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FRAMES WITH BLOCK SIZE FOUR

Dedicated to the memory of Haim Hanani

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ABSTRACT. We investigate the spectrum for frames with block size four, and discuss several applications to the construction of other combinatorial designs.

Our main result is that a frame of type h^u, having blocks of size four, exists if and only if $u \geq 5$, $h \equiv 0 \bmod 3$ and $h(u-1) \equiv 0 \bmod 4$, except possibly where

(i) $h = 9$ and $u \in \{13, 17, 29, 33, 93, 113, 133, 153, 173, 193\}$;
(ii) $h \equiv 0 \bmod 12$ and $u \in \{8, 12\}$,
$h = 36$ and $u \in \{7, 18, 23, 28, 33, 38, 43, 48\}$,
$h = 24$ or 120 and $u \in \{7\}$,
$h = 72$ and $u \in 2\mathbb{Z}^+ \cup \{n : n \equiv 3 \bmod 4 \text{ and } n \leq 527\} \cup \{563\}$; or
(iii) $h \equiv 6 \bmod 12$ and $u \in (\{17, 29, 33, 563\} \cup \{n : n \equiv 3 \text{ or } 11 \bmod 12 \text{ and } n \leq 527\} \cup \{n : n \equiv 7 \bmod 12 \text{ and } n \leq 259\})$, $h = 18$.

Additionally, we give a new recursive construction for resolvable group-divisible designs from frames: if there is a resolvable k-GDD of type g^u, a k-frame of type $(mg)^v$ where $u \geq m+1$, and a resolvable TD(k, mv) then there is a resolvable k-GDD of type $(mg)^{uv}$. We use this to construct some new resolvable GDDs with group size three and block size four.

1. **Introduction.** A *frame* is a group-divisible design (X, G, B) whose block set admits a partition into *holey* parallel classes, each holey parallel class being a partition of X-G_j for some group $G_j \in G$. The groups of a frame are usually referred to as *holes*. The *degree* of a hole G_j is the number of holey parallel classes that partition X-G_j. Perhaps the simplest example of a frame is a near-one-factorization of K_{2n+1}. (The groups all have size one and the holey parallel classes are the near-one-factors.) While the foregoing is the generally accepted definition of a frame, it is worth noting that the term frame originally meant a structure with an additional property, being used in the construction of Room squares (see *e.g.* [14]). Specifically, a *Room frame* is a group-divisible design, with blocks of size two, whose block set admits a pair of *orthogonal* holey resolutions, meaning that if H_i' is a holey parallel class with respect to hole G_i in the first resolution and H_j'' is a holey parallel class with respect to hole G_j in the second resolution then $|H_i' \cap H_j''| \leq 1$ if $i \neq j$, and $|H_i' \cap H_i''| = 0$.

We use as our standard design theory reference Beth, Jungnickel and Lenz [5]. Before proceeding, we will review some of the terminology and notation regarding incomplete designs. An *incomplete pairwise balanced design* $(v, w; K)$-IPBD is a triple (X, Y, B)

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where X is a v-set, Y is a w-subset of X (Y is called the *hole*) and B is a collection of subsets of X, each having cardinality from K, such that a given pair x_1, x_2 of points from X is contained in exactly one block from B unless both x_1, x_2 are in Y, in which case there is no block containing them both. Note then that (X, Y, B) is an IPBD if and only if $(X, B \cup \{Y\})$ is a PBD. Similarly, an *incomplete group-divisible design* (IGDD) is a quadruple (X, Y, G, B) where Y is a distinguished subset of X (called the *hole*) so that a given pair x_1, x_2 of points from X is contained either in one block from B or one group from G unless both x_1, x_2 are in Y, in which case there is no block containing them both. (There may be a group containing them both however.) We shall have occasion to speak about incomplete frames; to avoid ambiguity we will state at the outset that for these designs the members of G will be called groups and the set Y will be termed the hole. An *incomplete resolvable block design* $(v, w; \{k\})$-IRBIBD is a $(v, w; \{k\})$-IPBD (X, Y, B) whose block set is a disjoint union $P \cup Q$, where P can be partitioned into parallel classes on X and Q can be partitioned into holey parallel classes (with respect to the hole Y) on $X \setminus Y$. It is not difficult to show that if a $(v, w; \{k\})$-IRBIBD exists then $v \equiv w \equiv k \pmod{k(k-1)}$. Hence determining the spectrum for resolvable subdesigns in resolvable $(v, k, 1)$-BIBDs yields the spectrum for incomplete resolvable designs with block size k. The same is *not* true if we remove the resolvability criterion. For example there is an $(11, 5; \{3\})$-IPBD (just adjoin 5 infinite points to a one-factorization of K_6) but there is no $(5, 3, 1)$-BIBD.

A *transversal design* TD(k, n) is a group-divisible design in which there are k groups of size n and in which all blocks have size k. It is well-known that a TD(k, n) is equivalent to $k - 2$ MOLS of order n. Recent work of Abel [1] and Todorov [26] on the existence of 4 MOLS requires that we update [5]: there exists a TD$(6, n)$ if $n \geq 5$, $n \neq 6, 10, 14, 18, 22, 26, 30, 34, 42$. An *incomplete transversal design* ITD$\big(k, (n, m)\big)$ is an IGDD in which there are k groups of size n, having a hole Y which intersects each group in exactly m points, and in which all blocks have size k. These designs are also denoted by TD(k, n)-TD(k, m), to emphasize their equivalence to sets of $k - 2$ MOLS of order n 'missing' $k - 2$ sub-MOLS of order m. More generally, if M is a set of positive integers we will denote by ITD$\big(k, (n, M)\big)$ an incomplete transversal design having a mutually disjoint set of holes which, respectively, intersect each group in m points, $m \in M$. In particular if $\sum_{m \in M} m = n$ then the holes are said to be *spanning*. (Such designs are equivalent to sets of orthogonal partitioned incomplete latin squares (OPILS), which we will discuss in Section 5.) Incomplete transversal designs are of considerable interest because of their versatility in constructions for other types of combinatorial designs. For recent work (together with extensive bibliographies) on ITDs we refer the reader to Stinson and Zhu [25] and Zhu [28].

We will use the usual exponential notation for GDDs and frames. Thus a k-GDD of type $g_1^{t_1} g_2^{t_2} \cdots g_s^{t_s}$ is a group-divisible design in which the blocks have size k and in which there are t_i groups of size g_i, $i = 1, \ldots, s$. Alternatively we may say k-GDD of type S, where S is the multiset consisting of t_i copies of g_i, $i = 1, \ldots, s$. Similarly, a k-IGDD of type $(g_1, h_1)^{t_1} (g_2, h_2)^{t_2} \cdots (g_s, h_s)^{t_s}$ is an incomplete group-divisible design in which the blocks have size k and in which there are t_i groups of size g_i, each of which intersects the hole in h_i points. If some $h_j = 0$ we generally suppress it, writing just $g_j^{t_j}$. The following fundamental property of frames is proven in [23]:

LEMMA 1.1. *If (X, G, B) is a k-frame and $G_j \in G$, then the hole G_j has degree $|G_j|/(k-1)$. In particular, $|G_j| \equiv 0 \pmod{k-1}$.*

COROLLARY 1.2. *If there is a k-frame of type h^u (with $u > 1$), then $u \geq k+1$, $h \equiv 0 \pmod{k-1}$ and $h(u-1) \equiv 0 \bmod k$.*

PROOF. The first and third conditions follow from the definition of a holey parallel class, while the second follows from Lemma 1.1. ∎

The necessary conditions given by Corollary 1.2 are known to be sufficient to guarantee the existence of these k-frames for block sizes $k = 2$ or 3. The purpose of this paper is to investigate the case $k = 4$. Note that our concern here is only with establishing existence criteria for *uniform* frames, that is, frames in which all holes have the same size h.

