/\mathfrak{p}$ is not zero, so $\mathfrak{p}$ is not a factor of the ideal generated by α^2-1. As easily verified, if $\mathfrak{p}$ is a factor of $\mathfrak{q}_2$ (resp. $\mathfrak{q}_2'$, resp. $\mathfrak{q}_3$), then $\alpha^2-1\in\mathfrak{p}^3$ (resp. $\mathfrak{p}^2$, resp. $\mathfrak{p}$) for any $\mathfrak{p}$-unit α, and moreover $\alpha^2-1\notin\mathfrak{p}^4$ (resp. $\mathfrak{p}^3$, resp. $\mathfrak{p}^2$) for some $\mathfrak{p}$-unit α. We are done. q.e.d.

Following the standard notation, we define two subgroups of Γ_K associated with an ideal $\mathfrak{a}$ as follows;

$$\Gamma_0(\mathfrak{a})=\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma_K \mid \gamma\equiv 0 \bmod \mathfrak{a}\right\},$$

$$\Gamma(\mathfrak{a})=\left\{M\in\Gamma_K \mid M\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \bmod \mathfrak{a}\right\},$$

where the latter one is called the *principal congruence subgroup* associated with $\mathfrak{a}$.

PROPOSITION 1. *Let $\mathfrak{a}$ be a non-zero integral ideal. Then the commutator subgroup of $\Gamma_0(\mathfrak{a})$ contains matrices $\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}$ for any $\beta\in\mathfrak{q}_2^3\mathfrak{q}_2'^2\mathfrak{q}_3$.*

Proof. We invoke Serre [7] in which the congruence subgroup problem was solved affirmatively, particularly for a totally real algebraic number field of degree $n>1$. The commutator subgroup $\Gamma_0(\mathfrak{a})'$ of $\Gamma_0(\mathfrak{a})$ is a finite index in Γ_K (*loc. cit.* Cor. to Theorem 3), and hence $\Gamma_0(\mathfrak{a})'$ contains some principal congruence subgroup, say $\Gamma(\mathfrak{b})$, $\mathfrak{b}$ being a non-zero ideal contained in $\mathfrak{a}$ (*loc. cit.* Cor. 3 to Theorem 2). Let α be any $\mathfrak{b}$-unit. Then there are $\beta,\gamma\in\mathfrak{b}$ and a $\mathfrak{b}$-unit δ such that $M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma_K$. Then for $\zeta\in O_K$

$$\Gamma_0(\mathfrak{a})'\ni M\begin{pmatrix}1 & \zeta\\ 0 & 1\end{pmatrix}M^{-1}\begin{pmatrix}1 & \zeta\\ 0 & 1\end{pmatrix}^{-1}\equiv\begin{pmatrix}1 & (\alpha^2-1)\zeta\\ 0 & 1\end{pmatrix} \bmod \mathfrak{b}.$$

Since $\Gamma(\mathfrak{b})\subset\Gamma_0(\mathfrak{a})'$, $\begin{pmatrix}1 & (\alpha^2-1)\zeta\\ 0 & 1\end{pmatrix}$ is contained in $\Gamma_0(\mathfrak{a})'$, for any $\mathfrak{b}$-unit α and any $\zeta\in O_K$. The assertion follows from Lemma 1. q.e.d.

In the case of Γ_K, a further argument shows that the commutator subgroup contains $\Gamma(\mathfrak{q}_2^2\mathfrak{q}_2'^2\mathfrak{q}_3)$. Then the commutator factor group of Γ_K is easily calculated, which is, however, not necessary in the present paper. The commutator factor group is determined by Kirchheimer [**4**] in more general context.

For a group Γ in $SL_2(K)$, we denote by $U(\Gamma)$, the set $\left\{\beta \in K \,\middle|\, \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in \Gamma\right\}$, which is an additive group. Let χ be any complex valued function on Γ. Then χ_U denotes a function on $U(\Gamma)$ defined by $\chi_U(\beta) = \chi\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$. If χ is a character of Γ, then χ_U is a character of the additive group $U(\Gamma)$.

COROLLARY. *Let χ be a character of $\Gamma_0(\mathfrak{a})$. Then χ_U is a character of O_K with values in the set of 24th roots of unity. If the ideal (2) in O_K is unramified at any prime of degree one, then χ_U is valued in the set of sixth roots of unity.*

Proof. χ_U can be regarded as a character of $O_K/\mathfrak{q}_2^3\mathfrak{q}_2'^2\mathfrak{q}_3$. $O_K/\mathfrak{p}^3$ (resp. $O_K/\mathfrak{p}^2$, resp. $O_K/\mathfrak{p}$) is isomorphic to a cyclic group of order eight (resp. an abelian group of type (2, 2), resp. a cyclic group of order three) if a prime $\mathfrak{p} \supset \mathfrak{q}_2$ (resp. $\mathfrak{q}_2'$, resp. $\mathfrak{q}_3$). Our assertion is immediate from this. q.e.d.

