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FLOW CAUSED BY A POINT SINK IN A FLUID HAVING A FREE SURFACE

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Abstract

The flow caused by a point sink immersed in an otherwise stationary fluid is investigated. Low Froude number solutions are sought, in which the flow is radially symmetric and possesses a stagnation point at the surface, directly above the sink. A small-Froude-number expansion is derived and compared with the results of a numerical solution to the fully nonlinear problem. It is found that solutions of this type exist for all Froude numbers less than some maximum value, at which a secondary circular stagnation line is formed at the surface. The nonlinear solutions are reasonably well predicted by the small-Froude-number expansion, except for Froude numbers close to this maximum.

1. Introduction

This paper is concerned with the steady flow induced by a stationary point sink fixed beneath the free surface of an otherwise quiescent fluid of infinite depth. The fluid will be assumed to be ideal, in the sense that it is incompressible and inviscid and flows irrotationally. A surprising and counter-intuitive consequence of this assumption is that there is now no mechanism for distinguishing between the effects of a sink or a source submerged beneath the surface; the streamlines are identical in each case as is the shape of the free surface, and it is only the direction of flow along the streamlines which is affected. It is therefore possible that (at least) two solution types might exist to this problem, with one type perhaps corresponding to flow produced by a sink, and the other caused by a submerged source. In that case, some

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additional information concerning the flow type sought might need to be provided, before the solution can be commenced.

There is available some experimental evidence for the existence of two different flow types produced by a submerged sink or source. Imberger [9] identified a low-Froude-number type in which a stagnation point was present on the surface directly above the sink, and a high-Froude-number solution type in which the free surface was drawn downwards by the sink to form a cusp. His experiments apparently indicated that, at some critical value of the Froude number, the flow observed in the laboratory flume "jumped" from one solution type to the other. However, the experiments do *not* identify one solution type with a submerged source and the other with a sink, suggesting that the lack of distinction between a source or a sink evident in the mathematical model may actually be a feature of the physical problem.

In the case of *two dimensional* flow, in which the disturbance is a submerged *line* source or sink, there has been some recent success in computing solutions of the second type, where a vertical cusp is present at the surface. Tuck and Vanden-Broeck [16] obtained a numerical solution with a cusped free surface at a unique value of the Froude number, in the case when a line sink is present beneath the surface of a fluid of infinite depth. Their work was extended by Hocking [7] and Vanden-Broeck and Keller [17] to flows in the presence of either a flat or a sloping bottom boundary, and Collings [3], Vanden-Broeck and Keller [17], Hocking [8] and King and Bloor [10] have obtained closed-form solutions for cusped flow due to a submerged line sink at infinite Froude number (corresponding to the acceleration of gravity being exactly zero).

Rather less success has been achieved in the attempted computation of two-dimensional solutions of the first type, in which a stagnation point is present at the surface immediately above the submerged line sink. The problem seems first to have been addressed by Peregrine [14], who sought a solution in the form of a perturbation series in the Froude number based on the source strength and its submergence depth. He obtained plausible-looking solutions having the anticipated free-surface stagnation point, and noted that as the Froude number was increased, waves appeared near the stagnation point. The existence of a maximum Froude number was postulated, beyond which solutions of this type would not be possible. Peregrine's solution was re-examined by Vanden-Broeck, Schwartz and Tuck [18] in a somewhat more general context, and it was shown that the perturbation series in powers of the Froude number is in fact divergent everywhere, so that the solution obtained by Peregrine can at best only serve as an asymptotic approximation to the true solution (if one exists). Tuck and Vanden-Broeck [16] refer briefly to these solutions at the beginning of their paper, and describe preliminary results of their numerical computations, which again indicated the presence of small waves near the stagnation point and suggested that such solutions only existed for Froude numbers less than about two.

In this article we present numerical solutions of the first type, having a free-surface stagnation point, for the *three dimensional* problem involving a submerged point sink or source. In addition, we derive a low-Froude-number solution in Section 3, which is the three-dimensional counterpart to Peregrine's expansion.

Unlike the corresponding two dimensional problem described above, in which solutions with a stagnation point are evidently difficult to obtain, the numerical method to be detailed here has generated reliable axisymmetric solutions in a relatively straightforward manner, apparently for the first time. It is found that there exists a maximum Froude number beyond which it is apparently not possible to compute nonlinear solutions of this type; this breakdown of the numerical scheme is examined in Section 6, with reference to the experimental work of Lawrence and Imberger [12] and some recent numerical work by Blake and Kucera [2].

2. The governing equations

Consider a stationary fluid of density ρ acted upon by the downward acceleration of gravity g and having infinite depth. Locate a Cartesian co-ordinate system such that the z-axis points vertically, and the $x - y$ plane is coincident with the plane of the undisturbed surface of the fluid. It is assumed that the fluid is both incompressible and inviscid.

Suppose a point sink is now located a distance H beneath the origin of the Cartesian coordinate system. The sink has strength $m/4\pi$, so that it produces a total flux m (fluid volume per unit time). After waiting for transients to die away, a steady-state response will be achieved, in which the free surface of the fluid no longer occupies the plane $z = 0$, but instead will have some other shape to be determined. For the branch of solutions sought here, the free surface will possess a stagnation point at the origin of the coordinate system.

