15

De Sitter-like spacetimes

This chapter discusses the global existence and stability of de Sitter-like spacetimes, that is, vacuum spacetimes with a de Sitter-like value of the cosmological constant. This class of spacetimes admits a conformal extension with a spacelike conformal boundary; see Theorem 10.1. The construction of de Sitter-like spacetimes provides, arguably, the simplest application of the conformal field equations to the analysis of global properties of spacetimes. The original discussion of the analysis presented in this chapter was given in Friedrich (1986b). The results of this seminal analysis were subsequently generalised to the case of Einstein equations coupled to the Yang-Mills field in Friedrich (1991). The methods used in the proof of the stability of the de Sitter spacetime can be adapted to analyse the future non-linear stability of Friedman-Robertson-Walker cosmologies with a perfect fluid satisfying the equation of state of radiation; see Lübbe and Valiente Kroon (2013b).

The global existence and stability theorem proven in this chapter can be formulated as follows:

Theorem ***(global existence and stability of de Sitter-like spacetimes).*** *Small enough perturbations of initial data for the de Sitter spacetime give rise to solutions of the vacuum Einstein field equations which exist globally towards the past and the future. The solutions have the same global structure as the de Sitter spacetime. Thus, perturbations of the de Sitter spacetime are asymptotically simple.*

Intuitively, the last statement in the theorem can be read as saying that the resulting spacetimes have a Penrose diagram similar to the one of the de Sitter spacetime; see Figure 15.1. Accordingly, these spacetimes provide non-trivial (i.e. dynamic) examples of asymptotically simple spacetimes. A detailed formulation of the above result is given in the main text of the chapter; see Theorem 15.1.

To illustrate the comparative advantages of the hyperbolic reduction procedures discussed in Chapter 13, two versions of the proof are provided. The first one makes use of *gauge source functions* and follows the original proofs

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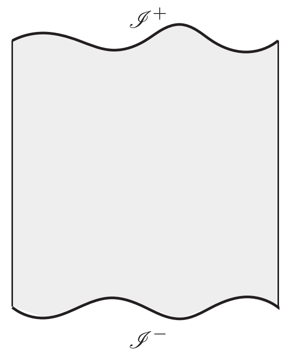

Figure 15.1 Penrose diagram of a de Sitter-like spacetime.

in Friedrich (1986b, 1991). The second proof makes use of *conformal Gaussian systems* and is based on the analysis given in Lübbe and Valiente Kroon (2009). In both approaches, and as a consequence of the use of the conformal field equations, it is possible to formulate initial value problems for the perturbed de Sitter-like spacetime not only on a standard initial hypersurface at a fiduciary finite time, but also on a hypersurface corresponding to the conformal boundary of the spacetime.

The basic strategy used in this chapter to analyse the global existence of solutions to the Einstein field equations had been previously used in Choquet-Bruhat and Christodoulou (1981) to establish the global existence of solutions to the Yang-Mills equations.

15.1 The de Sitter spacetime as a solution to the conformal field equations

The basic conformal properties of the de Sitter spacetime have already been discussed in Section 6.3. In this section the de Sitter spacetime is recast as a solution to the conformal field equations. This is a first step in the construction of an existence and global stability result.

15.1.1 Basic representation in the Einstein cosmos

For simplicity of the exposition, the cosmological constant will be assumed to take the value $\lambda = -3$ so that the conformal factor realising the embedding of the de Sitter spacetime (given in standard coordinates) into the Einstein static universe is given by

$$\mathring{\Xi} \equiv \Xi_{dS} = \cos\tau, \tag{15.1}$$

where τ, the affine parameter of the geodesics introduced in Equation (6.9), is used as a time coordinate. Here, and in the rest of the chapter, the symbol $\mathring{\ }$ is used to indicate that the associated object is treated as a *background field.*

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As a consequence of the conformal embedding, the geometry of the conformal de Sitter spacetime is given by the corresponding expressions for the Einstein static universe as discussed in Section 6.1.3. The various conformal fields on the Einstein cylinder $(\mathbb{R} \times \mathbb{S}^3, \boldsymbol{g}_{\mathscr{E}})$, with

$$\boldsymbol{g}_{\mathscr{E}} = \mathbf{d}\tau \otimes \mathbf{d}\tau - \boldsymbol{\hbar},$$

will be expressed in terms of an orthonormal frame $\{\mathring{\boldsymbol{e}}_{\boldsymbol{a}}\}$ such that

$$\mathring{\boldsymbol{e}}_0 = \boldsymbol{\partial}_\tau, \qquad \mathring{\boldsymbol{e}}_{\boldsymbol{i}} = \boldsymbol{c}_{\boldsymbol{i}}, \tag{15.2}$$

where $\{\boldsymbol{c}_{\boldsymbol{i}}\}$ denotes the globally defined frame on $\mathbb{S}^3$ discussed in Section 6.1.2; see Equations (6.2a)–(6.2c).

In what follows, the manifold $\mathcal{M}_{\mathscr{E}} \equiv \mathbb{R} \times \mathbb{S}^3$ will be described locally in terms of ***Gaussian coordinates*** (τ, x^α) where (x^α) are some local coordinates on $\mathbb{S}^3$ which are extended to coordinates on a subset $\mathcal{U} \subset \mathcal{M}_{\mathscr{E}}$ by requiring them to remain constant along the geodesics parametrised by τ. As a consequence of the $\boldsymbol{g}_{\mathscr{E}}$-orthonormality of the vector fields $\{\boldsymbol{\partial}_\tau,\ \boldsymbol{c}_{\boldsymbol{i}}\}$, it follows that

$$\mathring{\boldsymbol{e}}_{\boldsymbol{a}} = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}} \boldsymbol{c}_{\boldsymbol{b}} \equiv \mathring{e}_{\boldsymbol{a}}{}^{\boldsymbol{b}} \boldsymbol{c}_{\boldsymbol{b}}. \tag{15.3}$$

Using the structure equations – see Section 2.7.3 – on $\mathbb{S}^3$, it can be verified that the connection coefficients $\mathring{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}$ of the Levi-Civita connection $\boldsymbol{D}$ of the standard metric of $\mathbb{S}^3$, $\boldsymbol{\hbar}$, with respect to the *spatial frame* $\{\boldsymbol{c}_{\boldsymbol{i}}\}$ are given by

$$\mathring{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} = -\epsilon_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}, \tag{15.4}$$

where $\epsilon_{\boldsymbol{ijk}}$ denotes the components of the volume form on $\mathbb{S}^3$; see Section 6.1.2. Now, observing that $\mathring{\boldsymbol{e}}_0 = \boldsymbol{\partial}_\tau$ is a Killing vector of the Einstein cylinder, it follows that the connection coefficients associated to the frame $\{\mathring{\boldsymbol{e}}_{\boldsymbol{a}}\}$ are given by

$$\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = \epsilon_{\boldsymbol{0a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}. \tag{15.5}$$

The sign difference between Equations (15.4) and (15.5) arises from the fact that the Riemannian metric implied by $\boldsymbol{g}_{\mathscr{E}}$ on $\mathbb{S}^3$ is negative definite.

Using the expressions (6.8a) and (6.8b) for the Schouten tensor of the Einstein cylinder, it follows that, in terms of the frame $\{\mathring{\boldsymbol{e}}_{\boldsymbol{a}}\}$ described above, one has

$$\mathring{L}_{\boldsymbol{ab}} = \delta_{\boldsymbol{a}}{}^{\boldsymbol{0}} \delta_{\boldsymbol{b}}{}^{\boldsymbol{0}} - \frac{1}{2}\eta_{\boldsymbol{ab}}, \tag{15.6a}$$

$$\mathring{d}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} = 0. \tag{15.6b}$$

For later use, it is also observed that the components of the trace-free Ricci tensor are given by

$$\mathring{\Phi}_{\boldsymbol{ab}} = \delta_{\boldsymbol{a}}{}^{\boldsymbol{0}} \delta_{\boldsymbol{b}}{}^{\boldsymbol{0}} - \frac{1}{4}\eta_{\boldsymbol{ab}}.$$

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Finally, a further computation yields that

$$\mathring{\Sigma} = -\sin\tau, \qquad \mathring{\Sigma}_{\boldsymbol{i}} = 0, \tag{15.7a}$$

$$\mathring{s} = -\frac{1}{4}\cos\tau. \tag{15.7b}$$

Spinorial expressions

To compute the spinorial counterpart of the fields discussed in the previous section let $\tau^{AA'}$ denote the spinorial counterpart of the vector $\sqrt{2}\boldsymbol{\partial}_\tau$ so that one has the normalisation $\tau_{\boldsymbol{AA'}}\tau^{\boldsymbol{AA'}} = 2$.

The spinorial counterpart of the frame coefficients $\mathring{e}_{\boldsymbol{a}}{}^{\boldsymbol{b}} = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}$ – compare Equation (15.3) – is given by

$$\mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{b}} = \sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{b}},$$

where $\sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}$ denotes the Infeld-van der Waerden symbols; see Section 3.1.9. In general, the coefficients $\mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}$ can be decomposed as

$$\mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{a}} = \frac{1}{2}\tau_{\boldsymbol{AA'}}\mathring{e}^{\boldsymbol{a}} - \tau^{\boldsymbol{Q}}{}_{\boldsymbol{A'}}\mathring{e}_{(\boldsymbol{AQ})}{}^{\boldsymbol{a}},$$

with

$$\mathring{e}^{\boldsymbol{a}} \equiv \tau^{\boldsymbol{AA'}}\mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}, \qquad \mathring{e}_{\boldsymbol{AB}}{}^{\boldsymbol{a}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}.$$

By construction it follows that

$$\mathring{e}^{\boldsymbol{0}} = 1, \qquad \mathring{e}_{(\boldsymbol{AB})}{}^{\boldsymbol{0}} = 0,$$
$$\mathring{e}^{\boldsymbol{i}} = 0, \qquad \mathring{e}_{(\boldsymbol{AB})}{}^{\boldsymbol{i}} = \sigma_{\boldsymbol{AB}}{}^{\boldsymbol{i}},$$

with $\sigma_{\boldsymbol{AB}}{}^{\boldsymbol{i}}$ the spatial Infeld-van der Waerden symbols; see Section 4.2.2.