The case $k = 3$ was done by Stinson [24]. Due to a typesetting error that was not noticed in the proof-reading, a part of the frame of type 6^6 is missing in that paper. We present the whole frame here:

$$\begin{array}{ll} \text{Points:} & \big(\big((\mathbb{Z}_5 \times \{1,2\}) \cup \{\infty\}\big) \times \mathbb{Z}_3\big) \cup \{\alpha_1, \alpha_2, \alpha_3\}. \\ \text{Holes:} & \{(0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1),(0,2,2)\} \pmod{5, -; -} \\ & \{(\infty,1),(\infty,2),(\infty,3),\alpha_1,\alpha_2,\alpha_3\}. \end{array}$$

Holey parallel classes:

$$\begin{array}{lll} (\infty,1),\ (1,1,0)\ ,\ (3,1,2) & \alpha_3\ \ ,\ (4,1,1)\ ,\ (2,2,0) & \\ (\infty,2),\ (4,2,1)\ ,\ (3,2,2) & (2,1,1)\ ,\ (3,1,1)\ ,\ (4,1,2) & \\ (\infty,0),\ (3,1,0)\ ,\ (2,2,1) & (1,2,2)\ ,\ (3,2,1)\ ,\ (4,2,2) & \pmod{5,-;3} \\ \alpha_1\ \ ,\ (1,1,1)\ ,\ (2,2,2) & (4,1,0)\ ,\ (2,1,0)\ ,\ (1,2,0) & \\ \alpha_2\ \ ,\ (2,1,2)\ ,\ (1,2,1) & (3,2,0)\ ,\ (4,2,0)\ ,\ (1,1,2) & \end{array}$$

and

$$(4,1,2),\ (3,1,0)\ ,\ (0,2,2) \quad (2,2,0)\ ,\ (4,2,1)\ ,\ (1,1,1) \pmod{5,-;3}.$$

The case $k = 2$ falls under the category of 'folklore'. We are not aware of any simple constructions for these frames appearing in the literature, and so we will, for the sake of completeness, present such a set of constructions here. These are based largely on the following lemma:

LEMMA 1.3. *The additive group $\mathbb{Z}_n$ admits a partition P into pairs such that*

$\{\pm(a-b) : (a,b) \in P\} = \mathbb{Z}_n \backslash \{0\}$ *if and only if* $n \equiv 0$ *or* $2 \bmod 8$.

PROOF. We first show necessity. Suppose that $n \equiv 0 \bmod 4$. Then the difference $\frac{n}{2}$ is even and, without loss of generality, appears in P as $\{0, \frac{n}{2}\}$. This leaves $\frac{n}{2}$ odd residues and $\frac{n}{2} - 2$ even residues to be covered by the remaining pairs in P; this can occur only if the number $\left|\left\{\{a,b\} \in P : a-b \text{ is an even residue mod } n \text{ and } \{a,b\} \neq \{0, \frac{n}{2}\}\right\}\right|$ is odd. But this number is $\frac{n}{4} - 1$, that is, $\frac{n}{4}$ must be even, which means $n \equiv 0 \bmod 8$. A similar argument shows that if $n \equiv 2 \bmod 4$ then in fact we must have $n \equiv 2 \bmod 8$.

We now show sufficiency by direct construction.

(i) $n \equiv 0 \bmod 8$

$\mathbb{Z}_8$: $0,4 \quad 1,3 \quad 2,5 \quad 6,7$

$\mathbb{Z}_{16}$: $0,8 \quad 4,11 \quad 3,5 \quad 2,6 \quad 1,7 \quad 10,15 \quad 9,12 \quad 13,14$

$$\begin{array}{lllll}
\mathbb{Z}_n, n \geq 24: & 0, \frac{n}{2} & 1, \frac{n}{2}-1 & \frac{n}{2}+2, n-1 & \frac{5n}{8}+1, \frac{7n}{8}-2 \\
 & \frac{n}{4}, \frac{3n}{4}-1 & \vdots & \vdots & \vdots \\
 & \frac{7n}{8}-1, \frac{7n}{8} & \vdots & \vdots & \vdots \\
 & \frac{n}{2}+1, \frac{3n}{4} & \frac{n}{4}-1, \frac{n}{4}+1 & \frac{5n}{8}, \frac{7n}{8}+1 & \frac{3n}{4}-2, \frac{3n}{4}+1
\end{array}$$

(ii) $n \equiv 2 \bmod 8$

$\mathbb{Z}_2$: $0,1$

$\mathbb{Z}_{10}$: $0,5 \quad 1,2 \quad 3,6 \quad 4,8 \quad 7,9$

$$\begin{array}{lllll}
\mathbb{Z}_n, n \geq 18: & 0, \frac{n}{2} & 1, \frac{n}{2}-1 & \frac{n+14}{8}, \frac{3n+2}{8} & \frac{n}{2}+2, n-1 \\
 & \frac{n-2}{8}, \frac{n+6}{8} & \vdots & \vdots & \vdots \\
 & \frac{n+2}{4}, \frac{n}{2}+1 & \vdots & \vdots & \vdots \\
 & \frac{n+6}{4}, \frac{3n+2}{4} & \frac{n-10}{8}, \frac{3n+10}{8} & \frac{n-2}{4}, \frac{n+10}{4} & \frac{3n-2}{4}, \frac{3n+6}{4}
\end{array}$$

This completes the proof of Lemma 1.3. ∎

In the terminology of frame starters, Lemma 1.3 concerns frame starters in $\mathbb{Z}_n \backslash \{0, \frac{n}{2}\}$. The necessary condition $n \equiv 0$ or $2 \bmod 8$ was shown by Anderson [1a]. Furthermore, Rosa has pointed out that these starters can in fact be generated by Skolem sequences $\{(p_r, q_r) : r = 1, \ldots, m\}$ where $q_r - p_r = r$ and where $p_m = 1$. Such sequences were constructed by Rosa for all $m \equiv 0$ or $1 \pmod 4$ in [20]. (In fact our starters correspond *precisely* to Rosa's Skolem sequences when $n \equiv 0 \pmod 8$ (*i.e.* $m \equiv 0 \bmod 4$). For $n \equiv 2 \pmod 8$ we can add 1 to each symbol in our starters to obtain Skolem sequences with $m = \frac{1}{2}n$ which are non-isomorphic to Rosa's sequences.)

THEOREM 1.4. *There exists a 2-frame of type h^u if and only if $u \geq 3$ and $h(u-1) \equiv 0 \bmod 2$.*

PROOF. Necessity is given by Corollary 1.2. If u is odd the construction is quite simple; start with a near-one-factorization of K_u (*i.e.* 2-frame of type 1^u) and use weight h.

We now suppose that u is even, so that h must be even. It therefore suffices to show that there exist 2-frames of type 2^u for each even $u \geq 4$ (the desired frame is then obtained by applying weight $h/2$). If $u \equiv 0 \bmod 4$ we use the frame starter on $\mathbb{Z}_{2u}$ given in Lemma 1.3 and develop modulo $2u$ (the holes in the frame are $\{0,u\}, \{1,u+1\},\ldots,\{u-1,2u-1\}$). If $u \equiv 2 \bmod 4$ we begin with the frame starter on $\mathbb{Z}_{2u-2}$ and add two ideal points ∞_1 and ∞_2. Replace the pair $\left\{\frac{n-2}{8},\frac{n+6}{8}\right\}$ by $\left\{\infty_1,\frac{n-2}{8}\right\}$ and $\left\{\infty_2,\frac{n+6}{8}\right\}$ and develop modulo $2u-2$; this will give the holey parallel classes with respect to the holes $\{0,u-1\}$, $\{1,u\},\ldots,\{u-2,2u-3\}$. The holey parallel classes with respect to the hole $\{\infty_1,\infty_2\}$ are $0,1\ 2,3\cdots 2u-4, 2u-3$ and $1,2\ 3,4\cdots 2u-3, 0$.

