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CHARACTERIZATIONS OF MAXIMAL TOPOLOGIES

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Abstract

New characterizations of maximal topologies are given for a class of topologies including the compact, Lindelöf and m-compact topologies.

In what follows, let λ represent a property of topological spaces. If a space (X, T) has property λ, we will say that T is a *λ-topology* on X and call the space a *λ-space.* For certain choices of λ, characterizations have been given for maximal elements in the set of λ-topologies ordered by inclusion. For example, the following has been proved where λ is replaced by either of the words, "compact," "Lindelöf," or "countably compact," Joseph (1969), Raha (1973), Smythe and Wilkins (1963).

THEOREM. *The following statements are equivalent for a space (X, T).*

1) *T is a maximal λ-topology on X.*

2) *The set of λ-subspaces of $X =$ the set of closed subsets of X.*

3) *Any continuous bijection from a λ-space to X is a homeomorphism.*

Recently, the following result has been obtained, Joseph (to appear):

THEOREM. *The following statements are equivalent to the statements in the above theorem with λ replaced by "compact."*

4) *Any continuous surjection from a compact space to X is a closed quotient map.*

5) *Any function with a compact graph from X to a space is continuous.*

6) *Any function with a compact graph from a space to X is closed when T is a compact topology.*

We note that the following statements are true of a λ-space (X, T) with λ replaced throughout by either "compact," "Lindelöf," "countably compact," or "m-compact" for an infinite cardinal, m. These properties are readily verified by arguments paralleling those in Raha (1973).

(*) 1) *A closed subspace of X is a λ-space.*

2) *A continuous image of X is a λ-space.*

3) *For each λ-subspace, A of X, the supremum, $Q(T, A)$, of T and $\{X, X - A, \varnothing\}$ is a λ-space.*

In this paper, we prove the following:

THEOREM. *If λ is a property of topological spaces such that any λ-space satisfies the properties listed in (*) the following statements are equivalent for a space (X, T).*

1) *T is a maximal λ-topology on the set X.*

2) *The set of λ-subspaces of X = the set of closed subsets of X.*

3) *Any continuous surjection from a λ-space to X is a closed quotient map.*

4) *Any continuous bijection from a λ-space to X is a homeomorphism.*

5) *Any function with a λ-graph from X is continuous.*

6) *Any function with a λ-graph into X is closed when T is a λ-topology on X.*

PROOF. Let π_x and π_y be the projections of $X \times Y$ onto X and Y respectively. To show that 1) implies 2) we need show only that each λ-subspace of X is closed. This is clear since for each λ-subspace, A, of $X, T \subset Q(T, A)$ and $Q(T, A)$ is a λ-topology on X forcing $T = Q(T, A)$. Assuming 2), let Y be a λ-space and let $g: Y \to X$ be a continuous surjection. If A is closed in Y, then A is a λ-subspace of Y, so $g(A)$ is a λ-subspace of X and thus is closed in X. Now, let T^* be any topology on X for which g is continuous and $T \subset T^*$. Then $X = g(Y)$, so (X, T^*) is a λ-space. Let A be T^*-closed in X. Then A is a T^*-λ-subspace of X, so A is a T-λ-subspace of X since the identity function from (X, T^*) to (X, T) is continuous. So, A is T-closed, $T = T^*$, and g is a quotient map. It is immediate that 3) implies 4). To show that 5) follows from 4), let Y be a space and suppose $g: X \to Y$ has a λ-graph, $G(g)$. The restriction, π_x^*, of π_x to $G(g)$ is a continuous bijection and thus is a homeomorphism. Since $g = \pi_y \circ \pi_x^{*-1}$, g is continuous. To prove that 5) implies 6) let $g: Y \to X$ be a function with $G(g)$ a λ-subset of $Y \times X$ and let $A \subset Y$ be closed. Then for the restriction, π_y^*, we have $\pi_y^{*-1}(A)$ is closed in $G(g)$ and thus is a λ-subspace of $Y \times X$. Then $g(A) = \pi_x(\pi_y^{*-1}(A))$ is a λ-subspace in X. Since X is a λ-space, the identity function, i, from (X, T) to $(X, Q(T, g(A)))$ has a λ-graph because the function, h, from (X, T) to $X \times X$ defined by $h(x) = (x, x)$ is continuous $(T \subset Q(T, g(A))$, $Q(T, g(A)) \subset Q(T, g(A))$ and $h(X) = G(i))$; so i is continuous. Thus $g(A)$ is T-closed since $g(A)$ is $Q(T, g(A))$-closed. Finally, to verify that 6) implies 1), let T^* be a λ-topology on X and suppose $T \subset T^*$. The identity function from (X, T^*) to

(X, T) renders T a λ-topology and, with the same reasoning as above, has a λ-graph; and is thus closed. Therefore, $T = T^*$. This completes the proof.

The theorem is true when λ is replaced throughout by either of the words "compact," "Lindelöf," or "m-compact" for any infinite cardinal m.

Using the result of the theorem for m-compactness and the known fact that m-compact subsets are closed in a Hausdorff topological space with an open base of cardinality $\leqq m$ at each point, we may prove the following generalizations of corollaries 5 and 6 in Raha (1973).

COROLLARY 1. *If T is an m-compact Hausdorff topology which contains an open base of cardinality $\leqq m$ at each point, then T is maximal m-compact.*

COROLLARY 2. *If T is an m-compact Hausdorff topology which contains an open base of cardinality $\leqq m$ at each point, then T is a minimal element in the set of Hausdorff topologies which contain an open base of cardinality $\leqq m$ at each point.*

PROOF. Any topology which is contained in T would be m-compact; if in addition, the topology is Hausdorff and contains an open base of cardinality $\leqq m$ at each point, the topology must be maximal m-compact. Since T is m-compact, this topology must be T. This completes the proof.

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