2. Let Γ be a subgroup of $SL_2(K)$ commensurable with Γ_K. An automorphy factor J of Γ is the function on $\Gamma \times H^n$ with values in $\mathbb{C} - \{0\}$ such that (*i*) $J(M, z)$ is holomorphic in z for any $M \in \Gamma$, and (*ii*) $J(LM, z) = J(L, Mz)J(M, z)$ for $L, M \in \Gamma$, and (*iii*) $J(-M, z) = J(M, z)$ if $\pm M \in \Gamma$. J is said to be a *trivial* automorphy factor if $J(M, z) = u(Mz)/u(z)$ for some invertible holomorphic function u on H^n, $n > 1$. In the following we consider exclusively the automorphy factor of the form

$$J(M, z) = v(M) \prod_{i=1}^{n} (\gamma^{(i)} z_i + \delta^{(i)})^{k_i}, \quad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma,$$

where the weight $(k_1, \ldots, k_n)$ is a vector in $\mathbb{Q}^n$, and $v(M)$ is a complex number. Here $(\gamma^{(i)} z_i + \delta^{(i)})^{k_i}$ is defined to be $\exp(k_i \log(\gamma^{(i)} z_i + \delta^{(i)}))$, where

$$\log(\gamma^{(i)} z_i + \delta^{(i)}) = \log|\gamma^{(i)} z_i + \delta^{(i)}| + \sqrt{-1} \arg(\gamma^{(i)} z_i + \delta^{(i)})$$

with

$$-\pi < \arg(\gamma^{(i)} z_i + \delta^{(i)}) \leqq \pi.$$

By definition, v_U is a character of $U(\Gamma)$. Let m be the non-zero rational integer for which mk_i $(1 \leqq i \leqq n)$ are integral. Then v^m is a character of Γ, and hence some power of it is trivial since the commutator factor group of Γ is finite (Serre [**7**]). This implies that v_U is a finite character. So there is a non-zero integer ν in O_K for which v_U is trivial on $\nu O_k \cap U(\Gamma)$.

LEMMA 2. *Let J, J' be two automorphy factors for Γ with the same weight. Then $J = \chi J'$ for some character χ of Γ.*

Proof. J/J' is independent of $z \in H^n$ by the assumption. Then the definition (*ii*) of an automorphy factor implies that it is a character. q.e.d.

Now let us consider an automorphy factor J for $\Gamma_0(\mathfrak{a})$, $\mathfrak{a}$ being a non-zero integral ideal. Let ν be a totally positive integer in O_K. Then we make from J, automorphy factors for $\Gamma_0(\nu\mathfrak{a})$ of two kinds. One is given simply by restricting to $\Gamma_0(\nu\mathfrak{a})$, the group applied to J, which we denote again by J. Let m_ν be the automorphism of H^n given by

$$z = (z_1, \ldots, z_n) \rightarrow \nu z = (\nu^{(1)} z_1, \ldots, \nu^{(n)} z_n).$$

Let $M_\nu = \begin{pmatrix} \sqrt{\nu}^{-1} & 0 \\ 0 & \sqrt{\nu} \end{pmatrix}$. Since $m_\nu(Mz) = (M_\nu^{-1} M M_\nu) m_\nu(z)$ for any $M \in \mathrm{SL}_2(K)$, the action of $\Gamma_0(\mathfrak{a})$ is translated to that of $M_\nu^{-1}\Gamma_0(\mathfrak{a})M_\nu$ by m_ν. Noting that $M_\nu^{-1}\Gamma_0(\nu\mathfrak{a})M_\nu \subset \Gamma_0(\mathfrak{a})$, let us put

$$J'(M, z) = J(M_\nu^{-1} M M_\nu, m_\nu(z)) \quad \text{for} \quad M \in \Gamma_0(\nu\mathfrak{a}),$$

which is the pull back of an automorphy factor J via m_ν. Then J' is an automorphy factor for $\Gamma_0(\nu\mathfrak{a})$. If $\gamma^{(i)} z_i + \delta^{(i)}$ satisfies the condition that $-\pi < \arg(\gamma^{(i)} z_i + \delta^{(i)}) \leqq \pi$, then $\gamma^{(i)} \nu^{(i)} z_i + \delta^{(i)}$ does. A simple calculation shows that $J'(M, z) = v'(M) \prod_{i=1}^{n} (\gamma^{(i)} z_i + \delta^{(i)})^{k_i}$ with

$$v'\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = v\begin{pmatrix} \alpha & \nu\beta \\ \nu^{-1}\gamma & \delta \end{pmatrix}$$