This completes the proof of Theorem 1.4. ■

As a final note, we remark that frames with more than one block size have been considered by Rees in connection with $g^k(1,2;v)$ problem, which asks for the smallest number of blocks possible in a pairwise balanced design on v points in which the largest block has size k (see [15]). Another variation of frames which has received considerable attention recently is frames for α-resolvable designs (see Furino [9]). A design is called *α-resolvable* if its block set admits a partition into α-parallel classes, each α-parallel class being a set B' of blocks with the property that each point is contained in exactly α blocks of B'. Note that a 1-resolvable design is just a resolvable design in the usual sense.

2. **Some preliminaries.** We concentrate our efforts now on the construction of frames with block size four. Henceforth the term frame will, unless specifically indicated otherwise, mean a uniform 4-frame. Corollary 1.2 gives us the necessary conditions for the existence of our frames:

LEMMA 2.1. *If a frame of type h^u exists, then $u \geq 5$, $h \equiv 0 \bmod 3$ and $h(u-1) \equiv 0 \bmod 4$.*

We'll now briefly review some of the more common constructions for frames, many of which can be found in [24].

CONSTRUCTION 2.2 (INFLATING BY TDS). If there is a frame of type h^u and a resolvable TD$(4,m)$ then there is a frame of type $(mh)^u$.

CONSTRUCTION 2.3 (GDD CONSTRUCTION I). Let (X,G,B) be a GDD, and let $w\colon X \longrightarrow \mathbb{Z}^+ \cup \{0\}$ be a weight function on X. Suppose that for each block $b \in B$ there is a 4-frame of type $\{w(x) : x \in b\}$. Then there is a 4-frame of type $\{\sum_{x \in G_j} w(x) : G_j \in G\}$.

CONSTRUCTION 2.3A (GDD CONSTRUCTION II). Let (X,G,B) be a GDD, and let $w\colon X \longrightarrow \mathbb{Z}^+ \cup \{0\}$ be a weight function on X. Suppose that for each block $b \in B$ there is a 4-frame of type $\{w(x) : x \in b\}$ and that for each group $G_j \in G$ there is a frame of type $h^{s_j/h+\delta}$, where $s_j = \sum_{x \in G_j} w(x)$ and $\delta = 0$ or 1. Then there is a frame of type $h^{s/h+\delta}$, where $s = \sum_{x \in X} w(x)$.

As a corollary to Construction 2.3a, we get:

LEMMA 2.4 (PBD CLOSURE). *For each fixed hole size h, the set* $\{u : \exists$ *frame of type* $h^u\}$ *is* PBD-*closed.*

We will use, as a starting point for many of our frames, the following fundamental result of Hanani, Ray-Chaudhuri and Wilson [10]:

THEOREM 2.5. *There exists a resolvable* $(v, 4, 1)$-BIBD *if and only if* $v \equiv 4 \bmod 12$.

COROLLARY 2.6. *There exists a frame of type* 3^u *if and only if* $u \equiv 1 \bmod 4$.

PROOF. Necessity is given by Lemma 2.1; for sufficiency we delete a point from a resolvable $(3u + 1, 4, 1)$-BIBD.

We will make extensive use of pairwise balanced designs with block sizes from the set $H^4 = \{k : k \equiv 1 \bmod 4\}$. These designs have been studied extensively, particularly with a view towards determining the minimal finite basis for H^4 (see [12], [13], [14] and [27]). The following result is from [8].

THEOREM 2.7.

$$B(5, 9^*) \supseteq \{v : v \equiv 9, 17 \bmod 20, v \neq 17, 29\} \setminus \{49\}$$
$$B(5, 13^*) = \{v : v \equiv 13 \bmod 20, v \neq 33\}$$

As usual the presence of an asterisk indicates exactly one occurrence of a block of the indicated size. We will also use the following terminology, adapted from [14]. If (X, B) is a pairwise balanced design with block sizes from the set H^4 and there is a point $x \in X$ which is contained exclusively in blocks of size 5, then we will say that x is a *5-head* of (X, B) and we will write $v = |X| \in B(\bar{5}, H^4)$. Thus, for example, the only known construction that puts $49 \in B(5, 9)$ (see [12]) does not yield a 5-head, and so it is not yet known whether or not $49 \in (\bar{5}, 9)$. Note that $B(5, 9^*) \setminus \{9\} \subseteq B(\bar{5}, 9)$. It is worth noting that $v \in B(\bar{5}, H^4)$ if and only if there is an H^4-GDD of type $4^{(v-1)/4}$, *i.e.* $v - 1 \in \mathrm{GD}(H^4, 4)$. Lindner and Stinson [12a] investigated the set $\mathrm{GD}(\{5, 9, 13, 17, 29, 49\}, 4)$ and showed that $4u \in \mathrm{GD}(\{5, 9, 13, 17, 29, 49\}, 4)$ whenever $u \geq 5$, $u \neq 7, 8, 12, 14, 18, 19, 23, 24, 33, 34$. From Theorem 2.7 we see that the values $u = 14, 18, 19, 23, 24, 33$, and 34 can be removed from their list of exceptions, leaving only $u = 12$ open (the cases $u = 7, 8$ are of course real exceptions).

We will now work towards determining the set $B(\bar{5}, 9)$. In consideration of Theorem 2.7 we need only concentrate our efforts on the fibre $u \equiv 13 \pmod{20}$.

LEMMA 2.8. $B(\bar{5}, 9) \supseteq \{v : v \equiv 13 \bmod 20$ *and* $v \geq 213\}$.

PROOF. We first observe that $53 \in B(\bar{5}, 9)$ (a 5-GDD of type $4^7 8^3$ is given in the Appendix), and that $73 \in B(9)$ (take the projective plane of order 8).

Now let $v \geq 213$ be given. Let $m = \frac{1}{20}(v - 53)$.

If $m \equiv 0$ or $1 \bmod 5$, $m \geq 15$, $m \neq 26$ or 30 then take a TD$(6, m)$ and truncate a group to 13 points. Now apply Wilson's Fundamental Construction (WFC) [27], using weight 4 and using 5-GDDs of types 4^5 and 4^6, to obtain a 5-GDD of type $(4m)^5 52^1$. Then adjoin one ideal point, filling in $(4m+1,5,1)$-BIBDs and a $(53,\{5,9\})$-PBD.

If $m = 26$ or 30, start with a TD$(6, m-1)$, truncating a group to 17 points. Use the WFC to yield a 5-GDD of type $(4m-4)^5 68^1$, and adjoin five ideal points, filling in $(4m+1,5,1)$-BIBDs (in which the ideal points form a block) and a $(73,\{9\})$-PBD.

If $m \equiv 4 \bmod 5$, $m \geq 19$, $m \neq 34$ then we proceed as above, except that we truncate a group to 12 points before applying the WFC, and that we use five ideal points, filling $(4m+5,5,1)$-BIBDs (in which the ideal points form a block) and a $(53,\{5,9\})$-PBD.