The spinorial counterpart of the trace-free Ricci tensor is given by

$$\mathring{\Phi}_{\boldsymbol{AA'BB'}} = \frac{1}{2}\tau_{\boldsymbol{AA'}}\tau_{\boldsymbol{BB'}} - \frac{1}{4}\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}},$$

so that its space spinor version $\mathring{\Phi}_{\boldsymbol{ABCD}} = \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\tau_{\boldsymbol{D}}{}^{\boldsymbol{C'}}\mathring{\Phi}_{\boldsymbol{AA'CC'}}$ is given by

$$\mathring{\Phi}_{\boldsymbol{ABCD}} = \frac{1}{2}\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{CD}} - \frac{1}{4}\epsilon_{\boldsymbol{AC}}\epsilon_{\boldsymbol{BD}}.$$

From this last expression is can be verified that the irreducible components of $\mathring{\Phi}_{\boldsymbol{ABCD}}$ are given by

$$\mathring{\Phi}_{(\boldsymbol{ABCD})} = 0, \qquad \mathring{\Phi}_{\boldsymbol{AB}} = 0, \qquad \mathring{\Phi} = -\frac{3}{4};$$

compare the expressions in Equation (13.32). The rescaled Weyl spinor is trivially given by

$$\mathring{\phi}_{\boldsymbol{ABCD}} = 0.$$

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Let $\mathring{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}}$ denote the spinorial counterpart of the connection coefficients $\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$. Using the spinorial expression of the spacetime volume form, Equation (3.25), one finds that

$$\begin{aligned}\mathring{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}} &= \frac{1}{\sqrt{2}}\tau^{\boldsymbol{DD'}}\epsilon_{\boldsymbol{DD'AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}},\\ &= \frac{\mathrm{i}}{\sqrt{2}}\big(\tau_{\boldsymbol{C}}{}^{\boldsymbol{B'}}\epsilon_{\boldsymbol{C'A'}}\delta_{\boldsymbol{A}}{}^{\boldsymbol{B}} - \tau^{\boldsymbol{B}}{}_{\boldsymbol{C'}}\epsilon_{\boldsymbol{CA}}\delta_{\boldsymbol{A'}}{}^{\boldsymbol{B'}}\big).\end{aligned}$$

The reduced spin connection coefficients can be obtained from the expression $\mathring{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} = \frac{1}{2}\mathring{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{BQ'}}{}_{\boldsymbol{CQ'}}$. One obtains

$$\mathring{\Gamma}_{\boldsymbol{AA'BC}} = \frac{\mathrm{i}}{\sqrt{2}}\epsilon_{\boldsymbol{A}(\boldsymbol{B}}\tau_{\boldsymbol{C})\boldsymbol{A'}}.$$

The space spinor version of the above expression is given by

$$\mathring{\Gamma}_{\boldsymbol{ABCD}} = \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\mathring{\Gamma}_{\boldsymbol{AA'CD}} = -\frac{\mathrm{i}}{\sqrt{2}}h_{\boldsymbol{ABCD}},$$

where it is recalled that $h_{\boldsymbol{ABCD}} \equiv -\epsilon_{\boldsymbol{A}(\boldsymbol{C}}\epsilon_{\boldsymbol{D})\boldsymbol{B}}$. It can be verified that $\mathring{\Gamma}^{\dagger}_{\boldsymbol{ABCD}} = -\mathring{\Gamma}_{\boldsymbol{ABCD}}$; that is, the spin connection coefficients are the components of an *imaginary spinor*. From here it follows that

$$\mathring{\xi}_{\boldsymbol{ABCD}} = -\mathrm{i}h_{\boldsymbol{ABCD}}, \qquad \mathring{\chi}_{\boldsymbol{ABCD}} = 0.$$

Gauge source functions

The gauge source functions associated to the considered conformal representation of the de Sitter spacetime can be computed from the expressions given in the previous section.

Treating the frame component $\mathring{e}_{\boldsymbol{a}}{}^{\boldsymbol{b}}$ as the component of a covariant tensor one finds that

$$\begin{aligned}\mathring{\nabla}^{\boldsymbol{b}}\mathring{e}_{\boldsymbol{b}}{}^{\boldsymbol{a}} &= \eta^{\boldsymbol{cb}}\mathring{\boldsymbol{e}}_{\boldsymbol{c}}(\delta_{\boldsymbol{b}}{}^{\boldsymbol{a}}) - \eta^{\boldsymbol{cb}}\mathring{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\delta_{\boldsymbol{e}}{}^{\boldsymbol{a}}\\ &= -\eta^{\boldsymbol{cb}}\mathring{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = -\eta^{\boldsymbol{cb}}\epsilon_{\boldsymbol{0c}}{}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = 0.\end{aligned}$$

It follows that the *coordinate gauge source function* is given by

$$\mathring{F}^{\boldsymbol{a}}(x) = \mathring{\nabla}^{\boldsymbol{AA'}}\mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{a}} = 0.$$

Similarly, treating the connection coefficients as the components of a $(1,2)$-tensor one has

$$\begin{aligned}\eta^{\boldsymbol{da}}\mathring{\nabla}_{\boldsymbol{d}}\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} &= \eta^{\boldsymbol{da}}\boldsymbol{e}_{\boldsymbol{d}}(\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}) + \eta^{\boldsymbol{da}}\mathring{\Gamma}_{\boldsymbol{d}}{}^{\boldsymbol{b}}{}_{\boldsymbol{e}}\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{c}} - \eta^{\boldsymbol{da}}\mathring{\Gamma}_{\boldsymbol{d}}{}^{\boldsymbol{e}}{}_{\boldsymbol{a}}\mathring{\Gamma}_{\boldsymbol{e}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} - \eta^{\boldsymbol{da}}\mathring{\Gamma}_{\boldsymbol{d}}{}^{\boldsymbol{e}}{}_{\boldsymbol{c}}\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{e}}\\ &= \eta^{\boldsymbol{da}}\epsilon_{\boldsymbol{0d}}{}^{\boldsymbol{b}}{}_{\boldsymbol{e}}\epsilon_{\boldsymbol{0a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{c}} - \eta^{\boldsymbol{da}}\epsilon_{\boldsymbol{0d}}{}^{\boldsymbol{e}}{}_{\boldsymbol{c}}\epsilon_{\boldsymbol{0a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{e}} - \eta^{\boldsymbol{da}}\epsilon_{\boldsymbol{0d}}{}^{\boldsymbol{e}}{}_{\boldsymbol{a}}\epsilon_{\boldsymbol{0e}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = 0.\end{aligned}$$

It follows that the *frame gauge source functions* for the present representation of the de Sitter spacetime are given by

$$\mathring{F}_{\boldsymbol{BC}}(x) = \mathring{\nabla}^{\boldsymbol{AA'}}\mathring{\Gamma}_{\boldsymbol{AA'BC}} = 0.$$

Finally, the *conformal gauge source function* is given by the value of the Ricci scalar. That is, one has

$$\mathring{R}(x) = -6.$$

Summary

The results from the previous analysis are summarised in the following:

Lemma 15.1 (***de Sitter spacetime as a solution to the conformal Einstein field equations***) *The fields*

$$(\mathring{\Xi},\ \mathring{\Sigma},\ \mathring{\Sigma}_{\boldsymbol{i}},\ \mathring{e}_{\boldsymbol{a}}{}^{\boldsymbol{b}},\ \mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}},\ \mathring{L}_{\boldsymbol{ab}},\ \mathring{d}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}})$$

as given by Equations (15.1)–(15.5), (15.6a), (15.6b), (15.7a) *and* (15.7b) *or, respectively, their spinorial counterparts*

$$(\mathring{\Xi},\ \mathring{\Sigma},\ \mathring{\Sigma}_{\boldsymbol{AA'}},\ \mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{b}},\ \mathring{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}},\ \mathring{\Phi}_{\boldsymbol{AA'BB'}},\ \mathring{\phi}_{\boldsymbol{ABCD}})$$

defined over the Einstein cylinder $\mathbb{R}\times\mathbb{S}^3$ *constitute a solution to the standard frame vacuum conformal Einstein field Equations* (8.32a) *and* (8.32b) *and, respectively, the spinorial vacuum conformal Einstein field Equations* (8.38a) *and* (8.38b). *The gauge source functions associated to this solution are given by*

$$\mathring{F}^{\boldsymbol{a}}(x) = 0, \qquad \mathring{F}_{\boldsymbol{AB}}(x) = 0, \qquad \mathring{R}(x) = -6.$$

15.1.2 Representation using conformal Gaussian systems

In Section 6.1.3 it has been shown that an alternative conformal representation of the de Sitter spacetime is given by the conformal metric

$$\bar{\boldsymbol{g}}_{\mathscr{E}} = \mathbf{d}\bar{\tau}\otimes\mathbf{d}\bar{\tau} - \left(1+\frac{\bar{\tau}^2}{4}\right)^2\boldsymbol{\hbar},$$

where $\bar{\tau}$ is an affine parameter of the $\tilde{\boldsymbol{g}}_{dS}$-conformal geodesics as in Equation (6.32); that is, $\dot{\boldsymbol{x}} = \boldsymbol{\partial}_{\bar{\tau}}$. The associated covector is given by

$$\boldsymbol{\beta}_{dS}(\bar{\tau}) = -\frac{2\bar{\tau}}{4-\bar{\tau}^2}\mathbf{d}\bar{\tau}.$$

This covector is exact, thus indicating that the Weyl connection $\hat{\boldsymbol{\nabla}} = \tilde{\boldsymbol{\nabla}} + \boldsymbol{S}(\boldsymbol{\beta}_{dS})$ is, in fact, a Levi-Civita connection. Now, the metric $\bar{\boldsymbol{g}}_{\mathscr{E}}$ is related to the physical de Sitter metric $\tilde{\boldsymbol{g}}_{dS}$ via

$$\bar{\boldsymbol{g}}_{\mathscr{E}} = \Theta_{dS}^2\tilde{\boldsymbol{g}}_{dS}, \qquad \Theta_{dS} = 1-\frac{\bar{\tau}^2}{4},$$

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so that a calculation shows that

$$\boldsymbol{\beta}_{dS} = \Theta_{dS}^{-1}\mathbf{d}\Theta_{dS}.$$

That is, *the Weyl connection associated to the congruence of conformal geodesics* (6.32) *coincides with the Levi-Civita connection* $\bar{\boldsymbol{\nabla}}$ *of the metric* $\bar{\boldsymbol{g}}_{\mathscr{E}}$. Recalling that $\boldsymbol{g}_{\mathscr{E}}$ and $\bar{\boldsymbol{g}}_{\mathscr{E}}$ are related to each other by $\bar{\boldsymbol{g}}_{\mathscr{E}} = \bar{\Theta}^2 \boldsymbol{g}_{\mathscr{E}}$ with

$$\bar{\Theta} \equiv 1 + \frac{\bar{\tau}^2}{4},$$

one finds that an adapted $\bar{\boldsymbol{g}}_{\mathscr{E}}$-orthonormal frame $\{\bar{\boldsymbol{e}}_{\boldsymbol{a}}\}$ is given by

$$\bar{\boldsymbol{e}}_{\boldsymbol{0}} = \boldsymbol{\partial}_{\bar{\tau}} = \bar{\Theta}^{-1}\boldsymbol{\partial}_{\tau}, \qquad \bar{\boldsymbol{e}}_{\boldsymbol{i}} = \bar{\Theta}^{-1}\boldsymbol{c}_i.$$

This frame can be verified to be *Weyl propagated*. It follows that the frame coefficients $\bar{e}_{\boldsymbol{i}}{}^{\boldsymbol{b}}$, with $\bar{\boldsymbol{e}}_{\boldsymbol{i}} = \bar{e}_{\boldsymbol{i}}{}^{\boldsymbol{b}}\boldsymbol{c}_{\boldsymbol{b}}$, are given by

$$\bar{e}_{\boldsymbol{i}}{}^{\boldsymbol{b}} = \frac{4}{4+\bar{\tau}^2}\delta_{\boldsymbol{i}}{}^{\boldsymbol{b}}. \tag{15.8}$$

In terms of the above, the components of the covector $\bar{\boldsymbol{d}} = \Theta_{dS}\boldsymbol{\beta}_{dS}$ with respect to the frame $\{\bar{\boldsymbol{e}}_{\boldsymbol{a}}\}$ are given by

$$\bar{d}_{\boldsymbol{0}} = \dot{\Theta}_{dS} = -\frac{\bar{\tau}}{2}, \qquad \bar{d}_{\boldsymbol{i}} = 0.$$