PROPOSITION 2. *Let v be as above, the multiplier of an automorphy factor J for $\Gamma_0(\mathfrak{a})$. Then v_U is valued in the set of 24th roots of unity. If the ideal (2) in O_K is unramified at any prime ideal of degree one, then v_U is valued in the set of sixth roots of unity.*

Proof. For a totally real integer ν of O_K, we have two automorphy factors J, J' as above. By Lemma 2, there is a character χ of $\Gamma_0(\nu\mathfrak{a})$ for which the equality $J = \chi J'$ holds. Let us take as ν, a sufficiently divisible integer so that v_U is trivial on νO_K. Then for the multiplier v' of J', v'_U is trivial on $O_K = U(\Gamma_0(\nu\mathfrak{a}))$. So v_U equals χ_U on $U(\Gamma_0(\nu\mathfrak{a})) = U(\Gamma_0(\mathfrak{a}))$, and our assertion follows from Corollary to Proposition 1. q.e.d.

In a final step, we reduce the argument to the elliptic modular case, and so we make preparations for it. We refer to Rankin [6] for the detail. Let $\Delta(\tau)$, $\tau \in H$, be the cusp form of weight twelve for $\Gamma_{\mathbb{Q}} = \mathrm{SL}_2(\mathbb{Z})$ with a trivial multiplier. Since $\Delta(\tau)$ vanishes at no points of H, $\Delta(\tau)^k$ is a well-defined modular form for any complex number k. Let

$$w(M)(c\tau + d)^{12k}, \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\mathbb{Q}},$$

be its automorphy factor, where the branch of $(c\tau + d)^{12k}$ is determined in the above manner. Then the identity

$$w_U(b) = \exp(2\sqrt{-1}\,\pi k b/6), \qquad b \in \mathbb{Z},$$

holds, which implies that the multiplier determines the weight mod 12. Since the commutator factor group of $\Gamma_{\mathbb{Q}}\Big/\left\{\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}$ is of order six, by Lemma 2 and by the definition (*iii*) of an automorphy factor there are just six automorphy factors if the weight k is fixed. At any rate if J_1 is an automorphy factor for $\Gamma_{\mathbb{Q}}$ with a multiplier w', then the weight of J'^6 is determined by $(w'^6)_U$ mod 12. In particular the weight of J'^6 is the integer divisible by 12 if and only if $(w'^6)_U$ is trivial.

THEOREM. *Let Γ_K be the Hilbert modular group associated with a totally real algebraic number field K of degree >1. Let $(k_1, \ldots, k_n) \in \mathbb{Q}^n$ be the weight of an automorphy factor. Then $2\sum_{i=1}^{n} k_i$ is integral. If the ideal (2) in the maximal order of K is unramified at any prime of degree one, then $\frac{1}{2}\sum_{i=1}^{n} k_i$ is integral.*

Proof. Let us identify the upper half plane H with the image embedded diagonally into H^n;

$$\tau \to (\tau, \ldots, \tau) \in H^n.$$

The stabilizer subgroup at H equals $\Gamma_{\mathbb{Q}} \subset \Gamma_K$. If J_1 denotes the restriction of J to H, then J_1 is an automorphy factor for $\Gamma_{\mathbb{Q}}$ of weight $\sum k_i$. Proposition 2 shows that $(v^{24}|_{\Gamma_{\mathbb{Q}}})_U$ is trivial, and that $(v^6|_{\Gamma_{\mathbb{Q}}})_U$ is trivial if the ramification condition on the ideal (2) is satisfied. Then $24\sum k_i$ is divisible by 12, and $6\sum k_i$ is if the ramification condition is satisfied. q.e.d.

Let us suppose that K is a real quadratic field. If d_k denotes the discriminant of K, then the ideal (2) is unramified at any prime of degree one if and only if $d_K \not\equiv 1 \bmod 8$. We have the following corollary (compare with Gundlach [**3**, Theorem 4.1]);

COROLLARY. *Let K be a real quadratic field. Then $2\sum_{i=1}^{2} k_i$ is integral. If the discriminant d_k of K satisfies $d_k \not\equiv 1 \bmod 8$, then $\frac{1}{2}\sum_{i=1}^{2} k_i$ is integral.*

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