For $m \equiv 2 \bmod 5$, $m \geq 12$, $m \neq 22, 42$ we can proceed as in the $m \equiv 0$ or $1 \bmod 5$ case, truncating a group to 11 points before applying the WFC and using 9 ideal points, filling in $(4m+9,\{5,9^*\})$-PBDs (in which the ideal points form a block) and a $(53,\{5,9\})$-PBD (Theorem 2.7). If $m = 22$ or 42, we start with a TD$(6, m-1)$, truncating a group to 18 points and using one ideal point upon applying the WFC. Now we fill in $(4m-3,5,1)$-BIBDs and a $(73,\{9\})$-PBD.

Finally, if $m \equiv 3 \bmod 5$, $m \geq 13$, $m \neq 18$ we start with a TD$(6, m)$, truncating a group to 12 points and removing a block of size 5 to obtain a $\{5,6\}$-IGDD of type $(m,1)^5 12^1$. Now apply the WFC, yielding a 5-IGDD of type $(4m,4)^5 48^1$. Use 5 ideal points, filling in $(4m+5,9;\{5\})$-IPBDs ($=(4m+5,\{5,9^*\})$-PBDs in which the block of size 9 forms the hole, see Theorem 2.7), a $(25,5,1)$-BIBD in which the ideal points form a block, and a $(53,\{5,9\})$-PBD. For $m = 18$ use a TD$(6,17)$, truncating a group to 16 points and using 9 ideal points after applying the WFC; now fill in $(77,\{5,9^*\})$-PBDs (in which the ideal points form a block) and a $(73,\{9\})$-PBD.

There remain the values $v = 213, 233, 253, 273, 333$ and 733.

Now $213 = 4 \times 53 + 1$; since $53 \in B(5,9)$ and $4x+1 \in B(\bar{5},9)$ for $x \in B(5,9)$ it follows that $213 \in B(\bar{5},9)$. The case $v = 733$ can be done similarly. If we take a TD$(11,17)$ and remove two points from each of two groups we get $183 \in B(9,10,11,15,17)$. Since for each $x \in \{9,10,11,15,17\}$ we have $4x+1 \in B(\bar{5},9)$, we therefore have $4(183)+1 = 733 \in B(\bar{5},9)$.

For $v = 253$, start with a TD$(6,5)$ and apply the WFC with weight 8, filling in 5-GDDs of type 8^6 (which can be obtained by removing the hole from a TD$(6,10)$-TD$(6,2)$, see [4]). Now adjoin 13 ideal points, filling in $(53,\{5,13^*\})$-PBDs (in which the ideal points form a block) and a $(53,\{5,9\})$-PBD. For $v = 273$, proceed similarly, applying weight 4 to a TD$(6,11)$ and adjoining 9 ideal points. Fill in $(53,\{5,9\})$-PBDs in which the ideal points form a block.

Finally for $v = 233$ start with a TD$(6,10)$-TD$(6,2)$ (see [4]) and remove two points from the hole (but from different groups); apply weight 4 to get a 5-IGDD of type $(36,4)^2(40,8)^4$. Now adjoin one ideal point, filling in $(37,5;\{5,9^*\})$-IPBDs, $(41,9;\{5,9\})$-IPBDs and a $(41,5,1)$-BIBD. (The first designs exist by Theorem 2.7 and the second can be obtained by fusing the groups of a TD$(5,8)$ with a new point.) This can be done so that the ideal point becomes a 5-head in the finished design. For $v = 333$ take TD$(9,9)$ and truncate a group to one point. Now give all points on a long group weight 8, and all remaining points weight 4, applying the WFC with 5-GDDs of type

$4^7 8^1$ (obtained by deleting a point from the block of size 9 in a $(37, \{5, 9^*\})$-PBD) and $\{5,9\}$-GDDs of type $4^8 8^1$ (obtained by deleting a point from the $(41, \{5,9\})$-PBD previously referred to). Use 5 ideal points, filling in $(41,5;\{5\})$-IPBDs, a $(77,5;\{5,9^*\})$-IPBD and a $(9,\{9\})$-PBD. Note that the 77-point filler will have 5-heads in the finished design.

This completes the proof of Lemma 2.8. ∎

The reader will have noticed that in all but a few of the cases in Lemma 2.8 we showed that $v \in B(\bar{5},9)$ by first showing that $v \in B(5,53^*)$ or $v \in B(5,73^*)$. Anticipating that this result will be useful in applications other than that for which we used it here, we'll isolate it:

LEMMA 2.9. $B(5,53^*) \supseteq \{v : v \equiv 13 \bmod 20$ *and* $v \geq 213\} \backslash \{233, 273, 333, 413, 493, 573, 653, 733, 893\}$.

Of the possible exceptions, we have $\{413, 493, 573, 653, 893\} \subseteq B(5,73^*)$.

If we collect the results of Lemma 2.8 and Theorem 2.7 we get:

THEOREM 2.10. $B(\bar{5},9) \supseteq \{v : v \equiv 1 \bmod 4, v \geq 21, v \neq 29, 33\} \backslash \{49, 73, 93, 113, 133, 153, 173, 193\}$.

Before proceeding to our constructions we point out a useful equivalence between frames and incomplete group divisible designs which we shall have occasion to employ. If (X, G, B) is a frame (of type h^u) then from Lemma 1.1 each hole has degree $\frac{1}{3}h$. We can therefore adjoin $\frac{1}{3}h$ new points to each hole, each new point completing a holey parallel class, to obtain a 5-IGDD of type $(\frac{4}{3}h, \frac{1}{3}h)^u$. The construction is, of course, reversible.

3. Frames with block size four.

We will begin our consideration with the easiest case, namely $h \equiv 3 \bmod 6$. In this case we must have $u \equiv 1 \bmod 4$ (Lemma 2.1). Corollary 2.6 and Construction 2.2 now give the following result.

THEOREM 3.1. *Let* $h \equiv 3 \bmod 6$, $h \neq 9$. *There is a frame of type* h^u *if and only if* $u \equiv 1 \bmod 4$.

There remains the case $h = 9$ to be dealt with. Frames of types 9^5 and 9^9 appear, respectively, in [23] and in the Appendix. From Lemma 2.4, therefore, $B(5,9) \subseteq \{u : \exists$ frame type $9^u\}$. We get the following result.

THEOREM 3.2. *There exists a frame of type* 9^u *if and only if* $u \equiv 1 \bmod 4$, *except possibly when* $u \in \{13, 17, 29, 33, 93, 113, 133, 153, 173, 193\}$.

PROOF. Theorem 2.10, together with the fact that 49 and 73 are in $B(5,9)$. ∎

We turn our attention now to frames with hole size $h \equiv 0 \bmod 12$. Here there are no congruential conditions on u (Lemma 2.1). Note that by Construction 2.2 it is necessary only to consider $h = 12, 24, 36, 72$ and 120. We'll start with $h = 12$, using as our main tool the following.

LEMMA 3.3. *If* $v \in B(\bar{5}, H^4)$ *then there is a frame of type* 12^u *where* $u = \frac{1}{4}(v-1)$.

PROOF. Remove a 5-head from a $(v; \bar{5}, H^4)$-PBD to obtain an H^4-GDD of type 4^u. Now apply Construction 2.3, with weight 3, and appeal to Corollary 2.6. ∎

The following is an immediate consequence of Theorem 2.7.

LEMMA 3.4. $B(\bar{5}, 9, 13) \supseteq \{v : v \equiv 1 \bmod 4, v \geq 21, v \neq 29, 33\} \backslash \{49\}$.