The computation of the connection coefficients $\bar{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ requires a certain amount of care. Recalling that the connections $\hat{\boldsymbol{\nabla}} = \bar{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$ are related to each other via $\hat{\boldsymbol{\nabla}} - \boldsymbol{\nabla} = \boldsymbol{S}(\bar{\boldsymbol{\Upsilon}})$ with $\bar{\boldsymbol{\Upsilon}} \equiv \bar{\Theta}^{-1}\mathbf{d}\bar{\Theta}$, it follows by definition that

$$\begin{aligned}\bar{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} &= \bar{\omega}^{\boldsymbol{b}}{}_c \bar{e}_{\boldsymbol{a}}{}^a \bar{\nabla}_a \bar{e}_{\boldsymbol{c}}{}^c \\ &= \bar{\omega}^{\boldsymbol{b}}{}_c \bar{e}_{\boldsymbol{a}}{}^a \nabla_a \bar{e}_{\boldsymbol{c}}{}^c + \bar{\omega}^{\boldsymbol{b}}{}_c \bar{e}_{\boldsymbol{a}}{}^a \bar{e}_{\boldsymbol{c}}{}^d S_{ad}{}^{ec}\Upsilon_e,\end{aligned}$$

where $\bar{\Upsilon}_{\boldsymbol{a}} \equiv \langle \bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{e}}_{\boldsymbol{a}}\rangle$. Using that $\bar{e}_{\boldsymbol{a}}{}^a = \bar{\Theta}^{-1}e_{\boldsymbol{a}}{}^a$ one computes

$$\begin{aligned}\bar{\omega}^{\boldsymbol{b}}{}_c \bar{e}_{\boldsymbol{a}}{}^a \nabla_a \bar{e}_{\boldsymbol{c}}{}^c &= -\bar{\Theta}^{-1}\omega^{\boldsymbol{b}}{}_c e_{\boldsymbol{a}}{}^a e_{\boldsymbol{c}}{}^c \nabla_a \bar{\Theta} + \bar{\Theta}^{-1}\omega^{\boldsymbol{b}}{}_c e_{\boldsymbol{a}}{}^a \nabla_a e_{\boldsymbol{c}}{}^c \\ &= -\bar{\Upsilon}_{\boldsymbol{a}}\delta_{\boldsymbol{c}}{}^{\boldsymbol{b}} + \bar{\Theta}^{-1}\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}\end{aligned}$$

and

$$\bar{\Theta}^{-1}\bar{\omega}^{\boldsymbol{b}}{}_c \bar{e}_{\boldsymbol{a}}{}^a \bar{e}_{\boldsymbol{c}}{}^d S_{ad}{}^{ec}\nabla_e\bar{\Theta} = \bar{\Upsilon}_{\boldsymbol{a}}\delta_{\boldsymbol{c}}{}^{\boldsymbol{b}} + \bar{\Upsilon}_{\boldsymbol{c}}\delta_{\boldsymbol{a}}{}^{\boldsymbol{b}} - \eta_{\boldsymbol{ac}}\bar{\Upsilon}^{\boldsymbol{b}}.$$

Accordingly, one concludes that

$$\bar{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = \bar{\Theta}^{-1}\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} + \big(\Upsilon_{\boldsymbol{c}}\delta_{\boldsymbol{a}}{}^{\boldsymbol{b}} - \eta_{\boldsymbol{ac}}\Upsilon^{\boldsymbol{b}}\big).$$

Using

$$\Upsilon_{\boldsymbol{a}} = \frac{2\bar{\tau}}{4+\bar{\tau}^2}\delta_{\boldsymbol{a}}{}^{\boldsymbol{0}},$$

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it can be verified that $\bar{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = 0$, as one would expect from the Weyl connection associated to a congruence of conformal geodesics. Moreover,

$$\bar{f}_{\boldsymbol{a}} = \frac{1}{4}\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{b}} = 0. \tag{15.9}$$

A direct computation shows that

$$R[\bar{\boldsymbol{g}}_{\mathscr{E}}] = -\frac{36}{4+\bar{\tau}^2},$$
$$\boldsymbol{Schouten}[\bar{\boldsymbol{g}}_{\mathscr{E}}] = \frac{1}{2}\left(1+\frac{1}{4}\bar{\tau}^2\right)\boldsymbol{\hbar},$$
$$\boldsymbol{Weyl}[\bar{\boldsymbol{g}}_{\mathscr{E}}] = 0,$$

where the last expression follows simply by the conformal invariance of the Weyl tensor. The components of the Schouten tensor with respect to the frame $\{\bar{\boldsymbol{e}}_{\boldsymbol{a}}\}$ are given by

$$\bar{L}_{\boldsymbol{0a}} = 0, \qquad \bar{L}_{\boldsymbol{ij}} = \frac{2}{4+\bar{\tau}^2}\delta_{\boldsymbol{ij}}. \tag{15.10}$$

Furthermore, one has that

$$\bar{d}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} = 0. \tag{15.11}$$

Spinorial expressions

In what follows, let $\bar{\tau}^{AA'}$ denote the spinorial counterpart of the vector $\sqrt{2}\boldsymbol{\partial}_{\bar{\tau}}$. One has the normalisation $\bar{\tau}_{AA'}\bar{\tau}^{AA'} = 2$. Denoting the spinorial counterpart of the frame coefficients by $\bar{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}$ and making use of the standard space spinor decomposition

$$\bar{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{a}} = \frac{1}{2}\bar{\tau}_{\boldsymbol{AA'}}\bar{e}^{\boldsymbol{a}} - \bar{\tau}^{\boldsymbol{Q}}{}_{\boldsymbol{A'}}\bar{e}_{(\boldsymbol{AQ})}{}^{\boldsymbol{a}},$$

one obtains

$$\bar{e}^{\boldsymbol{0}} = 1, \qquad \bar{e}_{(\boldsymbol{AB})}{}^{\boldsymbol{0}} = 0, \tag{15.12a}$$
$$\bar{e}^{\boldsymbol{i}} = 0, \qquad \bar{e}_{(\boldsymbol{AB})}{}^{\boldsymbol{i}} = \frac{4}{4+\bar{\tau}^2}\sigma_{\boldsymbol{AB}}{}^{\boldsymbol{i}}. \tag{15.12b}$$

Now, let $\bar{L}_{\boldsymbol{AA'BB'}}$ denote the spinorial counterpart of the components of the Schouten tensor $\bar{L}_{\boldsymbol{ab}}$. Setting $\bar{L}_{\boldsymbol{ABCD}} \equiv \bar{\tau}_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\bar{\tau}_{\boldsymbol{D}}{}^{\boldsymbol{C'}}\bar{L}_{\boldsymbol{AA'CC'}}$ one finds

$$\bar{\Theta}_{\boldsymbol{CD}} = 0, \qquad \bar{\Theta}_{\boldsymbol{ABCD}} = -\frac{2}{4+\bar{\tau}^2}h_{\boldsymbol{ABCD}},$$

with $\bar{\Theta}_{\boldsymbol{CD}} \equiv \bar{L}_{\boldsymbol{Q}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{CD}}$ and $\bar{\Theta}_{\boldsymbol{ABCD}} \equiv \bar{L}_{(\boldsymbol{AB})(\boldsymbol{CD})}$. For the rescaled Weyl spinor one has that

$$\bar{\phi}_{\boldsymbol{ABCD}} = 0.$$

To obtain a spinorial expression for the connection coefficients one observes that the spinorial counterpart of $\zeta_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} \equiv \delta_{\boldsymbol{c}}{}^{\boldsymbol{0}}\delta_{\boldsymbol{a}}{}^{\boldsymbol{b}} - \eta_{\boldsymbol{ac}}\delta_{\boldsymbol{0}}{}^{\boldsymbol{b}}$ is given by

$$\zeta_{\boldsymbol{AA}'}{}^{\boldsymbol{BB}'}{}_{\boldsymbol{CC}'} = \frac{1}{\sqrt{2}}\big(\delta_{\boldsymbol{A}}{}^{\boldsymbol{B}}\delta_{\boldsymbol{A}'}{}^{\boldsymbol{B}'}\bar{\tau}_{\boldsymbol{CC}'} - \epsilon_{\boldsymbol{AC}}\epsilon_{\boldsymbol{A}'\boldsymbol{C}'}\bar{\tau}^{\boldsymbol{BB}'}\big),$$

so that the associated reduced coefficients are

$$\begin{aligned}\zeta_{\boldsymbol{AA}'}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} &\equiv \frac{1}{2}\zeta_{\boldsymbol{AA}'}{}^{\boldsymbol{BQ}'}{}_{\boldsymbol{CQ}'}\\ &= \frac{1}{2\sqrt{2}}\big(\delta_{\boldsymbol{A}}{}^{\boldsymbol{B}}\bar{\tau}_{\boldsymbol{CA}'} + \epsilon_{\boldsymbol{AC}}\bar{\tau}^{\boldsymbol{B}}{}_{\boldsymbol{A}'}\big).\end{aligned}$$

The space spinor version $\zeta_{\boldsymbol{ABCD}} \equiv \bar{\tau}_{\boldsymbol{B}}{}^{\boldsymbol{A}'}\zeta_{\boldsymbol{AA}'\boldsymbol{CD}}$ takes the form

$$\zeta_{\boldsymbol{ABCD}} = -\frac{1}{\sqrt{2}}h_{\boldsymbol{ABCD}}.$$

From the expressions computed in the previous paragraph it follows that

$$\bar{\Gamma}_{\boldsymbol{ABCD}} = -\frac{2(\bar{\tau}+2\mathrm{i})}{\sqrt{2}(1+4\bar{\tau}^2)}h_{\boldsymbol{ABCD}}$$

and, consequently,

$$\bar{\xi}_{\boldsymbol{ABCD}} = -\frac{4\mathrm{i}}{4+\bar{\tau}^2}h_{\boldsymbol{ABCD}}, \tag{15.13a}$$

$$\bar{\chi}_{\boldsymbol{ABCD}} = \frac{2\bar{\tau}}{4+\bar{\tau}^2}h_{\boldsymbol{ABCD}}, \tag{15.13b}$$

$$f_{\boldsymbol{AB}} = 0. \tag{15.13c}$$

To keep track of the behaviour of the conformal Gaussian gauge system, one considers separation fields measuring the deviation of the congruence of conformal geodesics. The separation fields are governed by Equations (13.67a) and (13.67b). Assume, without loss of generality, a separation vector field $\boldsymbol{z}$ that is spatial on the fiduciary hypersurface $\mathcal{S}_\star$ described by the condition $\bar{\tau} = 0$, so that

$$z_{\boldsymbol{AA}'\star} = -\tau^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}z_{(\boldsymbol{AQ})_\star}.$$

Using Equations (15.13b) and (15.13c) one can integrate Equations (13.67a) and (13.67b) to find

$$z = 0, \qquad z_{(\boldsymbol{AB})} = \left(1+\frac{1}{4}\bar{\tau}^2\right)z_{(\boldsymbol{AB})\star}. \tag{15.14}$$

Observe that $z_{(\boldsymbol{AB})} \neq 0$ for all $\bar{\tau}$. Thus, the congruence of conformal geodesics remains non-singular. This observation is key to ensure the non-singular behaviour of the gauge in the perturbed spacetime.

Summary

The results of the analysis of the last two sections are summarised in the following:

Lemma 15.2 (***de Sitter spacetime as a solution to the extended conformal Einstein field equations***) *The fields*

$$(\bar{\Theta}\,, \bar{d}_{\boldsymbol{a}}, \, \bar{e}_{\boldsymbol{a}}{}^{\boldsymbol{b}}, \, \bar{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \, \bar{L}_{\boldsymbol{ab}}, \, \bar{d}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}})$$

as given by Equations (15.8)–(15.11) *or, equivalently, their spinorial counterparts*

$$(\bar{\Theta}, \, \bar{d}_{\boldsymbol{AA'}}, \, \bar{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}, \, \bar{\Gamma}_{\boldsymbol{AA'BC}}, \, \bar{L}_{\boldsymbol{AA'BB'}}, \, \bar{\phi}_{\boldsymbol{ABCD}})$$

defined over $\mathbb{R}\times\mathbb{S}^3$ *constitute a solution to the extended conformal Einstein field Equations* (8.46) *and the associated gauge constraints* (8.48) *and, respectively, the spinorial vacuum conformal Einstein field Equations* (8.54a) *and* (8.54b) *and* (8.55).