It is convenient to define dimensionless variables immediately, and these will be used exclusively from now on. All lengths are made dimensionless with respect to the submergence depth H of the source beneath the origin, and velocities are referred to the scale m/H^2. Because the fluid is assumed to be ideal, it may be taken to flow irrotationally, and so the fluid velocity vector may be written as the gradient of a velocity potential Φ; this function is non-dimensionalised with respect to the quantity m/H. It is apparent that solutions to this problem depend only on the single dimensionless parameter

$$F^2 = m^2/gH^5, \tag{2.1}$$

which is the square of the Froude number F.

The anticipated solution to this problem is required to possess axial symmetry, and for this reason it is natural to introduce cylindrical polar coordinates (r, θ, z) at this stage. These are related to the Cartesian coordinates x and y by the usual formulae $x = r\cos\theta$, $y = r\sin\theta$. The incompressibility of the fluid leads at once to the requirement that the velocity potential $\Phi(r, z)$ satisfy Laplace's equation

$$\nabla^2\Phi = \Phi_{rr} + (1/r)\Phi_r + \Phi_{zz} = 0 \tag{2.2}$$

in which the subscript variables denote differentiation with respect to that variable. Equation (2.2) holds everywhere within the fluid region, except at the submerged sink where Φ becomes singular according to

$$\Phi \to \frac{1}{4\pi}\frac{1}{[r^2 + (z+1)^2]^{1/2}} \text{ as } (r, z) \to (0, -1). \tag{2.3}$$

Let the shape of the free surface be given by $z = \zeta(r)$; then the statement that the fluid is not free to cross its own surface leads to the kinematic condition

$$\Phi_z = \Phi_r\zeta_r \qquad \text{on } z = \zeta(r). \tag{2.4}$$

In addition, Bernoulli's equation within the fluid coupled with the requirement that the pressure on the surface be everywhere equal to atmospheric pressure gives rise to the dynamic free-surface condition

$$\frac{1}{2}F^2(\Phi_r^2 + \Phi_z^2) + z = 0 \qquad \text{on } z = \zeta(r), \tag{2.5}$$

where F is the Froude number defined in (2.1). The system of equations (2.2)–(2.5) coupled with the statement that Φ and its first derivatives vanish at infinity represents the complete mathematical model of the phenomenon. As explained in the introduction, however, such a model may not possess a unique solution.

We now derive an integral equation for the velocity potential Φ at the free surface. Let point Q on the free surface of the fluid be a fixed point with coordinates $(x, y, \zeta(x, y))$ in the Cartesian system and $(r, \theta, \zeta(r))$ in cylindrical polars. Define a second point P which is free to move about within the volume V shown in Figure 1; in Cartesians, P has coordinates (ξ, η, μ) and in cylindrical polars, it is represented as $P(\rho, \beta, \mu)$, where the usual relations $\xi = \rho\cos\beta$, $\eta = \rho\sin\beta$ apply. The volume V encompasses the entire fluid region, with the exception of the surface point Q which is excluded by a small hemispherical surface S_Q centred at Q. The sink at the

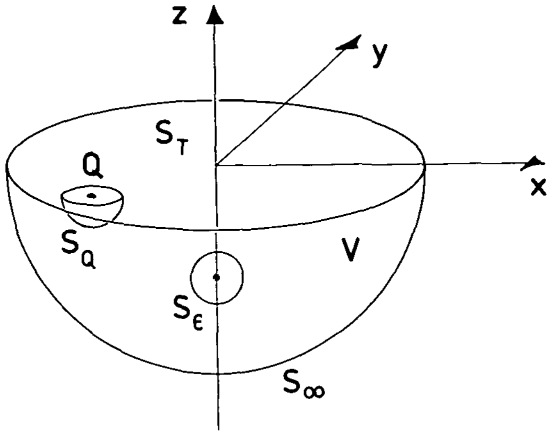

FIGURE 1. Volume V and its bounding surfaces used in the derivation of the integral equation.

point $(x, y, z) = (0, 0, -1)$ is also excluded from volume V by the small spherical surface S_ε which is centered at this point. If the distance between points P and Q is written R_{PQ}, then the function

$$\begin{aligned}\frac{1}{R_{PQ}} &= \frac{1}{[(x-\xi)^2 + (y-\eta)^2 + (z-\mu)^2]^{1/2}} \\ &= \frac{1}{[r^2 + \rho^2 - 2r\rho\cos(\beta - \theta) + (z-\mu)^2]^{1/2}}\end{aligned} \tag{2.6}$$

is thus harmonic within the volume V shown in Figure 1. Therefore, by Green's second identity, we may write

$$\iint_{\partial V} \left[\Phi \frac{\partial}{\partial n}\left(\frac{1}{R_{PQ}}\right) - \frac{1}{R_{PQ}} \frac{\partial \Phi}{\partial n}\right] dS = 0, \tag{2.7}$$

where $\mathbf{n}$ denotes the unit normal to the boundary surface ∂V of volume V, chosen to point out of the fluid region. The closed boundary surface ∂V may be written

$$\partial V = S_T + S_Q + S_\varepsilon + S_\infty, \tag{2.8}$$

in which S_T denotes the entire fluid free surface with a small circular disc about the point Q excluded, S_Q is a small hemispherical surface centred at Q, S_ε is a small spherical surface centred at the sink point and S_∞ is a hemispherical surface of arbitrarily large radius centred at the origin of the Cartesian coordinate system. These surfaces are all displayed in Figure 1.