THEOREM 3.5. *There is a frame of type* 12^u *for all* $u \geq 5$, *except possibly for* $u = 8, 12$.

PROOF. Apply Lemmas 3.3 and 3.4. There remains the value $u = 7$ to be dealt with; a frame of type 12^7 is constructed in the Appendix. ∎

We consider now hole size 36. We construct these frames in similar fashion as for hole size 12, except that a weight of 9 is applied instead of weight 3 (see Lemma 3.3).

THEOREM 3.6. *There is a frame of type* 36^u *for all* $u \geq 5$, *except possibly for* $u \in \{7, 8, 12, 18, 23, 28, 33, 38, 43, 48\}$.

PROOF. Let u be given and let $v = 4u + 1$. By Theorem 2.10 we have $v \in B(\bar{5}, 9)$, whence there is a $\{5, 9\}$-GDD of type 4^u. Apply Construction 2.3 to this GDD, using weight 9 (frames of types 9^5 and 9^9 exist by Theorem 3.2). ∎

Hole sizes 24 and 120 are dealt with analogously.

THEOREM 3.7. *There are frames of types* 24^u *and* 120^u *for all* $u \geq 5$, *except possibly for* $u \in \{7, 8, 12\}$.

PROOF. There are frames of type 6^5, 6^9 and 6^{13} (the first can be found in [23]; the second and third are constructed in the Appendix). Let u be given and let $v = 4u + 1$. By Lemma 3.4 we have $v \in B(\bar{5}, 9, 13)$ so that there is a $\{5, 9, 13\}$-GDD of type 4^u. Applying Construction 2.3 to this GDD, using weight 6, gives a frame of type 24^u. If we apply Construction 2.2 (with $m = 5$) to this resulting frame we get a frame of type 120^u. ∎

There remains hole size 72 (in the fibre $h \equiv 0 \bmod 12$); we will postpone discussion on this until the end of the section.

We consider now the case $h \equiv 6 \bmod 12$; in this fibre the number of holes u must be odd (Lemma 2.1). By Construction 2.2 we need consider only $h = 6, 18$. We begin with $h = 6$.

THEOREM 3.8. *There exists a frame of type* 6^u *for all* $u \equiv 1 \bmod 4$, $u \geq 5$, *except possibly for* $u = 17, 29, 33$.

PROOF. Frames of types 6^9 and 6^{13} are constructed in the Appendix, and a frame of type 6^5 can be found in [23]. By Lemma 2.4, $\{u : \exists \text{ frame of type } 6^u\} \supseteq B(5, 9, 13)$. Then from Lemma 3.4 we see that only the value $u = 49$ need be considered; now $49 \in B(5, 9)$ (see [12]), and so the proof is complete. ∎

Frames of type 6^u, where $u \equiv 3 \bmod 4$, seem to be difficult to construct directly. The only 'small' frame that we have is of type 6^{31}, and is obtained by applying Construction 2.3 to a 5-GDD of type 2^{31}. This latter design was kindly provided to us by W. H. Mills (see Appendix). We are nonetheless, able to obtain a reasonable bound on these frames via the following adaptation of Construction 2.3a:

LEMMA 3.9. *Suppose that $m > 8$, that there is a* TD(7, 2*m*)*, and that* $0 \le t \le m$ *where* $t \notin \{4, 7, 8\}$. *Then there is a frame of type* 6^u, *where* $u = 20m + 4t + 31$.

PROOF. Truncate one group in the TD to 15 points and a second to $2t$ points. Now apply Construction 2.3a with $h = 6, \delta = 1$ and weight 12; frames of types 12^5, 12^6 and 12^7 exist by Theorem 3.5, while frames of types 6^{4m+1} and 6^{4t+1} exist by Theorem 3.8. A frame of type 6^{31} exists from the discussion preceding the Lemma. ∎

THEOREM 3.10. *There exists a frame of type* 6^u *for all* $u \ge 531$ *with* $u \equiv 3 \bmod 4$, *except possibly for* $u = 547, 559, 563$.

PROOF. We apply Lemma 3.9 with $m \in S = \{25, 28, 29\} \cup \{n : n \ge 32\}$. It is a simple exercise to verify that given $u \ge 531$, $u \ne 547, 559, 563$, we can always write $u - 31 = 20m + 4t = 4(5m + t)$ for some $m \in S$ and $0 \le t \le m$ with $t \ne 4$, 7 or 8. Since $2m \in$ OA(7) (see [5]), the proof is complete. ∎

In order to obtain some more frames with hole size 6 we can start with a resolvable $(v, 5, 1)$-BIBD, adjoin r points at infinity ($0 \le r < \frac{1}{4}(v-1)$) to obtain a $\{5, 6\}$-GDD of type $5^{v/5}r^1$ and apply Construction 2.3a with $h = 6, \delta = 1$ and weight 36. Frames of types 36^5 and 36^6 exist by Theorem 3.6, while a frame of type 6^{31} exists, as noted previously. All that is needed is a frame of type 6^{6r+1}. If r is even, these frames all exist by Theorem 3.8. Thus we have:

LEMMA 3.11. *If* $v \in$ RB(5) *and* $0 \le 2m < \frac{1}{4}(v-1)$ *then there is a frame of type* 6^u, *where* $u = 6v + 12m + 1$.

REMARK. Note that $u \equiv 7 \pmod{12}$ in the foregoing lemma.

COROLLARY 3.12. *There is a frame of type* 6^u, *where* $u \in \{31, 151, 163, 175, 391, 403, 415, 427, 439, 451, 463, 475, 511, 523, 535, 547, 559\}$.

PROOF. Apply Lemma 3.11 with $v = 5, 25, 65$ and 85. ∎

Note that the foregoing construction admits to a simple generalization. If there is a K-GDD of type $5^t(2m)^1$ and a frame of type 36^k exists for each $k \in K$ then there is a frame of type 6^u, where $u = 30t + 12m + 1$. By employing resolvable transversal designs RTD(5, t) for $t = 9, 11$ and 15 we get the following further values for u in the fibre 7 (mod 12):

t	u
9	271, 283, 295, 307, 319
11	331, 343, 355, 367, 379
15	487, 499

TABLE 3.1

We'll collect the values from Table 3.1 together with those from Corollary 3.12 into the following.

THEOREM 3.13. *There is a frame of type* 6^u *for* $u \in \{31, 151, 163, 175\} \cup \{n : n \equiv 7 \pmod{12}$ *and* $n \geq 271\}$.

PROOF. We need only point out that frames with $u \geq 571$ exist by Theorem 3.10. ∎

Finally, we consider hole sizes 18 and 72. We can obtain the following partial result for hole size 72:

LEMMA 3.14. *There exists a frame of type* 72^u *for all odd* $u \geq 5$ *except possibly for* $u \in \{n : n \equiv 3 \bmod 4$ *and* $n \leq 527\} \cup \{563\}$.

PROOF. If $u \equiv 1 \bmod 4$ apply Construction 2.2, with $m = 24$, to a frame of type 3^u (Corollary 2.6). If $u \equiv 3 \bmod 4$ apply Construction 2.2, with $m = 12$, to a frame of type 6^u (Theorem 3.10 and Corollary 3.12). ∎

There is not much that we can say about hole size 18 at this time; the only frames that we have are of types 18^{13} and 18^{31}. Consequently, we are unable to make headway with hole size 72 where the number of holes is even (72 bears the same relationship to 18 as did 36 to 9 and 24 to 6).

A frame of type 18^{13} is obtained by applying Construction 2.3, with weight 3, to a $\{5, 13\}$-GDD of type 6^{13} (see Appendix), while a frame of type 18^{31} is obtained by applying the same construction, but using weight 9, to Mills' 5-GDD of type 2^{31} (see Appendix).