15.2 Perturbations of initial data for the de Sitter spacetime

This section clarifies the notion of perturbations of initial data for the de Sitter spacetime. In what follows, let $\mathcal{S}$ denote a three-dimensional manifold with the topology of $\mathbb{S}^3$. On $\mathcal{S}$ one considers a solution to the vacuum conformal Hamiltonian and momentum constraint equations $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \Omega, \Sigma)$ with a de Sitter-like value of the cosmological constant, that is, Equations (11.15a) and (11.15b) with $\varrho = 0$ and $j_k = 0$.

Remark. For conceptual clarity it is often convenient to distinguish between the 3-manifold $\mathcal{S}$ and its embedding, $\mathcal{S}_\star$, in the spacetime arising as the development of the initial data set $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \Omega, \Sigma)$.

15.2.1 Initial data on a standard initial hypersurface

Using the procedure described in Section 11.4.3, the tensor fields $\boldsymbol{h}$ and $\boldsymbol{K}$ can be used to construct a solution to the vacuum conformal constraint Equations (11.35a)–(11.35j). As the 3-manifold $\mathcal{S}$ is assumed to be compact, one can, without loss of generality, assume that $\Omega = 1$ and $\Sigma = 0$.

As $\mathcal{S} \approx \mathbb{S}^3$, there exists a diffeomorphism $\psi : \mathcal{S} \to \mathbb{S}^3$ which can be used to pull back coordinates $\underline{x} = (x^\alpha)$ in $\mathbb{S}^3$ to $\mathcal{S}$. In this way one obtains a system of coordinates $\underline{x}' \equiv \underline{x} \circ \psi$ on $\mathcal{S}$ and can write $\underline{x}' = (x'^\alpha)$. The diffeomorphism ψ can be used to push forward the vector fields $\{\boldsymbol{c}_{\boldsymbol{i}}\}$ on $T(\mathbb{S}^3)$ to vector fields $\{\psi_*^{-1}\boldsymbol{c}_{\boldsymbol{i}}\}$ on $T(\mathcal{S})$ and to pull back their dual covectors $\{\boldsymbol{\alpha}^{\boldsymbol{i}}\}$ on $T^*(\mathbb{S}^3)$ to covectors $\{\psi_*\boldsymbol{\alpha}^{\boldsymbol{i}}\}$ on $T^*(\mathcal{S})$. *For simplicity of the presentation, in a slight abuse of notation, the vectors and covectors* $\{\psi_*^{-1}\boldsymbol{c}_{\boldsymbol{i}}\}$ *and* $\{\psi_*\boldsymbol{\alpha}^{\boldsymbol{i}}\}$ *will be written (except for the next subsection) as* $\{\boldsymbol{c}_{\boldsymbol{i}}\}$ *and* $\{\boldsymbol{\alpha}^{\boldsymbol{i}}\}$*, respectively.*

Gauge fixing

The construction described in the previous paragraph depends strongly on the particular choice of the diffeomorphism ψ. This ***gauge freedom*** can be fixed by considerations similar to those used in the discussion of the *coordinate gauge source functions* of Section 13.2.1.

Given an $\boldsymbol{h}$-orthonormal frame $\{\boldsymbol{e_i}\}$ on $\mathcal{S}$, one can write $\boldsymbol{e_i} = e_{\boldsymbol{i}}{}^{\boldsymbol{j}}(\psi_*^{-1}\boldsymbol{c_j})$ and use the frame coefficients $e_{\boldsymbol{i}}{}^{\boldsymbol{j}}$ to introduce a ***spatial coordinate gauge source function*** $F^{\boldsymbol{i}}(\underline{x}')$ via the relation

$$D^{\boldsymbol{j}} e_{\boldsymbol{j}}{}^{\boldsymbol{i}} = F^{\boldsymbol{i}}(\underline{x}'),$$

where $\boldsymbol{D}$ denotes the Levi-Civita covariant derivative of $\boldsymbol{h}$. Writing

$$\psi_* \boldsymbol{\alpha}^{\boldsymbol{i}} = (\psi_* \boldsymbol{\alpha}^{\boldsymbol{i}})_\alpha \mathbf{d}x'^\alpha$$

and noticing that $e_{\boldsymbol{i}}{}^{\boldsymbol{j}} = \langle \psi_* \boldsymbol{\alpha}^{\boldsymbol{j}}, \boldsymbol{e_i}\rangle$ one finds that $D^{\boldsymbol{j}} e_{\boldsymbol{j}}{}^{\boldsymbol{i}} = D^\beta (\psi_* \boldsymbol{\alpha}^{\boldsymbol{i}})_\beta$. Expressing the coordinates $\underline{x}$ in $\mathbb{S}^3$ in terms of the coordinates $\underline{x}'$ on $\mathcal{S}$ in the form $x^\alpha = x^\alpha(\underline{x}')$ one finds, by a calculation similar to the one discussed in Section 13.2.1, that the diffeomorphism $\psi : \mathcal{S} \to \mathbb{S}^3$ is a ***harmonic map***. That is, one has that

$$D^\beta D_\beta x^\alpha = 0,$$

if

$$h^{\alpha\beta} \not{D}_\gamma \alpha^{\boldsymbol{i}}{}_\delta \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'^\beta} = F^{\boldsymbol{i}}(\underline{x}'),$$

where $\not{D}$ denotes the Levi-Civita connection of the metric $\boldsymbol{\hbar}$ on $\mathbb{S}^3$ and $\boldsymbol{\alpha}^{\boldsymbol{i}} = \alpha^{\boldsymbol{i}}{}_\alpha \mathbf{d}x^\alpha$. Finally, if one lets $x'^\alpha = x'^\alpha(\underline{x})$ be the identity map so that $\underline{x}' = \underline{x}$, one concludes that

$$F^{\boldsymbol{i}}(\underline{x}) = \delta^{\boldsymbol{jk}} \mathring{\gamma}_{\boldsymbol{j}}{}^{\boldsymbol{i}}{}_{\boldsymbol{k}} = 0,$$

where the last equality follows from (15.4). This construction and the resulting spatial gauge source function fixes the gauge freedom in the diffeomorphism ψ; see Figure 15.2.

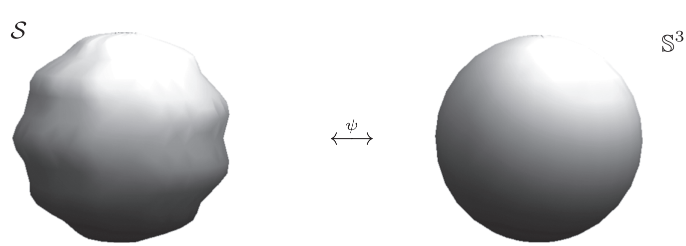

Figure 15.2 Construction of coordinates on a compact three-dimensional manifold describing perturbations of standard de Sitter initial data. The identification of the 3-manifolds $\mathcal{S}$ and $\mathbb{S}^3$ is realised through a harmonic map; see main text for further details.

Parametrising the perturbation data

While the frame $\{\boldsymbol{c_i}\}$ is orthonormal with respect to the standard metric $\boldsymbol{\hbar}$ of $\mathbb{S}^3$, in general, this will not be the case with respect to the 3-metric $\boldsymbol{h}$ on $\mathcal{S}$. Now, let $\{\boldsymbol{e_i}\}$ denote an $\boldsymbol{h}$-orthonormal frame over $T(\mathcal{S})$ and let $\{\boldsymbol{\omega^i}\}$ denote its corresponding cobasis. In what follows, it will be assumed that one can write

$$\boldsymbol{e_i} = \boldsymbol{c_i} + \breve{\boldsymbol{e}}_{\boldsymbol{i}}, \qquad (15.15)$$

for some vectors $\{\breve{\boldsymbol{e}}_{\boldsymbol{i}}\}$. This is essentially equivalent to saying that one has introduced coordinates $\underline{x} = (x^\alpha)$ on $\mathcal{S}$ such that

$$\boldsymbol{h} = \boldsymbol{\hbar} + \breve{\boldsymbol{h}}.$$

It is important to emphasise that the above statement depends on the gauge.

From the split in Equation (15.15), it follows that the solution to the conformal constraint equations implied by $(\Omega = 1, \Sigma = 0, \boldsymbol{h}, \boldsymbol{K})$ on $\mathcal{S}$ can be written as

$$e_{\boldsymbol{a}}{}^{\boldsymbol{b}} = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}} + \breve{e}_{\boldsymbol{a}}{}^{\boldsymbol{b}},$$
$$\gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} = \epsilon_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} + \breve{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}, \qquad \chi_{\boldsymbol{ij}} = \breve{K}_{\boldsymbol{ij}},$$
$$L_{\boldsymbol{ij}} = \delta_{\boldsymbol{ij}} + \breve{L}_{\boldsymbol{ij}}, \qquad L_{\boldsymbol{i}} = \delta_{\boldsymbol{i}}{}^{\boldsymbol{0}} + \breve{L}_{\boldsymbol{i}},$$
$$d_{\boldsymbol{ij}} = \breve{d}_{\boldsymbol{ij}}, \qquad d^*_{\boldsymbol{ij}} = \breve{d}^*_{\boldsymbol{ij}},$$

where the components of the various fields are expressed as components with respect to the frame $\{\boldsymbol{e_i}\}$ as given in (15.15) and one has

$$\breve{e}_{\boldsymbol{a}}{}^{\boldsymbol{b}} = 0, \qquad \breve{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} = 0, \qquad \breve{L}_{\boldsymbol{ij}} = 0, \qquad \breve{L}_{\boldsymbol{i}} = 0, \qquad \breve{d}_{\boldsymbol{ij}} = 0, \qquad \breve{d}^*_{\boldsymbol{ij}} = 0,$$

if and only if

$$\breve{\boldsymbol{e}}_{\boldsymbol{i}} = 0, \qquad \boldsymbol{K} = 0.$$

Accordingly, the fields topped with a $\breve{}$ together with $K_{\boldsymbol{ij}}$ describe the deviation of a solution to the conformal constraint equations from data for the exact de Sitter spacetime. It is important to observe that as

$$\breve{e}_{\boldsymbol{a}}{}^{\boldsymbol{b}}, \qquad \breve{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}, \qquad \breve{K}_{\boldsymbol{ij}}, \qquad \breve{L}_{\boldsymbol{ij}}, \qquad \breve{L}_{\boldsymbol{i}}, \qquad \breve{d}_{\boldsymbol{ij}}, \qquad \breve{d}^*_{\boldsymbol{ij}}$$

are scalars, by virtue of the diffeomorphism $\psi : \mathcal{S} \to \mathbb{S}^3$, they can be considered as fields over $\mathbb{S}^3$. As such, for $m \geq 0$, one defines the Sobolev norms

$$\| \breve{e}_{\boldsymbol{a}}{}^{\boldsymbol{b}} \|_{\mathcal{S},m} \equiv \sum_{\boldsymbol{a},\boldsymbol{b}} \| \breve{e}_{\boldsymbol{a}}{}^{\boldsymbol{b}} \|_{\mathbb{S}^3,m}, \qquad \| \breve{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} \|_{\mathcal{S},m} \equiv \sum_{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}} \| \breve{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} \|_{\mathbb{S}^3,m},$$

and, similarly, for the other fields – the sums in the previous expressions are carried out over the independent components of the particular field under consideration. In terms of these norms, it will be said that the initial data for

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the conformal field equations are ε***-close*** in the norm $\parallel \quad \parallel_{\mathcal{S},m}$ to initial data for the de Sitter spacetime if

$$\parallel \breve{e}_{\boldsymbol{a}}{}^{\boldsymbol{b}} \parallel_{\mathcal{S},m} + \parallel \breve{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} \parallel_{\mathcal{S},m} + \parallel \breve{K}_{\boldsymbol{ij}} \parallel_{\mathcal{S},m} + \parallel \breve{L}_{\boldsymbol{ij}} \parallel_{\mathcal{S},m} \\ + \parallel \breve{L}_{\boldsymbol{i}} \parallel_{\mathcal{S},m} + \parallel \breve{d}_{\boldsymbol{ij}} \parallel_{\mathcal{S},m} + \parallel \breve{d}^{*}_{\boldsymbol{ij}} \parallel_{\mathcal{S},m} < \varepsilon. \tag{15.16}$$

This notion of closeness to initial data is gauge dependent. Nevertheless, it is the appropriate one to exploit the existence and stability theorems of Chapter 12.