It is necessary to evaluate the contribution to the integral in (2.7) from each of the surfaces in (2.8). Far away from the point sink, the velocity potential must behave like the function $1/2\pi R$ as $R \to \infty$, which follows from (2.3) and the method of images, since we are seeking a solution with a free surface stagnation point. Accordingly, the contribution from surface S_∞ becomes

$$\lim_{R\to\infty} 2\pi R^2 \left[-\frac{1}{2\pi R}\left(\frac{1}{R^2}\right)+\frac{1}{R}\left(\frac{1}{2\pi r^2}\right)\right],$$

which is zero. A similar argument shows that, as the hemispherical surface S_Q is allowed to shrink to the point Q, the contribution to the integral from this surface becomes $2\pi\Phi(Q)$, and that the contribution from surface S_ε becomes $-[r^2+(z+1)^2]^{-1/2}$ as the radius of this surface is allowed to approach zero. This then leads to the desired integral equation for Φ at the free surface, in the form

$$2\pi\Phi(Q) = \frac{1}{[r^2+(z+1)^2]^{1/2}} - \iint_{S_T} \Phi(P)\frac{\partial}{\partial n_P}\left(\frac{1}{R_{PQ}}\right) dS_P, \tag{2.9}$$

in which the kinematic surface condition (2.4) has been used to set the quantity $\partial\Phi/\partial n$ to zero, on the punctured free surface S_T.

It is convenient at this stage to remove the singularity in the integrand of the integral appearing in (2.9). This is done in the usual way, by adding and subtracting a term which has the same degree of singularity as that already present in the integrand. Thus the integral on the right hand side of (2.9) is re-written in the form

$$\begin{aligned}\iint_{S_T} \Phi(P)\frac{\partial}{\partial n_P}\left(\frac{1}{R_{PQ}}\right) dS_P &= \iint_{S_T} [\Phi(P)-\Phi(Q)]\frac{\partial}{\partial n_P}\left(\frac{1}{R_{PQ}}\right) dS_P \\ &\quad + \Phi(Q)\iint_{S_T} \frac{\partial}{\partial n_P}\left(\frac{1}{R_{PQ}}\right) dS_P. \end{aligned}\tag{2.10}$$

A straightforward Taylor expansion shows that the integrand of the first integral on the right hand side of (2.10) is now nonsingular, as intended. The second integral may be evaluated in closed form, using a device based upon Gauss' flux theorem and apparently first proposed by Landweber and Macagno [11]. Since the function $1/R_{PQ}$ defined in (2.6) is harmonic within the volume V shown in Figure 1, we have

$$\iint_{\partial V} \frac{\partial}{\partial n_P}\left(\frac{1}{R_{PQ}}\right) dS_P = 0, \tag{2.11}$$

in which the boundary surface ∂V is made up of the four component surfaces given in (2.8). The contribution from each of these surfaces to the integral in (2.11) must again be evaluated. Proceeding as before, we find that the surfaces S_∞ and S_Q contribute amounts -2π and 2π respectively, as might be expected on physical grounds, and that the surface S_ε gives a zero contribution to the integral. Thus (2.11) shows that the integral over surface S_T is zero, so that the second integral on the right hand side of (2.10) may be deleted. This method of treating the singular integrals arising in a more general three-dimensional free-surface calculation has also been employed recently by Forbes [6].

In view of the fact that the integral in (2.9) has now been rendered nonsingular, the domain of integration may be taken to be the whole free surface, instead of the punctured surface S_T. The surface integral is then evaluated using the familiar result

$$dS_P = \frac{d\xi\, d\eta}{|\mathbf{n}.\mathbf{k}|} = \frac{\rho\, d\rho\, d\beta}{|\mathbf{n}.\mathbf{k}|},$$

in which the upward-pointing normal $\mathbf{n}$ to the free surface is given in cylindrical polar coordinates as

$$\mathbf{n} = \frac{-\zeta'(\rho)\mathbf{e}_\rho + \mathbf{k}}{[1 + (\zeta'(\rho))^2]^{1/2}}$$

and the symbol $\mathbf{e}_\rho$ denotes the unit vector in the radial direction ρ. After a little algebra, the integral equation (2.9) is obtained in the form

$$2\pi\Phi(Q) = \frac{1}{[r^2 + (z+1)^2]^{1/2}} - \int_0^\infty [\Phi(P) - \Phi(Q)]\mathbb{K}(a, b, c, d)\, d\rho, \quad (2.12a)$$

where the kernel is given by

$$\mathbb{K}(a, b, c, d) = \rho \int_0^{2\pi} \frac{a - b\cos(\beta - \theta)}{[c - d\cos(\beta - \theta)]^{3/2}}\, d\beta \qquad (2.12b)$$

and we have defined auxiliary functions

$$a = \rho\zeta_\rho(P) - (\zeta(P) - \zeta(Q)), \qquad b = r\zeta_\rho(P),$$
$$c = \rho^2 + r^2 + (\zeta(P) - \zeta(Q))^2, \qquad d = 2r\rho. \qquad (2.12c)$$

We show in the appendix that the kernel defined in (2.12b) may be re-written in a form more convenient for numerical computation as

$$\mathbb{K}(a, b, c, d) = \frac{4\rho}{d\sqrt{c+d}}\left[bK\left(\frac{2d}{c+d}\right) + \frac{ad - bc}{c - d}E\left(\frac{2d}{c+d}\right)\right], \quad (2.13)$$

where K and E are the complete elliptic integrals of the first and second kinds respectively, as defined by Abramowitz and Stegun [1, page 590].