We'll collect the relevant results of this section to form our main theorem:

THEOREM 3.15. *There is a frame of type* h^u *if and only if* $u \geq 5$, $h \equiv 0 \bmod 3$ *and* $h(u-1) \equiv 0 \bmod 4$, *except possibly where*

(i) $h = 9$ *and* $u \in \{13, 17, 29, 33, 93, 113, 133, 153, 173, 193\}$;

(ii) $h \equiv 0 \bmod 12$ *and* $u \in \{8, 12\}$,
$h = 36$ *and* $u \in \{7, 18, 23, 28, 33, 38, 43, 48\}$,
$h = 24$ *or* 120 *and* $u \in \{7\}$,
$h = 72$ *and* $u \in 2\mathbb{Z}^+ \bigcup \{n : n \equiv 3 \bmod 4$ *and* $n \leq 527\} \cup \{563\}$; *or*

(iii) $h \equiv 6 \bmod 12$ *and* $u \in \{17, 29, 33, 563\} \cup \{n : n \equiv 3$ *or* $11 \bmod 12$ *and* $n \leq 527\} \cup \{n : n \equiv 7 \bmod 12$ *and* $n \leq 259\}$,
$h = 18$.

4. **A new construction for resolvable group divisible designs.** One of the most important direct applications of frames is to the construction of resolvable group-divisible designs. Frames, together with resolvable GDDs, are then very useful in the consideration of certain subdesign problems (which we will discuss in Section 5). The following results are well-known (see [15a], [16], [2]):

THEOREM 4.1 (FOLKLORE). *A resolvable* 2-GDD *of type* g^u *exists if and only if* gu *is even.*

PROOF. Necessity is obvious. For sufficiency, suppose first that u is even. Apply weight g to a resolvable 2-GDD of type 1^u ($\equiv$ one-factorization of K_u). If u is odd then g must be even, and so we can apply weight $g/2$ to a resolvable 2-GDD of type 2^u ($\equiv$ one-factorization of K_{2u}). ∎

THEOREM 4.2. *A resolvable* 3-GDD *of type* g^u *exists if and only if* $gu \equiv 0 \bmod 3$ *and* $g(u-1) \equiv 0 \bmod 2$, *except when* $(g,u) = (2,3), (2,6)$ *or* $(6,3)$, *and possibly when* $(g,u) = (22,6)$.

To our knowledge the only systematic work done regarding resolvable GDDs with block size 4 is the following result, due to H. Shen [21], [22]:

THEOREM 4.3. *There is a resolvable* 4-GDD *of type* 3^u *if and only if* $u \equiv 0 \bmod 4$, $u \geq 8$, *except possibly for* $u \in \{28, 40, 44, 60, 72, 88, 104, 108, 124, 152, 184, 216, 220, 268, 284, 296\}$.

We will present a new construction for resolvable GDDs from frames and then use it to remove several of the exceptional values from Shen's list in Theorem 4.3. Our construction is a generalization of Zhu, Du and Zhang's Theorem 2.1 [29].

CONSTRUCTION 4.4. Suppose that there is a resolvable k-GDD of type g^u, and that there is a k-GDD of type $(mg)^v$ with the following property: its block set admits a partition into at most r_u+r_v colour classes, where $r_u = g(u-1)/(k-1)$ and $r_v = mg(v-1)/(k-1)$, and where each colour class constitutes a set of pairwise disjoint blocks which precisely covers some subcollection of groups. If there is a resolvable TD(k, mv) then there is a resolvable k-GDD of type $(mg)^{uv}$.

Briefly, the construction works as follows. Let (U, G, B) be the first (resolvable) GDD and let (V, H, L) be the second GDD. Let $G = \{G_1, G_2, \ldots, G_u\}$. The finished design will be on the point set $X = U \times \{1, 2, \ldots, mv\}$ with groups $\big\{G_i \times \{m(j-1)+1, m(j-1)+2, \ldots, mj\} : 1 \leq i \leq u, 1 \leq j \leq v\big\}$. We build a copy of (V, H, L) on each 'horizontal' line $G_i \times \{1, 2, \ldots, mv\}$; now replace each block in (U, G, B) by a transversal design so that (U, G, B) is copied onto each 'vertical' line $U \times \{j\} (1 \leq j \leq mv)$ in the grid. The parallel classes on X are of essentially two types. The first type exhausts the blocks in the horizontal GDDs. Specifically, for each colour class C in the colouring ξ of (V, H, L) we create a parallel class on X which looks like $\{C_1, C_2, \ldots, C_u\} \cup \{P_j : j \in S(C)\}$, where C_i is the copy of C on the i^{th} horizontal line and P_j is any (as yet unused) parallel class on the j^{th} vertical line. The set $S(C) = \big\{j \in \{1, 2, \ldots, mv\} : (G_i \times \{j\}) \cap c = \emptyset$ for every $c \in C_i\big\}$, that is, $S(C)$ is the set of indices for the vertical lines which are *not* crossed by any block of the colour class C. Note that the condition $|\xi| \leq r_u + r_v$ insures that no vertical line will be called upon more than r_u times to provide a parallel class. The second type of parallel class on X involves composing the parallel classes of the GDD (U, G, B) with those of the transversal design in the usual way; note that $r_u - (|\xi| - r_v)$ of these parallel classes will be vertical (and so will exhaust the blocks in the vertical GDDs).

If we set $g = m = 1$ in Construction 4.4 we retrive the Zhu, Du and Zhang construction for resolvable BIBDs.

COROLLARY 4.5. *If there is a resolvable* k-GDD *of type* g^u, *a* k*-frame of type* $(mg)^v$ *where* $u \geq m+1$, *and a resolvable* $\mathrm{TD}(k, mv)$ *then there is a resolvable* k-GDD *of type* $(mg)^{uv}$.

PROOF. Use Construction 4.4, noting that a k-frame of type $(mg)^v$ admits an allowable block-colouring of size $\frac{mg}{k-1} \times v$, which is $\leq r_u + r_v$ if and only if $u \geq m+1$. ■

If we specialize Corollary 4.5 to resolvable 4-GDDs of type 3^u we have the following.

LEMMA 4.6. *If there is a resolvable* 4-GDD *of type* 3^u *and* $v \equiv 1 \bmod 4$ *then there is a resolvable* 4-GDD *of type* 3^{uv}.

PROOF. Apply Corollary 4.5 with $k = 4$, $g = 3$ and $m = 1$. A 4-frame of type 3^v exists by Corollary 2.6. ■

Several of Shen's possible exceptions can be factored as $n = u \times v$ where u and v satisfy the hypothesis of Lemma 4.6:

n	$u \times v$
40	8×5
60	12×5
72	8×9
104	8×13
108	12×9
216	24×9
296	8×37

TABLE 4.1

Corollary 4.5 will undoubtedly play an important role in future investigations into resolvable GDDs.

Related to resolvable designs and resolvable group-divisible designs are *resolvable coverings* and *resolvable packings*. A resolvable k-covering (resp. k-packing) on v points ($v \equiv 0 \bmod k$) is a collection of parallel classes of blocks of size k that, among them, covers each pair of the v-set at least (resp. at most) once, and that has minimum (resp. maximum) possible cardinality. Thus for example a resolvable $(v, k, 1)$-BIBD is both a resolvable k-covering and a resolvable k-packing.