15.2.2 Initial data on the conformal boundary

An important property of de Sitter-like spacetimes is that the individual components of the conformal boundary can serve as Cauchy hypersurfaces of the unphysical spacetimes. Accordingly, it is possible to formulate for these spacetimes an ***asymptotic initial value problem*** where initial data are prescribed on a 3-manifold corresponding to, say, $\mathscr{I}^-$.

The solutions to the conformal constraint equations at the conformal boundary have been discussed in Section 11.4.4. In particular, it has been shown that one needs to prescribe on $\mathscr{I}^-$ a 3-metric $\boldsymbol{h}$, a symmetric trace-free and divergence-free tensor corresponding to the initial value of the electric part of the rescaled Weyl tensor and a function $\varkappa$. From these free data it is possible to compute the values of the remaining conformal fields. In the particular case of the exact de Sitter spacetime it can be verified that the asymptotic free data are given by

$$\boldsymbol{h} \simeq \boldsymbol{\hbar}, \qquad d_{\boldsymbol{ij}} \simeq 0, \qquad \varkappa \simeq 0,$$

where components are expressed with respect to the $\boldsymbol{\hbar}$-orthonormal frame $\{\boldsymbol{c_i}\}$. From the above one finds

$$e_{\boldsymbol{i}}{}^{\boldsymbol{j}} \simeq \delta_{\boldsymbol{i}}{}^{\boldsymbol{j}}, \quad \gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} \simeq \epsilon_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}, \quad K_{\boldsymbol{ij}} \simeq 0, \quad L_{\boldsymbol{i}} \simeq 0, \quad L_{\boldsymbol{ij}} \simeq \frac{1}{2}\delta_{\boldsymbol{ij}}, \quad d^{*}_{\boldsymbol{ij}} \simeq 0.$$

Perturbations of the above asymptotic initial data for the de Sitter spacetime are discussed in a manner similar to that of perturbations of standard Cauchy data. Accordingly, assuming that $\mathscr{I}^- \approx \mathbb{S}^3$, one can make use of diffeomorphisms $\psi : \mathscr{I}^- \rightarrow \mathbb{S}^3$ to introduce coordinates on the conformal boundary and to pull back the components of the various conformal fields to $\mathbb{S}^3$. Initial data corresponding to perturbations of asymptotic de Sitter initial data will then be described in terms of fields

$$\boldsymbol{h} = \boldsymbol{\hbar} + \breve{\boldsymbol{h}}, \qquad d_{\boldsymbol{ij}} = \breve{d}_{\boldsymbol{ij}}, \qquad \breve{\varkappa},$$

where $\breve{d}_{\boldsymbol{ij}}$ are the components of a symmetric, $\boldsymbol{h}$-trace-free and $\boldsymbol{h}$-divergence-free tensor expressed in terms of the components of the $\boldsymbol{h}$-orthonormal frame $\{\boldsymbol{e_i}\} = \{\boldsymbol{c_i} + \breve{\boldsymbol{e}}_{\boldsymbol{i}}\}$. Mimicking the standard Cauchy case, the perturbation of asymptotic data for the de Sitter spacetime will be said to be ε***-close to exact asymptotic de Sitter data*** in the $\parallel \quad \parallel_m$-norm if the various conformal fields on $\mathscr{I}^-$ satisfy an inequality of the form of (15.16). In principle, it is possible to express this smallness requirement in terms of a smallness condition on the *basic perturbation data* $\breve{e}_{\boldsymbol{i}}{}^{\boldsymbol{j}}$, $\breve{d}_{\boldsymbol{ij}}$ and $\breve{\varkappa}$; this idea will not be further pursued here.

15.3 Global existence and stability using gauge source functions

In this section a first proof of the global existence and stability of de Sitter-like spacetimes is provided. This proof makes use of the hyperbolic reduction of the spinorial conformal field equations using gauge source functions as discussed in Section 13.2 and of the conformal representation of the de Sitter spacetime discussed in Section 15.1.1. This approach can be readily generalised to include trace-free matter. The discussion presented here follows the seminal work by Friedrich (1986b, 1991).

15.3.1 Gauge considerations

The first step in the construction of de Sitter-like spacetimes consists of the fixing of the gauge in the evolution equations. This *gauge fixing* allows one to relate, in an unambiguous manner, fields in the background de Sitter spacetime with fields in the perturbed spacetime; see Figure 15.3.

As the (unphysical) spacetime $(\mathcal{M}, \boldsymbol{g})$ to be constructed will be of the form $\mathcal{M} \approx [a, b] \times \mathbb{S}^3 \subset \mathbb{R} \times \mathbb{S}^3$ with a, $b \in \mathbb{R}$, it is natural to make use of the coordinates and frames in the background spacetime $(\mathbb{R} \times \mathbb{S}^3, \boldsymbol{g}_{\mathscr{E}})$ to coordinatise and construct a suitable gauge in the perturbed spacetime. Following the discussion of Section 13.2.1, coordinates $x = (\tau, x^\alpha)$ on the Einstein cylinder $\mathcal{M}_{\mathscr{E}} = \mathbb{R} \times \mathbb{S}^3$ can be regarded as coordinates on a perturbed spacetime $(\mathcal{M}, \boldsymbol{g})$ if one identifies the manifolds $\mathcal{M}$ and $\mathcal{M}_{\mathscr{E}}$. This coordinatisation is equivalent to the coordinate gauge source choice

$$F^{\boldsymbol{a}}(x) = -\eta^{\boldsymbol{bc}} \mathring{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{c}} = 0,$$

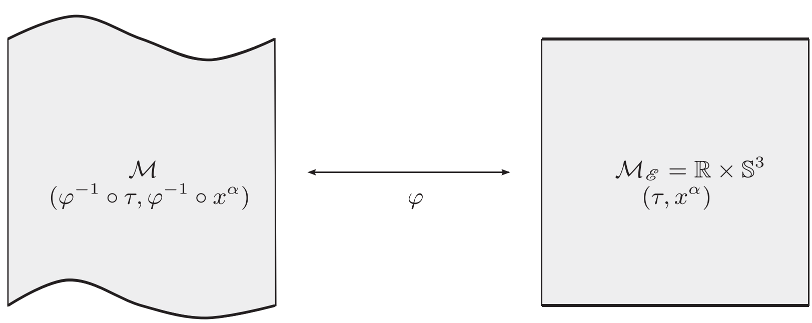

Figure 15.3 Schematic representation of the construction of coordinates on a perturbation of the de Sitter spacetime $(\mathcal{M}, \boldsymbol{g})$ using coordinates on the exact de Sitter spacetime $(\mathcal{M}_{\mathscr{E}}, \boldsymbol{g}_{\mathscr{E}})$ and a diffeomorphism $\varphi : \mathcal{M} \to \mathring{\mathcal{M}}$ as described in the main text. The particular realisation of the diffeomorphism identifies the manifolds $\mathcal{M}$ and $\mathcal{M}_{\mathscr{E}}$ in such a way that φ is a wave map.

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where the last equality follows from the discussion leading to Lemma 15.1. This particular choice of coordinate gauge source function makes the identification between $\mathcal{M}$ and $\mathcal{M}_{\mathscr{E}}$ a wave map; see Section 13.2.1.

By similar considerations, the vectors $\{\mathring{\boldsymbol{c}}_{\boldsymbol{a}}\} = \{\boldsymbol{\partial}_\tau, \mathring{\boldsymbol{c}}_{\boldsymbol{i}}\}$ originally defined on $\mathcal{M}_{\mathscr{E}}$ can be regarded as vectors on the perturbed spacetime $(\mathcal{M}, \boldsymbol{g})$ in terms of which the $\boldsymbol{g}$-orthonormal frame $\{\boldsymbol{e_a}\}$ can be expanded by writing $\boldsymbol{e_a} = \boldsymbol{e_a}^{\boldsymbol{b}}\mathring{\boldsymbol{c}}_{\boldsymbol{b}}$. In an analogous manner, the fields

$$\mathring{\mathbf{u}} = (\mathring{\Xi}, \mathring{\Sigma}, \mathring{\Sigma}_{\boldsymbol{AA'}}, \mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}, \mathring{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}, \mathring{\Phi}_{\boldsymbol{AA'BB'}}, \mathring{\phi}_{\boldsymbol{ABCD}}),$$

as given by Lemma 15.1, can be regarded as fields over $\mathcal{M}$. It is important to emphasise that all of the above fields (except for $\mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}$) are, in fact, components of tensors with respect to the background frame $\{\mathring{\boldsymbol{e}}_{\boldsymbol{a}}\} = \{\delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}\mathring{\boldsymbol{c}}_{\boldsymbol{b}}\}$.

The gauge fixing is completed by setting the frame gauge source function $F_{\boldsymbol{AB}}(x)$ and the conformal gauge source function $R(x)$ equal to their values in the background spacetime $(\mathbb{R} \times \mathbb{S}^3, \boldsymbol{g}_{\mathscr{E}})$. That is, one sets

$$F_{\boldsymbol{AB}}(x) = 0, \qquad R(x) = -6;$$

compare Lemma 15.1.

15.3.2 The evolution system

The hyperbolic reduction procedure discussed in Section 13.2 and summarised in Proposition 13.1, leads to an evolution system which, in terms of local coordinates $x = (\tau, x^\alpha)$ of an open domain $\mathcal{U} \subset \mathbb{R} \times \mathbb{S}^3$, takes the form

$$\partial_\tau \boldsymbol{\sigma} = \mathbf{G}(\boldsymbol{\Gamma})\boldsymbol{\sigma} + \mathbf{H}(\boldsymbol{\sigma}, \boldsymbol{\upsilon}), \tag{15.17a}$$

$$\big(\mathbf{I} + \mathbf{D}^0(\boldsymbol{e})\big)\partial_\tau \boldsymbol{\upsilon} + \mathbf{D}^\alpha(\boldsymbol{e})\partial_\alpha \boldsymbol{\upsilon} = \mathbf{E}(\boldsymbol{\Gamma})\boldsymbol{\upsilon} + \mathbf{F}(\boldsymbol{\sigma}, \boldsymbol{\upsilon}, \boldsymbol{\phi}), \tag{15.17b}$$

$$\big(\mathbf{I} + \mathbf{A}^0(\boldsymbol{e})\big)\partial_\tau \boldsymbol{\phi} + \mathbf{A}^\alpha(\boldsymbol{e})\partial_\alpha \boldsymbol{\phi} = \mathbf{B}(\boldsymbol{\Gamma})\boldsymbol{\phi}, \tag{15.17c}$$

where $\boldsymbol{\sigma}$ encodes the conformal factor Θ and the independent components of its concomitants; $\boldsymbol{\upsilon}$ collects the independent components of the frame components, the connection coefficients and the trace-free Ricci spinor and $\boldsymbol{\phi}$ groups the independent components of the rescaled Weyl spinor.