The numerical solution of this problem is accomplished most efficiently by associating with the surface point Q an arclength s along the surface, as described by Miksis, Vanden-Broeck and Keller [13] and Forbes [5]. We assume that $s = 0$ at $r = 0$, and define s according to the relation

$$\left(\frac{dr}{ds}\right)^2 + \left(\frac{d\zeta}{ds}\right)^2 = 1. \tag{2.14}$$

A surface velocity potential ϕ is now defined as $\phi(r(s)) = \Phi(r, \zeta(r))$, and it follows at once from the chain rule of calculus that

$$\frac{\partial\phi}{\partial r} = \Phi_r(r, \zeta) + \Phi_z(r, \zeta)\frac{d\zeta}{dr}. \tag{2.15}$$

The kinematic condition (2.4) is used to eliminate the velocity Φ_z from the Bernoulli equation (2.5) and the relation (2.15), which are then combined to give a single condition

$$\frac{1}{2}F^2\left(\frac{d\phi}{ds}\right)^2 + \zeta(s) = 0 \tag{2.16}$$

to be applied along the free surface. The final form of the integral equation (2.12) in terms of arclength is

$$2\pi\phi(s) = \frac{1}{[r^2(s) + (\zeta(s)+1)^2]^{1/2}} - \int_0^\infty [\phi(\sigma) - \phi(s)]\mathbb{K}(A, B, C, D)\,d\sigma, \tag{2.17a}$$

where σ is the arclength associated with surface point P and

$$A = r(\sigma)\zeta'(\sigma) - r'(\sigma)[\zeta(\sigma) - \zeta(s)], \qquad B = r(s)\zeta'(\sigma),$$
$$C = r^2(\sigma) + r^2(s) + (\zeta(\sigma) - \zeta(s))^2, \qquad D = 2r(s)r(\sigma). \tag{2.17b}$$

Equations (2.14), (2.16) and (2.17) thus represent a complete statement of the problem to be solved.

3. Small-Froude-number expansion

An asymptotic approximation valid for small Froude numbers is derived after the fashion of Peregrine [14], assuming the regular expansion in powers of the Froude number

$$\Phi(r, z) = \Phi_0(r, z) + F^2\Phi_1(r, z) + O(F^4)$$
$$\zeta(r) = F^2 Z_1(r) + F^4 Z_2(r) + O(F^6). \tag{3.1}$$

The expansions (3.1) are substituted into the system of equations (2.2)–(2.5) and at the first order, it is found that the potential Φ_0 satisfies the Laplace equation (2.2) and the normal-derivative condition $\partial\Phi_0/\partial z = 0$ on the plane

$z = 0$. The method of images, subject to the condition (2.3), at once yields the solution for Φ_0 in the form

$$\Phi_0(r, z) = \frac{1}{4\pi}\left[\frac{1}{[r^2 + (z+1)^2]^{1/2}} + \frac{1}{[r^2 + (z-1)^2]^{1/2}}\right], \tag{3.2}$$

and the first-order approximation to Bernoulli's equation gives

$$Z_1(r) = -\frac{1}{2}\Phi_{0_r}^2(r, 0) = -\frac{r^2}{8\pi^2(r^2+1)^3}. \tag{3.3}$$

At the second order, the potential function Φ_1 is again found to satisfy the Laplace equation (2.2) in the lower half-space $z < 0$, and the approximation to the kinematic condition (2.4) at this order becomes

$$\Phi_{1_z}(r, 0) = Z_1'(r)\Phi_{0_r}(r, 0) - Z_1(r)\Phi_{0_{zz}}(r, 0) = \frac{r^2(4 - 5r^2)}{16\pi^3[r^2+1]^{11/2}}. \tag{3.4}$$

It is clear that the solution of Laplace's equation with radial symmetry must involve the Hankel transform of order zero, in the form

$$\Phi_1(r, z) = \int_0^\infty M(k)J_0(kr)e^{kz}\,dk, \tag{3.5}$$

where J_0 denotes the Bessel function of order zero and $M(k)$ is a function to be determined. When (3.5) is substituted into (3.4), there results the equation

$$\int_0^\infty kM(k)J_0(kr)\,dk = \frac{r^2(4-5r^2)}{16\pi^3[r^2+1]^{11/2}}, \tag{3.6}$$

from which $M(k)$ may be determined immediately, using the fact that the Hankel transform and its inverse are symmetric. Therefore,

$$M(k) = \frac{1}{16\pi^3}\int_0^\infty r\left[-\frac{5}{[r^2+1]^{7/2}} + \frac{14}{[r^2+1]^{9/2}} - \frac{9}{[r^2+1]^{11/2}}\right]J_0(kr)\,dr, \tag{3.7}$$

where partial fraction decomposition of the right-hand side of (3.6) has been employed. It turns out that each of the integrals in (3.7) may be evaluated in closed form using Prudnikov *et al.* [15, page 179 formula 29], for example. The result involves modified Bessel functions of the second kind of negative half-integer order, but since these are elementary functions, some algebra yields the final solution

$$M(k) = \frac{e^{-k}(4k^2 + 4k^3 - k^4)}{1680\pi^3}. \tag{3.8}$$

The solution for Φ_1 is given by (3.5) and (3.8), and may be written

$$\Phi_1(r, z) = \frac{1}{1680\pi^3} \int_0^\infty [4k^2 + 4k^3 - k^4] e^{-k(1-z)} J_0(kr)\, dk. \tag{3.9}$$