Not surprisingly, frames are proving to be very useful in the construction of these designs. Assaf, Mendelsohn and Stinson [3] used Kirkman Frames to determine almost completely the spectrum of sizes of resolvable coverings by triples on $v \equiv 0 \pmod 6$ points:

THEOREM 4.7. *There is a resolvable covering of* $6n$ *points by* $3n$ *parallel classes of triples if and only if* $n \geq 3$, *except possibly when* $n \in \{6, 7, 8, 10, 11, 13, 14, 17, 22\}$.

Note that resolvable packings by triples on $v \equiv 0 \pmod 6$ points are just Nearly Kirkman Triple Systems when $v \geq 18$. When $v = 6$ or 12 a resolvable packing consists of 1 (resp. 4) triangle-factors. The resolvable 4-GDDs of type 3^u (Theorem 4.3) of Shen constitute resolvable 4-packings on $3u \equiv 0 \pmod{12}$ points. Resolvable 4-packings on $2u \equiv 8 \pmod{12}$ points will, except for some small cases, correspond to resolvable 4-GDDs of type 2^u; virtually nothing is known about the spectrum for these designs. Similarly we know of no systematic work concerning resolvable coverings by quadruples. It is anticipated that our current work on frames with block size four will be very helpful in the consideration of these problems.

5. **Other applications of frames.**

Incomplete block designs and incomplete resolvable block designs. As indicated in Section 4, frames (together with resolvable GDDs) have come to play an important role in certain sub-design problems, via the following construction, taken from [17]:

CONSTRUCTION 5.1. Suppose that (X, Y, G, B) is an incomplete GDD and that $w: X \longrightarrow Z^+ \cup \{0\}$ and $d: X \longrightarrow \mathbb{Z}^+ \cup \{0\}$ are functions on X (called *weight* and *degree* functions, respectively) where $d(x) \leq w(x)$ for all $x \in X$. Let a be a fixed non-negative integer and suppose that

(i) for each block $b \in B$ there is a K-IGDD of type $\{(w(x), d(x)) : x \in B\}$,

(ii) there exists a K-IGDD of type $\{(\sum_{x \in G_j \cap Y} w(x), \sum_{x \in G_j \cap Y} d(x)) : G_j \in G\}$, and

(iii) for each group $G_j \in G$ there is a K-GDD on $a + \sum_{x \in G_j} w(x)$ points having a group of size a and a group of size $\sum_{x \in G_j} d(x)$.

Then there is a K-GDD on $a + \sum_{x \in X} w(x)$ points having a group of size $\sum_{x \in X} d(x)$ (and a group of size a).

Incomplete block designs arise by setting $K = \{k\}$ in the above construction. Typically, frames and (resolvable) GDDs are required in the construction of designs satisfying criteria (ii) and (iii). Designs satisfying criterion (i) form a 'mixed bag' which must generally be constructed directly. Thus, for example in the case of $k = 4$ we had to construct 4-IGDDs of types $(9,3)^t 6^{5-t}$ and $(9,3)^s 6^{6-s}$ for all $0 \leq t \leq 5$, $0 \leq s \leq 6$ (see [17]). It is the group of size $\sum_{x \in X} d(x)$ (possibly with some ideal points) that becomes the hole in the incomplete block design.

Consider the following illustration. We will *suppose* that there is a 5-IGDD of type $(16,4)^5 12^1$. Take a transversal design TD(6, 5) and truncate a group to three points. Let y_1, y_2 be two points on the truncated group. Let

$$w(x) = \begin{cases} 16, & x \neq y_1 \text{ or } y_2 \\ 12, & x = y_1 \text{ or } y_2 \end{cases} \text{ and let } d(x) = \begin{cases} 4, & x \neq y_1 \text{ or } y_2 \\ 0, & x = y_1 \text{ or } y_2 \end{cases},$$

and set $a = 4$. Now

(i) there exist 5-IGDDs of types $(16,4)^5$ and $(16,4)^6$ (these are equivalent to frames of types 12^5 and 12^6), and by supposition there is a 5-IGDD of type $(16,4)^5 12^1$.

(ii) this condition is vacuous as $Y = \emptyset$ here.

(iii) there exist 5-GDDs of types $4^{16}20^1$ (adjoin 20 infinite points to a resolvable 4-GDD of type 4^{16}) and $4^{10}4^1$ (remove a point from a (45, 5, 1)-BIBD).

Therefore, there is a 5-GDD of type $4^{85}104^1$. If we now adjoin one ideal point to complete the groups we get a $(445, 105; \{5\})$-IPBD.

Incomplete resolvable block designs arise from Construction 5.1 when we add an extra condition:

(iv) for each $k \in K$ there is a frame of type h^k.

Let us again hypothesize the existence of a 5-IGDD of type $(16,4)^5 12^1$. Then the 5-GDD of type $4^{85}104^1$ constructed previously allows us to construct an incomplete resolvable design with block size four as follows. Apply the frame construction, using weight $h = 3$, to get a frame of type $12^{85}312^1$. Adjoin four ideal points and fill the holes with $(16, 4; \{4\})$-IRBIBDs to get a $(1336, 316; \{4\})$-IRBIBD.

As another illustration let us again start with our TD(6, 5) and truncate a group to three points, but this time assign weight 12 and degree 0 to just one of its points. To satisfy condition (iii) of Construction 5.1 we must find a GDD on 48 points with a group of size 4 and a group of size 8. If it is our objective to construct an incomplete design with block size 5 then we are out of luck, since no such GDD is (yet) known to exist. If instead our objective is to construct an incomplete resolvable design with block size 4 then we are in good shape; start with a 5-GDD of type 8^6 (see [4]) and add a new point to complete the groups, and then delete an old point to get a 5, 9-GDD of type $4^{10}8^1$. The result of Construction 5.1 is now a 5, 9-GDD of type $4^{85}108^1$ (with five blocks of size 9). Now apply weight $h = 3$, using the frames construction, and then adjoin four ideal points to yield a $(1348, 328; \{4\})$-IRBIBD. Note that this corresponds to a maximum embedding, that is, a resolvable $(1348, 4, 1)$-BIBD cannot have a resolvable sub-design on more than 328 points.

For many more examples of applications of Construction 5.1 we refer the reader to [17]. This construction proved instrumental in giving complete solutions for the spectrum of incomplete designs with block size 4 (see [19]) and the spectrum of incomplete resolvable designs with block size 3 (see [18]). The spectrum for incomplete designs with block size 3 had been determined some time previously by Doyen and Wilson [7], and Huang, Mendelsohn and Rosa [11].

We conclude this subsection by noting that Tianwen Cai has shown that a $(v, w; \{4\})$-IRBIBD exists whenever $v \equiv w \equiv 4 \pmod{12}$ and $v \geq 5w - 160$, where $w \geq 340$ [6]. Thus there remain the 'large' embeddings to look at.

*Orthogonal partitioned incomplete latin squares (*OPILS*)* . A Partitioned incomplete latin square (PILS) of order n is an $n \times n$ array A indexed by an n-set S which has been equipped with a partition $P = \{S_1, S_2, \ldots, S_m\}$ where $m \geq 2$. The following properties are to be satisfied:

(i) each cell of A is either empty or contains a symbol from S.

(ii) the subarrays indexed by $S_i \times S_i$ are empty for $1 \leq i \leq m$, (these subarrays are called *holes*) and

(iii) the symbols occuring in row (or column) $s \in S$ are exactly those of $S \backslash S_j$ where $s \in S_j$.

The *type* of a PILS is the multiset $T = \{|S_i| : 1 \leq i \leq m\}$, for which an exponential notation is often used (as for GDDs).