To apply the methods of the theory of hyperbolic partial differential equations (PDEs) discussed in Chapter 12 it is convenient to split the various field unknowns into a ***background part*** and a ***perturbation part***. More precisely, one sets

$$\Xi = \mathring{\Xi} + \breve{\Xi}, \quad \Sigma = \mathring{\Sigma} + \breve{\Sigma}, \quad \Sigma_{\boldsymbol{AB}} = \breve{\Sigma}_{\boldsymbol{AB}}, \quad s = \mathring{s} + \breve{s}, \tag{15.18a}$$

$$e^{\boldsymbol{0}} = \mathring{e}^{\boldsymbol{0}} + \breve{e}^{\boldsymbol{0}}, \quad e_{\boldsymbol{AB}}{}^{\boldsymbol{0}} = \breve{e}_{\boldsymbol{AB}}{}^{\boldsymbol{0}}, \tag{15.18b}$$

$$e^{\boldsymbol{i}} = \breve{e}^{\boldsymbol{i}}, \quad e_{\boldsymbol{AB}}{}^{\boldsymbol{i}} = \mathring{e}_{\boldsymbol{AB}}{}^{\boldsymbol{i}} + \breve{e}_{\boldsymbol{AB}}{}^{\boldsymbol{i}}, \tag{15.18c}$$

$$\Gamma_{\boldsymbol{AB}} = \breve{\Gamma}_{\boldsymbol{AB}}, \quad \Gamma_{(\boldsymbol{AB})\boldsymbol{CD}} = \mathring{\Gamma}_{(\boldsymbol{AB})\boldsymbol{CD}} + \breve{\Gamma}_{(\boldsymbol{AB})\boldsymbol{CD}}, \tag{15.18d}$$

$$\Phi_{(\boldsymbol{ABCD})} = \breve{\Phi}_{(\boldsymbol{ABCD})}, \quad \Phi_{\boldsymbol{AB}} = \breve{\Phi}_{(\boldsymbol{AB})}, \quad \Phi = \mathring{\Phi} + \breve{\Phi}, \tag{15.18e}$$

$$\phi_{\boldsymbol{ABCD}} = \breve{\phi}_{\boldsymbol{ABCD}}, \tag{15.18f}$$

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where

$$\mathring{\Xi}, \quad \mathring{\Sigma}, \quad \mathring{s}, \quad \mathring{e}^{\mathbf{0}}, \quad \mathring{e}_{\boldsymbol{AB}}{}^{\boldsymbol{i}}, \quad \mathring{\Gamma}_{(\boldsymbol{AB})\boldsymbol{CD}}, \quad \mathring{\Phi}$$

are the non-vanishing components of the fields describing the *background de Sitter solution* as discussed in Section 15.1.1, while

$$\breve{\Xi}, \quad \breve{\Sigma}, \quad \breve{\Sigma}_{\boldsymbol{AB}}, \quad \breve{s}, \quad \breve{e}^{\boldsymbol{a}}, \quad \breve{e}_{\boldsymbol{AB}}{}^{\boldsymbol{a}}, \quad \breve{\Gamma}_{\boldsymbol{AB}}, \quad \breve{\Gamma}_{(\boldsymbol{AB})\boldsymbol{CD}}, \tag{15.19a}$$
$$\breve{\Phi}_{(\boldsymbol{ABCD})}, \quad \breve{\Phi}_{(\boldsymbol{AB})}, \quad \breve{\Phi}, \quad \breve{\phi}_{\boldsymbol{ABCD}} \tag{15.19b}$$

describe the *perturbations away from the de Sitter solution*. The split between background and perturbations given by Equations (15.17a)–(15.17c) depends strongly on the choice of gauge.

By construction, the background fields are a solution to the conformal evolution Equations (15.17a)–(15.17c). Consequently, one has

$$\partial_\tau \mathring{\boldsymbol{\sigma}} = \mathbf{G}(\mathring{\boldsymbol{\Gamma}})\mathring{\boldsymbol{\sigma}} + \mathbf{H}(\mathring{\boldsymbol{\sigma}}, \mathring{\boldsymbol{\upsilon}}),$$
$$\big(\mathbf{I} + \mathbf{D}^0(\mathring{\boldsymbol{e}})\big)\partial_\tau \mathring{\boldsymbol{\upsilon}} + \mathbf{D}^\alpha(\mathring{\boldsymbol{e}})\partial_\alpha \mathring{\boldsymbol{\upsilon}} = \mathbf{E}(\mathring{\boldsymbol{\Gamma}})\mathring{\boldsymbol{\upsilon}}.$$

Accordingly, substituting now the ansatz (15.18a)–(15.18f) in the evolution system (15.17a)–(15.17c), one obtains equations for the independent components of the perturbation fields (15.19a) and (15.19b):

$$\begin{aligned}
\partial_\tau \breve{\boldsymbol{\sigma}} &= \mathbf{G}(\mathring{\boldsymbol{\Gamma}})\breve{\boldsymbol{\sigma}} + \mathbf{G}(\breve{\boldsymbol{\Gamma}})\mathring{\boldsymbol{\sigma}} + \mathbf{G}(\breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{\sigma}} \\
&\quad + \mathbf{H}(\mathring{\boldsymbol{\sigma}}, \breve{\boldsymbol{\upsilon}}) + \mathbf{H}(\breve{\boldsymbol{\sigma}}, \mathring{\boldsymbol{\upsilon}}) + \mathbf{H}(\breve{\boldsymbol{\sigma}}, \breve{\boldsymbol{\upsilon}}),
\end{aligned} \tag{15.20a}$$

$$\begin{aligned}
\big(\mathbf{I} + \mathbf{D}^0(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}})\big)\partial_\tau \breve{\boldsymbol{\upsilon}} + \mathbf{D}^\alpha(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}})\partial_\alpha \breve{\boldsymbol{\upsilon}} &= \mathbf{E}(\breve{\boldsymbol{\Gamma}})\mathring{\boldsymbol{\upsilon}} + \mathbf{E}(\mathring{\boldsymbol{\Gamma}})\breve{\boldsymbol{\upsilon}} + \mathbf{E}(\breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{\upsilon}} + \mathbf{F}(\mathring{\boldsymbol{\sigma}}, \mathring{\boldsymbol{\upsilon}}, \breve{\boldsymbol{\phi}}) \\
&\quad + \mathbf{F}(\mathring{\boldsymbol{\sigma}}, \breve{\boldsymbol{\upsilon}}, \breve{\boldsymbol{\phi}}) + \mathbf{F}(\breve{\boldsymbol{\sigma}}, \mathring{\boldsymbol{\upsilon}}, \breve{\boldsymbol{\phi}}) + \mathbf{F}(\breve{\boldsymbol{\sigma}}, \breve{\boldsymbol{\upsilon}}, \breve{\boldsymbol{\phi}}) \\
&\quad - \big(\mathbf{I} + \mathbf{D}^0(\breve{\boldsymbol{e}})\big)\partial_\tau \mathring{\boldsymbol{\upsilon}},
\end{aligned} \tag{15.20b}$$

$$\big(\mathbf{I} + \mathbf{A}^0(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}})\big)\partial_\tau \breve{\boldsymbol{\phi}} + \mathbf{A}^\alpha(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}})\partial_\alpha \breve{\boldsymbol{\phi}} = \mathbf{B}(\mathring{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{\phi}}. \tag{15.20c}$$

In view of the properties of the original conformal evolution Equation (15.17a)–(15.17c) the above equations constitute a symmetric hyperbolic evolution system for the components of $\breve{\mathbf{u}} = (\breve{\boldsymbol{\sigma}}, \breve{\boldsymbol{\upsilon}}, \breve{\boldsymbol{\phi}})$. Accordingly, the theory of hyperbolic PDEs, as discussed in Chapter 12, can be applied in domains of the form $[0, \tau_\bullet] \times \mathbb{S}^3$ with $\tau_\bullet > 0$, to guarantee the existence of solutions and to assert Cauchy stability. In particular, as the background solution $(\mathring{\boldsymbol{\sigma}}, \mathring{\boldsymbol{\upsilon}}, \mathring{\boldsymbol{\phi}})$ is well defined on the whole of $\mathbb{R} \times \mathbb{S}^3$ one obtains the following existence, uniqueness and Cauchy stability result:

Proposition 15.1 (***existence of solutions to the standard conformal evolution equations***) *Let $\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ denote de Sitter-like initial data for the conformal field equations prescribed on a 3-manifold $\mathcal{S} \approx \mathbb{S}^3$. Given $m \geq 4$ and $\tau_\bullet > \frac{3}{4}\pi$, then:*

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(i) *There exists* $\varepsilon > 0$ *such that if*

$$\| \breve{\mathbf{u}}_\star \|_m < \varepsilon,$$

then there exists a C^{m-2} *unique solution to the conformal evolution Equations* (15.20a)–(15.20c) *defined on* $[0, \tau_\bullet] \times \mathbb{S}^3$.

(ii) *Given a sequence of initial data* $\mathbf{u}_\star^{(n)} = \mathring{\mathbf{u}}_\star^{(n)} + \breve{\mathbf{u}}_\star^{(n)}$ *such that*

$$\| \breve{\mathbf{u}}_\star^{(n)} \|_m < \varepsilon \qquad \textit{and} \qquad \| \breve{\mathbf{u}}_\star^{(n)} \|_m \to 0 \qquad \textit{as} \qquad n \to \infty,$$

then for the corresponding solutions $\breve{\mathbf{u}}^{(n)} \in C^{m-2}([0, \tau_\bullet] \times \mathbb{S}^3)$ *one has that* $\| \breve{\mathbf{u}}^{(n)} \|_m \to 0$ *uniformly in* $\tau \in [0, \tau_\bullet]$ *as* $n \to \infty$.

Proof The above proposition is a direct consequence of Theorem 12.4. To apply this theorem it is necessary to ensure that both

$$\mathbf{I} + \mathbf{A}^0(\mathring{\boldsymbol{e}}_\star + \breve{\boldsymbol{e}}_\star) \qquad \text{and} \qquad \mathbf{I} + \mathbf{D}^0(\mathring{\boldsymbol{e}}_\star + \breve{\boldsymbol{e}}_\star) \tag{15.21}$$

are both positive definite away from zero in a uniform manner over $\mathbb{S}^3$. An explicit calculation shows that

$$\mathbf{I} + \mathbf{A}^0(\mathring{\boldsymbol{e}}_\star) \qquad \text{and} \qquad \mathbf{I} + \mathbf{D}^0(\mathring{\boldsymbol{e}}_\star)$$

are positive definite away from zero. Thus, by setting ε sufficiently small, condition (15.21) can be guaranteed. By further reducing ε, if necessary, one can ensure that all solutions with $\| \breve{\mathbf{u}}_\star \|_m < \varepsilon$ have a minimum existence time $\tau_\bullet > \frac{3}{4}\pi$. Taking into account the above, point (i) follows from points (i)–(iii) of Theorem 12.4 while point (ii) follows from point (iv) in the same theorem. □

Remark 1. The purpose of point (i) in Proposition 15.1 is to guarantee a minimum existence time of solutions to the evolution system (15.20a)–(15.20c) containing the conformal boundary of the perturbed solution.

Remark 2. Point (ii) in Proposition 15.1 is a statement of Cauchy stability. It ensures that data sufficiently close to data for the de Sitter spacetime give rise to solutions with an existence time similar to that of the background solution. Moreover, within the established existence time, the solutions are suitably close to the background solution. Observe, however, that this result makes no statement about whether a particular solution converges in time to the background solution. Thus, one has obtained only an ***orbital stability result*** for the conformal evolution Equations (15.20a)–(15.20c).