Each integral in this expression can again be evaluated in closed form, using Erdélyi *et al.* [4, page 182 formula 9], for example, and involves Legendre functions of the first kind of integer order. Since these functions are simply polynomials in this case, the solution may again be written in terms of elementary functions after some algebra, according to

$$\Phi_1(r, z) = \frac{1}{1680\pi^3} \left[-\frac{4}{R_1^3} + \frac{12(1-z)^2 - 36(1-z) - 9}{R_1^5} + \frac{60(1-z)^3 + 90(1-z)^2}{R_1^7} - \frac{105(1-z)^4}{R_1^9} \right], \tag{3.10}$$

where we have defined $R_1 = [r^2 + (1-z)^2]^{1/2}$ for convenience.

The second-order approximation to Bernoulli's equation (2.5) becomes

$$Z_2(r) = -\Phi_{0_r}(r, 0)\Phi_{1_r}(r, 0),$$

from which it is possible to compute the second-order term in the expansion for the surface elevation in a straightforward manner. Inserting the result into (3.1) gives the surface profile

$$\zeta(r) = -\frac{F^2 r^2}{8\pi^2 (r^2+1)^3} + \frac{F^4 r^2}{1120\pi^4} \left[\frac{4}{(r^2+1)^4} + \frac{55}{(r^2+1)^5} - \frac{350}{(r^2+1)^6} + \frac{315}{(r^2+1)^7} \right] + O(F^6). \tag{3.11}$$

It will be seen later that (3.11) provides a good estimate of the surface shape over much of the interval of values of F for which solutions of this type can be found.

4. Numerical methods

This section gives a brief summary of the numerical method used to solve the nonlinear system of equations (2.14), (2.16) and (2.17). The domain $0 \le s < \infty$ of the independent variable s is truncated to some finite interval over which N equally-spaced numerical grid-points $s_1 = 0, s_2, \ldots, s_N$ are placed, separated by uniform point spacing h. The dependent variables are represented by a set of discrete values at these grid points. At the first grid point, the condition $r_1 = 0$ at $s = s_1$ is imposed.

An initial guess is now made for the unknown values of the surface velocity potential, $\phi_1, \phi_2, \dots, \phi_N$ at each free-surface grid point, and these will eventually be updated by a Newtonian iteration scheme. All the other dependent variables are next computed, on the basis of this guess for the function ϕ. The derivatives $\phi'_1, \phi'_2, \dots, \phi'_N$ are obtained by exact differentiation of a cubic spline fitted through the assumed values for ϕ, and the surface condition (2.16) then yields the surface elevations $\zeta_1, \zeta_2, \dots, \zeta_N$. Notice that there is no special treatment given to the first grid point $s = s_1$ in spite of the fact that a stagnation point is expected there. For this reason, the present numerical method is not restricted just to solutions involving stagnation points, and should be capable of detecting other solution branches, if these exist.

The surface elevation values thus obtained are differentiated using cubic splines to give $\zeta'_1, \zeta'_2, \dots, \zeta'_N$, and (2.14) provides an immediate means for the calculation of $r'_1, r'_2, \dots, r'_N$. Finally, a cubic spline is fitted through these values and integrated exactly to generate the quantities $r_2, \dots, r_N$, using the additional requirement $r_1 = 0$ as an initial condition.

The initial estimate for the values of the surface velocity potential ϕ is now updated iteratively, using Newton's method to enforce the satisfaction of the integral equation (2.17a) at each of the mesh points $s_1, s_2, \dots, s_N$. The integral in (2.17a) is evaluated numerically by first truncating its domain of integration to the finite interval $0 \leq s \leq s_N$, and then approximating the resulting proper integral using exact integration of a cubic spline which interpolates values of the integrand at the grid points. Since the singularity has been removed from the integrand in (2.17a), using the device described in (2.10), no difficulty is therefore encountered when $\sigma = s$ in (2.17a). In fact, the kernel function $\mathbb{K}$ has a logarithmic singularity in the limit $\sigma \to s$, but this is multiplied by the quantity $\phi(\sigma) - \phi(s)$, so that the value of the integrand at $\sigma = s$ is zero. The integral equation (2.17) thus leads to a system of N equations in the N unknowns ϕ_j, $j = 1, 2, \dots, N$, which is solved by the damped Newtonian algorithm described by Forbes [5].