A pair of PILS on the same set-partition pair (S, P) are called *orthogonal* if their superposition yields every pair in $S^2 \backslash (\cup S_i^2)$, and a collection of PILS is termed orthogonal (and denoted a set of OPILS) if each pair in the collection is. OPILS arise naturally in the consideration of many design-theoretical problems; for further discussion on this we refer the reader to Stinson and Zhu [25].

A set of k OPILS of type T is equivalent to an incomplete transversal design $\mathrm{ITD}\big(k+2, (\sum_{x\in T} x, T)\big)$, that is, a transversal design with $k+2$ groups of side $\sum_{x\in T} x (= |S| =$ order of the squares), having a spanning disjoint set of holes of side $x, x \in T$. Now let G be a fixed group in the ITD. Since every block intersects G and the holes induce a partition of G, its removal yields an incomplete $(k+1)$-frame of type $\{(k+1)x : x \in T\}$ having a spanning disjoint set of holes (formerly the groups of the ITD, except G), $k+1$ in number, each hole intersecting each group H in the frame in exactly $|H|/(k+1)$ points (note that this number depends only on the group H, and is independent of the hole concerned). This construction is reversible. If we start with an incomplete $(k+1)$-frame of type T', where $y \equiv 0 \pmod{k+1}$ for each $y \in T'$, having a spanning disjoint $(k+1)$-set of holes each one of which intersects a given group in the same number of points, then we can adjoin sufficiently many points to each group to 'complete' the holey parallel classes corresponding to that group. (Here 'sufficiently many' is easily calculated. If H is a group, then because each hole intersects H in a fixed number (namely $|H|/(k+1)$) of points, the group H has degree $\frac{|H|}{k} - \frac{|H|/(k+1)}{k} = \frac{|H|}{k+1}$ (Lemma 1.1). This is because our incomplete frame can be regarded as a $(k+1)$-frame of type T' having (or missing) a spanning disjoint set of $(k+1)$-subframes of type $T = \{t'/(k+1) : t' \in T'\}$.) Thus the result of adjoining these new points is just an $\mathrm{ITD}\big(k+2, (\sum_{x\in T} x, T)\big)$, where $T = \{y/(k+1) : y \in T'\}$, which in turn give us k OPILS of type T.

The foregoing is the basis for the following construction of Stinson and Zhu [25, Theorem 2.8].

CONSTRUCTION 5.2. If there is an m-frame of type T and k MOLS of order m then there are k OPILS of type T.

PROOF. Apply weight $k+1$ to the frame, replacing each block in the frame by a resolvable $\mathrm{TD}(k+1, m)$. The result is an incomplete $(k+1)$-frame of $T' = \{(k+1)t : t \in T\}$, having a spanning disjoint $(k+1)$-set of holes where each hole intersects a group H of the frame in exactly $|H|/(k+1)$ points. The result follows from the discussion preceding the statement of the construction. ■

Construction 5.2 is similar to the following well-known construction due to Bose: if there is an incomplete resolvable design $(v, w; \{m\})$ IRBIBD and k MOLS of order m then there are k MOLS of order v 'missing' k sub-MOLS of order w (or equivalently, an $\mathrm{ITD}\big(k+2, (v, w)\big)$).

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APPENDIX

Frame of type 9^9 (given as a 5-IGDD of type $(12,3)^9$):

Points: $\{1,2,\ldots,108\}$
Hole: $\{82,83,\ldots,108\}$
Groups: $\{\{9x+y : 0 \le x \le 11\} : 1 \le y \le 9\}$
Blocks: Develop the following base blocks under the group generated by $(1\ 2 \cdots 81)(82\ 83 \cdots 108)$

1	2	9	48	87		1	5	21	42	88
1	3	13	26	83		1	6	23	54	102
1	4	15	30	92		1	7	31	50	99

Frame of type 12^7 (given as a 5-IGDD of type $(16,4)^7$):

Points: $\{1,2,\ldots,112\}$
Hole: $\{85,86,\ldots,112\}$
Groups: $\{\{7x+y : 0 \le x \le 15\} : 1 \le y \le 7\}$
Blocks: Develop the following base blocks under the group generated by $(1\ 2 \cdots 84)(85\ 86 \cdots 112)$

1	2	10	67	89		1	5	30	74	94
1	3	13	35	103		1	6	39	59	86
1	4	17	40	98		1	7	31	48	102

Frame of type 6^9 (given as a 5-IGDD of type $(8,2)^9$):

Points: $\{1,2,\ldots,72\}$
Hole: $\{55,56\ldots,72\}$
Groups: $\{\{9x+y : 0 \le x \le 7\} : 1 \le y \le 9\}$
Blocks: Develop the following base blocks under the group generated by $(1\ 2 \cdots 54)(55\ 56 \cdots 72)$

1	2	4	24	61		1	6	13	41	71
1	5	16	49	69		1	9	26	39	70

Frame of type 6^{13} (given as a 5-IGDD of type $(8,2)^{13}$):

Points: $\{1,2,\ldots,104\}$
Hole: $\{79,80\ldots,104\}$
Groups: $\{\{13x+y : 0 \le x \le 7\} : 1 \le y \le 13\}$
Blocks: Develop the following base blocks under the group generated by $(1\ 2 \cdots 78)(79\ 80 \cdots 104)$

1	2	9	30	90		1	5	19	52	99
1	3	12	56	96		1	6	23	42	80
1	4	16	36	86		1	7	17	55	104

A 5-GDD of type $4^7 8^3$:

Points: $\{1, 2, \ldots, 52\}$
Groups: $\big\{\{1,2,3,4\}, \{5+x, 11+x, 17+x, 23+x\}$
$\{29+y, 32+y, 35+y, 38+y, 41+y, 44+y, 47+y, 50+y\}$:
$0 \le x \le 5, 0 \le y \le 2\big\}$
Blocks: Develop the following base blocks under the group generated by
$(1\ 2\ 3\ 4)(5\ 6 \cdots 16)(17\ 18 \cdots 28)$
$(29\ 30 \cdots 40)(41\ 42 \cdots 52)$ and $(5\ 17)(6\ 18) \cdots$
$(16\ 28)(29\ 41)(30\ 42) \cdots (40\ 52)$

1	5	7	29	30
1	6	20	32	39
5	6	13	26	46
5	8	36	43	47
5	22	35	49	51

A 5-GDD of type 2^{31} (W. H. Mills):

Points: $\mathbb{Z}_{31} \times \{1, 2\}$
Groups: $\big\{\{(x,1), (x,2)\} : x \in \mathbb{Z}_{31}\big\}$
Blocks: Develop the following base blocks modulo 31:

(7, 1)	(14, 1)	(19, 1)	(25, 1)	(28, 1)
(0, 1)	(1, 1)	(4, 2)	(11, 2)	(15, 2)
(0, 1)	(2, 1)	(8, 2)	(22, 2)	(30, 2)
(0, 1)	(4, 1)	(16, 2)	(13, 2)	(29, 2)
(0, 1)	(8, 1)	(1, 2)	(26, 2)	(27, 2)
(0, 1)	(16, 1)	(2, 2)	(21, 2)	(23, 2).

A $\{5, 13\}$-GDD of type 6^{13}:

Points: $\mathbb{Z}_{78}$
Groups: $\big\{\{0+x, 13+x, 26+x, 39+x, 52+x, 65+x\} : 0 \le x \le 12\big\}$
Blocks: Develop the following base blocks modulo 78:
$\{6x : 0 \le x \le 12\}$

0	1	5	16	51
0	2	9	49	70
0	3	23	37	56

Remark: The blocks of size 13 form a parallel class.

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