The solutions $\breve{\mathbf{u}}$ provided by Proposition 15.1 give rise, in turn, to a solution to the conformal field equations. More precisely, one has:

Proposition 15.2 (***propagation of the constraints for the standard conformal evolution system***) *Given a solution* $\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}$ *to the conformal* *evolution Equations* (15.20a)–(15.20c) *on* $[0, \tau_\bullet] \times \mathbb{S}^3$ *such that the conformal constraint equations are satisfied on* $\mathcal{S}_\star$*, then*

$$Z_{\boldsymbol{a}} = 0, \qquad Z_{\boldsymbol{ab}} = 0, \qquad \Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0, \qquad \Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0,$$
$$\Delta_{\boldsymbol{cdb}} = 0, \qquad \Lambda_{\boldsymbol{bcd}} = 0,$$

on $[0, \tau_\bullet] \times \mathbb{S}^3$.

Proof From the discussion in Chapters 11 and 13 it follows that if the conformal constraint equations and the conformal evolution equations are satisfied on the initial hypersurface $\mathcal{S}_\star$, then one obtains

$$Z_{\boldsymbol{a}}|_{\mathcal{S}_\star} = 0, \qquad Z_{\boldsymbol{ab}}|_{\mathcal{S}_\star} = 0, \qquad \Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}|_{\mathcal{S}_\star} = 0, \qquad \Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}}|_{\mathcal{S}_\star} = 0,$$
$$\Delta_{\boldsymbol{cdb}}|_{\mathcal{S}_\star} = 0, \qquad \Lambda_{\boldsymbol{bcd}}|_{\mathcal{S}_\star} = 0.$$

Now, from Proposition 13.2 it follows that the above zero quantities satisfy a symmetric hyperbolic subsidiary evolution system. As the initial data for this evolution system vanish and the evolution system is homogeneous in the zero quantities, it follows from Corollary 12.1 that the zero quantities must vanish on $[0, \tau_\bullet) \times \mathbb{S}^3$ so that the result follows. □

Locating the conformal boundary

The existence of solutions to the evolution Equations (15.20a)–(15.20c) for a minimum existence interval $[0, \tau_\bullet) \supset [0, \frac{3}{4}\pi)$ provides room enough for the development of the conformal boundary. That this does indeed happen is crucial for the interpretation of the solution to the conformal evolution equations as a *global solution to the Einstein field equations*. This property is ensured by the following:

Lemma 15.3 (***structure of the conformal boundary***) *Given a solution* $\breve{\mathbf{u}}$*, as given by Proposition 15.1, with* $\| \breve{\mathbf{u}}_\star \|_m < \varepsilon$ *sufficiently small, there exists a function* $\tau_+ = \tau_+(\underline{x})$, $\underline{x} \in \mathbb{S}^3$ *such that* $0 < \tau_+(\underline{x}) < \tau_\bullet$ *and*

$$\Xi > 0 \qquad \text{on } \tilde{\mathcal{M}} \equiv \big\{(\tau, \underline{x}) \in \mathbb{R}^3 \mid 0 \leq \tau < \tau_+(\underline{x})\big\},$$
$$\Xi = 0 \quad \text{and} \quad \Sigma_{\boldsymbol{a}}\Sigma^{\boldsymbol{a}} = -\frac{1}{3}\lambda < 0 \qquad \text{on } \mathscr{I}^+ \equiv \big\{(\tau_+(\underline{x}), \underline{x}) \in \mathbb{R} \times \mathbb{S}^3\big\}.$$

Remark. The above lemma ensures the existence, at least for sufficiently small perturbations, of a complete spacelike component of the conformal boundary. Observe also that the function $\tau_+(\underline{x})$ provided by Lemma 15.3 defines a diffeomorphism between $\mathbb{S}^3$ and $\mathscr{I}^+$. Consequently $\mathscr{I}^+ \approx \mathbb{S}^3$.

Proof The key observation to prove this result is that $\mathring{\Xi}|_{\tau=3\pi/4} < 0$. Using Proposition 15.1 (ii), for sufficiently small $\varepsilon > 0$ one has $(\mathring{\Xi} + \breve{\Xi})|_{\tau=3\pi/4} < 0$. As $(\mathring{\Xi} + \breve{\Xi})|_{\tau=0} > 0$ there must exist a τ_+ for which $\Xi = 0$. By reducing ε further – if necessary – one has that τ is unique, and, hence, the function $\tau_+(\underline{x})$ is well defined. Now, from the conformal Equation (8.28e) it follows that

$$\nabla^{\boldsymbol{a}}\Xi\nabla_{\boldsymbol{a}}\Xi = -\frac{1}{3}\lambda > 0, \qquad \text{if } \Xi = 0.$$

Accordingly, $\tau = \tau_+(\underline{x})$ defines a regular spacelike hypersurface $\mathscr{I}^+$. □

The last step in the present analysis is to show that the obtained solutions to the conformal evolution Equations (15.20a)–(15.20c) give rise to a global solution to the vacuum Einstein field equations. One has the following:

Theorem 15.1 (***global existence and stability of de Sitter-like spacetimes: gauge source functions version***) *Given $m \geq 4$, a solution $\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ to the conformal constraint equations with de Sitter-like cosmological constant such that $\| \breve{\mathbf{u}}_\star \|_m < \varepsilon$ for $\varepsilon > 0$ suitably small gives rise to a unique C^{m-2} solution to the conformal Einstein field equations on*

$$\mathcal{M} \equiv \tilde{\mathcal{M}} \cup \mathscr{I}^+$$

with $\tilde{\mathcal{M}}$ and $\mathscr{I}^+$ as defined in Lemma 15.3. This solution implies, in turn, a solution $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$, to the Einstein field equations with de Sitter-like cosmological constant for which $\mathscr{I}^+$ represents conformal infinity.

Remark. The above theorem together with Propositions 15.1 and 15.2 and Lemma 15.3 constitute a technical version of the main theorem of this chapter. As the component of the conformal boundary obtained by this procedure is a spacelike hypersurface with the topology of $\mathbb{S}^3$, one concludes that the solution $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ to the Einstein field equations has the same global structure as the exact de Sitter spacetime; see Figure 15.4.

Proof From Proposition 8.2; it follows that a solution to the conformal Einstein field equations implies the existence of a metric $\tilde{\boldsymbol{g}}$ satisfying the Einstein field

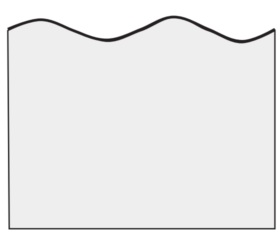

Figure 15.4 Penrose diagram of a perturbation of the de Sitter spacetime given by Theorem 15.1. The spacetime is obtained as a result of an initial value problem on the Cauchy hypersurface $\mathcal{S}_\star$.

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equations wherever $\Xi \neq 0$. The statement about the interpretation of the conformal boundary follows from Lemma 15.3. □

15.4 Global existence and stability using conformal Gaussian systems

This section provides an alternative proof of the main theorem of this chapter using the extended conformal Einstein field equations expressed in terms of a conformal Gaussian system. This alternative proof allows one to contrast the strengths and weaknesses of the two different hyperbolic reduction methods discussed in Chapter 13. As will be seen in the following, the use of properties of conformal geodesics greatly simplifies the analysis of the conformal boundary of the spacetime. Generalising this approach to include matter fields is, however, more complicated than if one were to use gauge source functions.

The details of the construction of a *conformal Gaussian system* for the extended conformal field equations have already been discussed in Section 13.4.1. To apply this general discussion to the analysis of perturbations of the de Sitter spacetime, one needs to specify the particular form of the conformal factor Θ and the covector $\boldsymbol{d}$ associated to the congruence of conformal geodesics. This is done in the following subsection.

15.4.1 A priori analysis of the structure of the conformal boundary of perturbations of the de Sitter spacetime

One of the advantages of the formulation of the conformal evolution equations in terms of conformal Gaussian systems is that it provides an a priori knowledge of the location and structure of the conformal boundary; that is, one has an explicit description of the locus of its points, even before knowing that a solution actually exists. This a priori knowledge provides valuable insight into the nature of the underlying initial value problem.

In what follows, let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denote a vacuum spacetime with de Sitter-like cosmological constant. It will be assumed that $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ can be covered with a non-intersecting congruence of conformal geodesics $(x(\bar{\tau}), \tilde{\boldsymbol{\beta}}(\bar{\tau}))$ with affine parameter $\bar{\tau}$ and that the data for the congruence is prescribed on a fiduciary spacelike hypersurface $\mathcal{S}_\star$ described by the condition $\bar{\tau} = 0$. From Proposition 5.1 it follows that, associated to this congruence of conformal geodesics, one has a *canonical conformal factor* Θ of the form

$$\Theta = \Theta_\star + \dot{\Theta}_\star \bar{\tau} + \frac{1}{2}\ddot{\Theta}_\star \bar{\tau}^2, \tag{15.22}$$

with the constraints

$$\dot{\Theta}_\star = \langle \boldsymbol{d}_\star, \dot{\boldsymbol{x}}_\star \rangle, \qquad \Theta_\star \ddot{\Theta}_\star = \frac{1}{2}\boldsymbol{g}^\sharp(\boldsymbol{d}_\star, \boldsymbol{d}_\star) + \frac{1}{6}\lambda, \tag{15.23}$$

where the coefficients $\Theta_\star$, $\dot{\Theta}_\star$ and $\ddot{\Theta}_\star$ are constant along a given conformal geodesic. The conformal factor Θ allows one to obtain a conformal extension

$(\bar{\mathcal{M}}, \bar{\boldsymbol{g}})$ of the physical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ with $\bar{\boldsymbol{g}} \equiv \Theta^2 \tilde{\boldsymbol{g}}$. The specific details of the conformal factor Θ depend on the location of the hypersurface $\mathcal{S}_\star$ with respect to the conformal boundary and give rise to two different initial value problems.

Standard Cauchy problem

First, consider a situation where the initial hypersurface $\mathcal{S}_\star$ does not coincide with the conformal boundary. As one is interested in the construction of spacetimes whose spatial sections have the topology of $\mathbb{S}^3$ it is natural to set, without loss of generality, that

$$\Theta_\star = 1, \qquad \dot{\Theta}_\star = 0,$$

so that no further distortion is introduced in the 3-manifold and the congruence of conformal geodesics is symmetric with respect to the initial hypersurface. Moreover, one can set

$$\tilde{\boldsymbol{\beta}}_\star = 0,$$

so that $\boldsymbol{d}_\star = \Theta_\star \tilde{\boldsymbol{\beta}}_\star = 0$ and the general expression for the conformal factor reduces to

$$\Theta = 1 + \frac{1}{12}\lambda \bar{\tau}^2.$$

Now, using Proposition 5.1 one finds that the components $d_{\boldsymbol{a}}$ of the covector $\boldsymbol{d}$ respect to a Weyl propagated frame $\{\boldsymbol{e_a}\}$ along the congruence of conformal geodesics and such that $\boldsymbol{e_0} = \dot{\boldsymbol{x}}$ are given by

$$d_{\boldsymbol{0}} = \dot{\Theta}, \qquad d_{\boldsymbol{i}} = 0.$$

A direct computation shows that the conformal factor Θ vanishes for

$$\bar{\tau}_\pm = \pm\sqrt{\frac{12}{|\lambda|}}.$$

The above expression gives the location of the conformal boundary. Accordingly, it is natural to define

$$\mathscr{I}^\pm \equiv \{\bar{\tau}_\pm\} \times \mathcal{S},$$

and one has $\mathscr{I}^\pm \approx \mathbb{S}^3$. Finally, recalling the constraint $\boldsymbol{d} = \Theta \boldsymbol{f} + \mathbf{d}\Theta$ and assuming that $\boldsymbol{f}$ is regular at $\mathscr{I}^\pm$ one finds

$$\boldsymbol{g}(\mathbf{d}\Theta, \mathbf{d}\Theta) = \eta^{\boldsymbol{ab}} d_{\boldsymbol{a}} d_{\boldsymbol{b}} = \dot{\Theta}^2 > 0 \qquad \text{at } \mathscr{I}^\pm.$$

Thus, if the conformal boundary exists and is regular, it must be a spacelike hypersurface.