5. Presentation of results

There is a one-parameter family of solutions to this problem, dependent only upon the Froude number F defined in (2.1). We have obtained numerical solutions for many different Froude numbers, and three of these are presented in Figure 2. Here, surface profiles are shown for the three Froude numbers $F = 4$, $F = 5$ and $F = 6.4$. These profiles show the presence

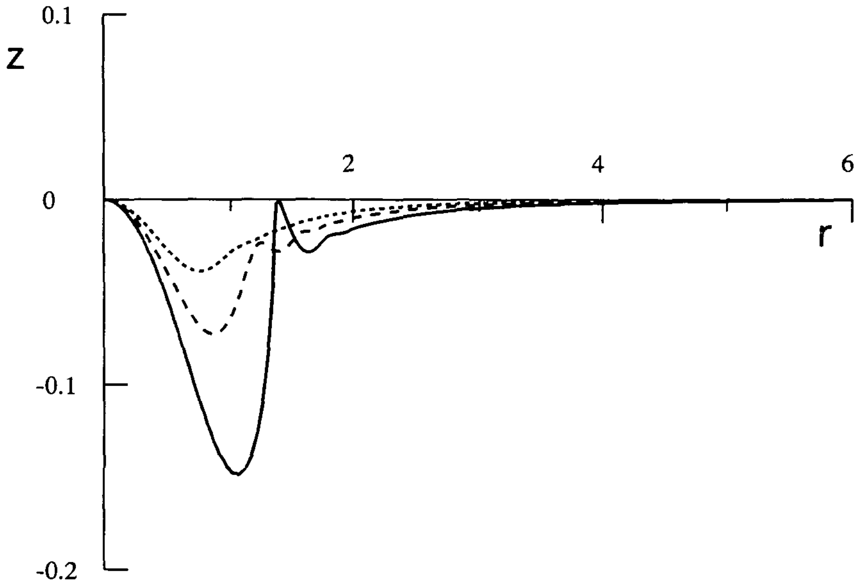

FIGURE 2. Three nonlinear solutions obtained numerically, for Froude numbers $F = 4$ (dotted line), $F = 5$ (dashed line) and $F = 6.4$ (solid line).

of the stagnation point at $r = 0$, and each possesses a local minimum at about $r = 1$, with an asymptotic return to the plane $z = 0$ as $r \to \infty$, as expected. When $F = 4$, the nonlinear solution is in excellent agreement with the predictions of the low-Froude-number expansion of Section 3, except for a small region near $r = 1$, in which a very small wavelet appears in the nonlinear profile. At $F = 5$, this subsidiary wavelet has grown in amplitude, and the agreement between the low-Froude-number expansion and the nonlinear solution is only moderate. As the Froude number F is increased, the amplitude of this secondary wavelet increases substantially, until the value $F = 6.4$ is reached, at which point the crest of the secondary wave is very close to the maximum value $z = 0$ allowed by Bernoulli's equation (2.5). Thus the solution shown in Figure 2 for $F = 6.4$ is evidently close to some limiting configuration, and represents the largest value of F for which the numerical scheme of Section 4 was capable of yielding a converged solution.

Figure 2 shows that steady solutions of this type, in which a stagnation point is present at the surface at $r = 0$, are only possible in the approximate interval $0 \leq F \lessapprox 6.4$. The physical significance of this result seems clear, since for the limiting profile obtained with $F \approx 6.4$, there is a subsidiary wave crest at about $r = 1.4$ at the maximum elevation $z = 0$. By Bernoulli's equation (2.5), the fluid must therefore form a circular stagnation line at about $r = 1.4$, where the fluid speed becomes zero. Any attempt to increase the Froude number beyond this maximum would presumably give rise to a circular breaking wave crest at about $r = 1.4$, which is an inherently unsteady phenomenon unable to be described by the present steady equations.

It is necessary to comment upon the effects of numerical error in the solution profiles shown in Figure 2. For moderate Froude numbers in the approximate interval $0 \leq F < 5$, the numerical method gives results of great accuracy over the entire computational domain. When the interval $0 \leq s \leq 6$ is chosen as the computational window, as in Figure 2, solutions obtained with $N = 61$ grid points typically agree to three significant figures with solutions obtained with $N = 121$ points, and these in turn typically agree to at least four significant figures with solutions obtained with $N = 241$ mesh points. At the very large Froude numbers close to the maximum, this high level of accuracy persists over most of the computational domain, except in a small region about the minimum surface elevation, where small waves appear at this surface trough. Similar waves were reported by Tuck and Vanden-Broeck [16] in the attempted numerical solution of the corresponding two-dimensional problem. However, it is clear in the present problem that these small waves near the trough have no physical significance, and their amplitude may be made arbitrarily small by decreasing the numerical grid spacing. For the solution shown in Figure 2 at the maximum Froude number $F = 6.4$, it was found necessary to employ $N = 241$ grid points to suppress these numerically-produced wavelets, although some evidence of them may still be visible near the trough in Figure 2.

In Figure 3, the solution obtained at the maximum Froude number $F = 6.4$ (drawn with a solid line) is contrasted with the surface profile predicted by the low-Froude-number expansion in (3.11) (sketched with a dashed line). For this extreme case, the agreement between the two sets of results is not good, in particular since the low-Froude-number expansion is not capable of predicting the subsidiary stagnation line formed at about $r = 1.4$ in the nonlinear profile. Nevertheless, the approximate solution (3.11) is clearly still accurate away from the submerged point sink, in the approximate region $r > 2$, even for this limiting case. We therefore conclude that the low-Froude-number expansion of Section 3 predicts the true nonlinear solution very well for Froude numbers in the interval $0 \leq F \leq 4$, and indeed gives an accurate approximation to the region away from the submerged point sink for all Froude numbers at which solutions of this type may be found.

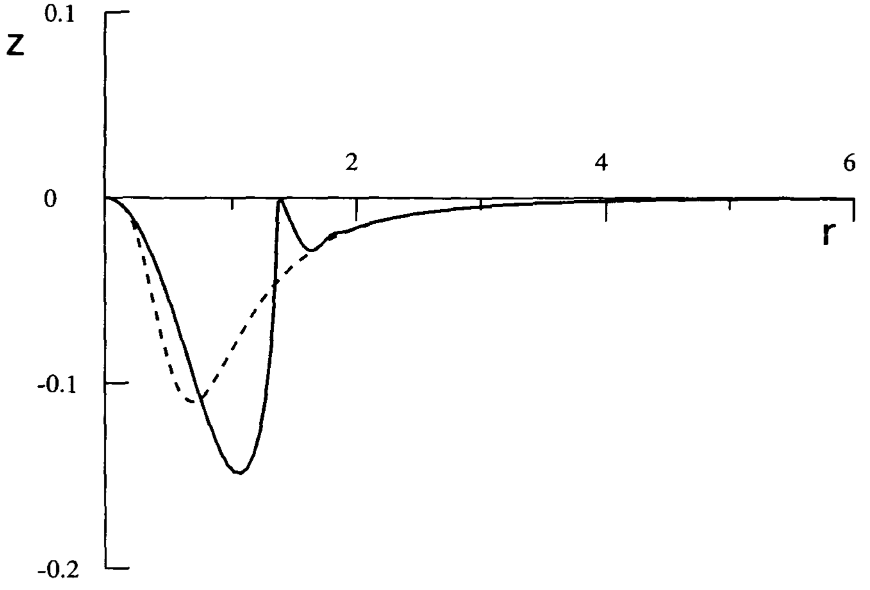

FIGURE 3. Comparison of the free-surface elevation predicted by the approximate solution (3.11) (shown with a dashed line) with the fully nonlinear solution (solid line), for the case $F = 6.4$.

A perspective view of the axi-symmetric fluid surface is given in Figure 4, for the limiting case $F = 6.4$. Here, we have amplified the vertical z scale by a factor of 20, for ease of viewing. In addition, portions of the surface hidden from view have not been drawn, to avoid the figure becoming excessively cluttered. Only the portion $0 \leq s \leq 2$ of the numerical solution shown in Figures 2 and 3 has been shown, in order to focus attention on the region of interest near the submerged point sink. The primary stagnation point at the origin is clearly visible, as is the secondary stagnation line formed at about $r = 1.4$.

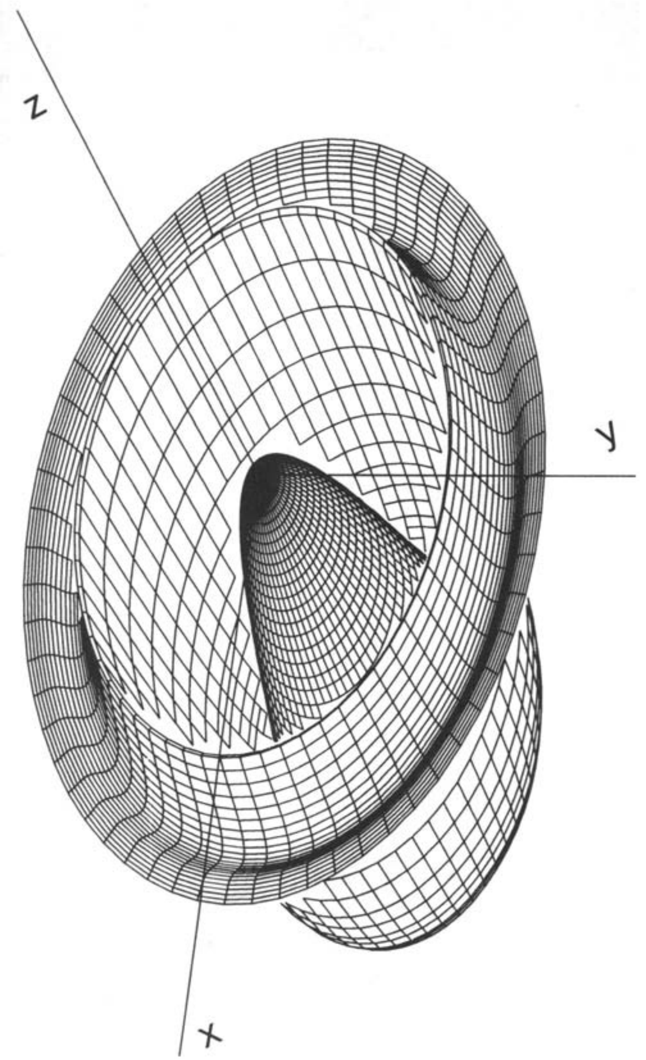

FIGURE 4. Perspective view of a portion of the surface, for the case $F = 6.4$. The vertical scale has been amplified by a factor of 20.

6. Summary and discussion

The flow produced in an otherwise stationary ideal fluid by a submerged point sink must surely represent one of the simplest of all free-surface problems, and yet its solution possesses a complicated nonlinear limiting behaviour. We have formulated the problem in terms of an integral equation, and presented a successful numerical method for its solution. In addition, an approximate solution has been derived in terms of elementary functions, that nevertheless gives good agreement with the nonlinear results for most Froude numbers, except those close to the maximum which occurs at about $F = 6.4$. This limiting Froude number is evidently associated with the formation of a circular line of stagnant fluid at the maximum elevation $z = 0$, at about $r = 1.4$.

Whether or not steady solutions exist for Froude numbers beyond $F = 6.4$ remains an open question. In the corresponding two-dimensional problem, in which a *line* sink is submerged beneath the surface, a second steady solution type is known at large Froude number. It possesses a vertical cusp at the surface, above the submerged sink; such solution types have even been found at infinite Froude number, as discussed in the introduction. In the present three-dimensional problem, however, solutions of this type have not been detected, even although the numerical method of Section 4 should be capable of computing them if they exist.

It may be the case that steady solutions possessing a cusped free surface do in fact exist for $F > 6.4$, although they have not yet been found with the present numerical solution scheme. In fact, Blake and Kucera [2] have recently computed solutions of this general type in a somewhat similar problem arising in the study of oil reservoirs. In that case, it might be expected that an experiment in which the Froude number was continuously increased through the value $F = 6.4$ would show a sudden "jump" from the solutions found in this paper to this other branch having a cuspoid at the free surface. Such behaviour is expected in the corresponding two-dimensional problem, as the pilot experiments of Imberger [9] indicate. This argument suggests that the value $F = 6.4$ is associated with a sudden transition from one stable steady solution branch to some other branch, similar to the "jump phenomenon" familiar from the study of dynamical systems. Such a proposition could be tested experimentally by seeking the value of the Froude number at which a particular solution type breaks down. This has been attempted by Lawrence and Imberger [12] using a system of two fluid layers of different density. They measured the Froude number at which a withdrawal type solution failed, so that the interface between the fluid layers was drawn into the sink, along with fluid from each fluid layer. Seven experiments were performed, from which we have computed the value of the critical Froude number, and we find it to lie between 2.3 and 4.3, with wide scatter in the results. These results are clearly inconclusive, due at least in part to the influence of finite interface thickness upon the experimental values; however, these results by no means invalidate the above proposition, and further experimental work is indicated.

Of course, *unsteady* solutions are always a possibility for $F > 6.4$. Indeed, the results of the present investigation suggest that one such unsteady type involves a circular breaking wave at about $r = 1.4$. Another possibility is that the surface may be drawn down into the sink in an unsteady manner. In the present problem, it is unclear whether this unsteady type of withdrawal flow is the only permissible outcome for large Froude number, in contrast to the corresponding two-dimensional problem described in the introduction. Work is currently in progress to compute both steady and unsteady solutions of the withdrawal type, having a cuspoid at the free surface, and the results will be reported elsewhere.

7. Acknowledgements

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Appendix—transformation of the kernel integral

We show here that the kernel given by the integral (2.12b) may be expressed in terms of complete elliptic integrals according to (2.13). To begin, the argument $\beta - \theta$ of the cosine function may be replaced simply with β, because of the 2π-periodicity of this trigonometric function. This formally eliminates the dependence on the angle θ, as expected from the axi-symmetric assumption. Since the integrand is now even, we have

$$\mathbb{K}(a, b, c, d) = 2\rho \int_0^{\pi} \frac{a - b\cos\beta}{[c - d\cos\beta]^{3/2}}\, d\beta. \tag{A-1}$$

The change of integration variable $u^2 = c - d\cos\beta$ in (A-1) gives the form

$$\mathbb{K}(a, b, c, d) = 4\rho[bI_1([p, q) + (ad - bc)I_2(p, q)]/d, \tag{A-2}$$

where

$$I_1(p, q) = \int_p^q \frac{du}{[(q^2 - u^2)(u^2 - p^2)]^{1/2}}$$
$$I_2(p, q) = \int_p^q \frac{du}{u^2[(q^2 - u^2)(u^2 - p^2)]^{1/2}} \tag{A-3}$$

and we have defined $p^2 = c - d$ and $q^2 = c + d$ for convenience.

In the first integral in (A-3), the change of variable $t^2 = (q^2 - u^2)/(q^2 - p^2)$ easily yields the result

$$I_1(p, q) = \frac{1}{q} K\left(\frac{q^2 - p^2}{q^2}\right), \tag{A-4}$$

in which K denotes the complete elliptic integral of the first kind as defined by Abramowitz and Stegun [1, page 590]. The substitution

$$t^2 = u^{-2}q^2(u^2 - p^2)/(q^2 - p^2)$$

in the second integral in (A-3) gives, after some algebra, the simplification

$$I_2(p, q) = \frac{1}{p^2 q} E\left(\frac{q^2 - p^2}{q^2}\right), \tag{A-5}$$

where the complete elliptic integral of the second kind is denoted by the symbol E. Equation (2.13) in the text now follows from (A-2), (A-4) and (A-5), after the intermediate functions p and q are eliminated in favour of c and d.