Asymptotic Cauchy problem at $\mathscr{I}^-$

The conformal field equations allow the formulation of an alternative initial value problem in which initial data are prescribed on a spacelike hypersurface $\mathcal{S} \approx \mathbb{S}^3$ representing one of the components of the conformal boundary, say, $\mathscr{I}^-$ – an ***asymptotic initial value problem***. In this spirit, it is natural to prescribe the initial data for the congruence of conformal geodesics directly at the conformal boundary. This is made possible by the conformal invariance of the conformal geodesic equations.

By assumption, on an asymptotic initial value problem one has that $\Theta = 0$ on $\mathscr{I}^-$. Thus, one necessarily has that $\Theta_\star = 0$ and the conformal factor takes the form

$$\Theta = \dot{\Theta}_\star \bar{\tau} + \frac{1}{2}\ddot{\Theta}_\star \bar{\tau}^2.$$

The second expression in Equation (15.23) implies that $\boldsymbol{g}^\sharp(\boldsymbol{d}_\star, \boldsymbol{d}_\star) = -\lambda/3 > 0$ so that $\boldsymbol{d}_\star$ must be timelike. Now, taking into account the further constraint $\boldsymbol{d} = \Theta \boldsymbol{f} + \mathbf{d}\Theta$ and requiring $\dot{\boldsymbol{x}}_\star$ to be normal to $\mathscr{I}^-$, it follows that

$$d_{\boldsymbol{0}\star} = \dot{\Theta}_\star = \sqrt{\frac{|\lambda|}{3}}, \qquad d_{\boldsymbol{i}\star} = 0,$$

where the positive root has been chosen so that Θ is positive in the future of $\mathscr{I}^-$. Accordingly, off $\mathscr{I}^-$ one has

$$d_{\boldsymbol{0}} = \dot{\Theta} = \left(\sqrt{\frac{|\lambda|}{3}} + \ddot{\Theta}_\star \bar{\tau} \right), \qquad d_{\boldsymbol{i}} = 0.$$

So far, the coefficient $\ddot{\Theta}_\star$ has remained unspecified. Accordingly, it will be considered as free data. These data are, in fact, related to value of the Friedrich scalar

$$s \equiv \frac{1}{4}\nabla_a \nabla^a \Theta + \frac{1}{24} R\Theta$$

on $\mathscr{I}^-$. On the one hand, a calculation gives

$$\begin{aligned} s &\simeq \frac{1}{4}\left(\boldsymbol{e_a}\Sigma^{\boldsymbol{a}} + \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{a}}{}_{\boldsymbol{b}} \Sigma^{\boldsymbol{b}} \right) \\ &\simeq \frac{1}{4}\left(\ddot{\Theta}_\star + \dot{\Theta}_\star \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{a}}{}_{\boldsymbol{0}} \right), \end{aligned}$$

where the last equality follows from the fact that $\Sigma_{\boldsymbol{i}}$ vanishes at $\mathscr{I}^-$. On the other hand, the solution to the conformal constraints at the conformal boundary, as discussed in Section 11.4.4, shows that $s_\star \simeq \dot{\Theta}_\star \varkappa$ where $\varkappa$ is a scalar field over $\mathscr{I}^-$; compare Equation (11.40). A further calculation using a frame adapted to $\mathscr{I}^-$ readily yields

$$\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{a}}{}_{\boldsymbol{0}} \simeq \Gamma_{\boldsymbol{i}}{}^{\boldsymbol{i}}{}_{\boldsymbol{0}} \simeq \chi_{\boldsymbol{i}}{}^{\boldsymbol{i}} \simeq \varkappa \delta_{\boldsymbol{i}}{}^{\boldsymbol{i}} \simeq 3\varkappa.$$

One thus concludes that

$$\ddot{\Theta}_\star = \varkappa \dot{\Theta}_\star.$$

In practice, it is convenient to set $\varkappa$ to be constant on $\mathscr{I}^-$. The choice $\varkappa = 0$ gives a representation in which $\mathscr{I}^-$ is a hypersurface with vanishing extrinsic curvature; see Equation (11.41). This representation does not involve a second component of the conformal boundary.

To have a second component of the conformal boundary one hence needs $\varkappa \neq 0$. Adopting the simple choice $\ddot{\Theta}_\star = -1/2$, the conformal factor vanishes at

$$\bar{\tau}_- = 0, \qquad \bar{\tau}_+ = 4\sqrt{\frac{|\lambda|}{3}}.$$

In this conformal representation, the two components of the conformal boundary of the de Sitter-like spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ are given by the sets

$$\mathscr{I}^- = \{0\} \times \mathcal{S}, \qquad \mathscr{I}^+ = \left\{ 4\sqrt{\frac{|\lambda|}{3}} \right\} \times \mathcal{S}.$$

More generally, keeping $\ddot{\Theta}_\star$ unspecified, one finds that the location of $\mathscr{I}^+$ is determined by the free data $\ddot{\Theta}_\star$. Finally, on $\mathscr{I}^+$ one has

$$\boldsymbol{g}(\mathbf{d}\Theta, \mathbf{d}\Theta) = \eta^{\boldsymbol{ab}} d_{\boldsymbol{a}} d_{\boldsymbol{b}} = -\frac{\lambda}{3} > 0,$$

so that both $\mathscr{I}^\pm$, if they exist, are spacelike hypersurfaces.

Remark. In what follows, the analysis of both the standard and the asymptotic initial value problems will be simplified by making use of the choice $\lambda = -3$ for the cosmological constant.

15.4.2 The extended conformal evolution system

Once the conformal factor Θ and the covector $\boldsymbol{d}$ associated to the conformal Gaussian system have been specified, one can proceed to the formulation of an initial value problem. In Proposition 13.3 it has been shown that the extended conformal Einstein field equations expressed in terms of a conformal Gaussian system imply a symmetric hyperbolic evolution system of the form

$$\partial_{\bar{\tau}} \hat{\boldsymbol{\upsilon}} = \mathbf{K}\hat{\boldsymbol{\upsilon}} + \mathbf{Q}(\hat{\boldsymbol{\Gamma}})\hat{\boldsymbol{\upsilon}} + \mathbf{L}(x)\boldsymbol{\phi}, \tag{15.24a}$$

$$\big(\mathbf{I} + \mathbf{A}^0(\boldsymbol{e})\big)\partial_{\bar{\tau}}\boldsymbol{\phi} + \mathbf{A}^\alpha(\boldsymbol{e})\partial_\alpha \boldsymbol{\phi} = \mathbf{B}(\hat{\boldsymbol{\Gamma}})\boldsymbol{\phi}, \tag{15.24b}$$

for $\hat{\mathbf{u}} = (\hat{\boldsymbol{\upsilon}}, \boldsymbol{\phi})$ where $\hat{\boldsymbol{\upsilon}}$ encodes the independent components of the frame, the connection coefficients and the Schouten tensor, while $\boldsymbol{\phi}$ contains the components of the rescaled Weyl spinor.

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Mimicking the analysis of Section 15.3, one considers solutions of the form

$$e_{\boldsymbol{AB}}{}^0 = \breve{e}_{\boldsymbol{AB}}{}^0, \quad e_{\boldsymbol{AB}}{}^\alpha = \bar{e}_{\boldsymbol{AB}}{}^\alpha + \breve{e}_{\boldsymbol{AB}}{}^\alpha,$$
$$\xi_{\boldsymbol{ABCD}} = \bar{\xi}_{\boldsymbol{ABCD}} + \breve{\xi}_{\boldsymbol{ABCD}}, \quad \chi_{\boldsymbol{ABCD}} = \bar{\chi}_{\boldsymbol{ABCD}} + \breve{\chi}_{\boldsymbol{ABCD}}, \quad f_{\boldsymbol{AB}} = \breve{f}_{\boldsymbol{AB}},$$
$$\Theta_{\boldsymbol{ABCD}} = \bar{\Theta}_{\boldsymbol{ABCD}} + \breve{\Theta}_{\boldsymbol{ABCD}}, \quad \phi_{\boldsymbol{ABCD}} = \breve{\phi}_{\boldsymbol{ABCD}},$$

where

$$\bar{e}_{\boldsymbol{AB}}{}^\mu, \qquad \bar{\xi}_{\boldsymbol{ABCD}}, \qquad \bar{\chi}_{\boldsymbol{ABCD}}, \qquad \bar{\Theta}_{\boldsymbol{ABCD}}$$

are the values of the *exact de Sitter solution* expressed in a conformal Gaussian system as discussed in Section 15.1.2; see, in particular, Proposition 15.2. Accordingly, one can write

$$\hat{\boldsymbol{\upsilon}} = \bar{\boldsymbol{\upsilon}} + \breve{\boldsymbol{\upsilon}}, \qquad \boldsymbol{\phi} = \breve{\boldsymbol{\phi}}, \tag{15.25a}$$
$$\boldsymbol{e} = \bar{\boldsymbol{e}} + \breve{\boldsymbol{e}}, \qquad \hat{\boldsymbol{\Gamma}} = \bar{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}}. \tag{15.25b}$$

On the initial hypersurface $\mathcal{S}_\star$ one has

$$\hat{\boldsymbol{\upsilon}}_\star = \bar{\boldsymbol{\upsilon}}_\star + \breve{\boldsymbol{\upsilon}}_\star, \qquad \boldsymbol{\phi} = \breve{\boldsymbol{\phi}}_\star,$$

where $\mathring{\mathbf{u}}_\star = (\bar{\boldsymbol{\upsilon}}_\star, \mathbf{0})$ is the exact de Sitter data discussed in Section 15.2 and $\breve{\mathbf{u}}_\star = (\breve{\boldsymbol{\upsilon}}_\star, \breve{\boldsymbol{\phi}}_\star)$.

As the conformal factor Θ and the covector $\boldsymbol{d}$ are universal – that is, they possess the same form for either the exact de Sitter data or the perturbations thereof – one has

$$\partial_{\bar{\tau}} \bar{\boldsymbol{\upsilon}} = \mathbf{K} \bar{\boldsymbol{\upsilon}} + \mathbf{Q}(\bar{\boldsymbol{\Gamma}}) \bar{\boldsymbol{\upsilon}}.$$

Substituting the ansatz (15.25a) and (15.25b) into Equations (15.24a) and (15.24b) yields the following evolution equations for $\breve{\mathbf{u}} = (\breve{\boldsymbol{\upsilon}}, \breve{\boldsymbol{\phi}})$:

$$\partial_{\bar{\tau}} \breve{\boldsymbol{\upsilon}} = \mathbf{K} \breve{\boldsymbol{\upsilon}} + \mathbf{Q}(\bar{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}}) \breve{\boldsymbol{\upsilon}} + \mathbf{Q}(\breve{\boldsymbol{\Gamma}}) \bar{\boldsymbol{\upsilon}} + \mathbf{L}(x) \breve{\boldsymbol{\phi}}, \tag{15.26a}$$
$$\big(\mathbf{I} + \mathbf{A}^0(\bar{\boldsymbol{e}} + \breve{\boldsymbol{e}})\big) \partial_{\bar{\tau}} \breve{\boldsymbol{\phi}} + \mathbf{A}^\alpha(\bar{\boldsymbol{e}} + \breve{\boldsymbol{e}}) \partial_\alpha \breve{\boldsymbol{\phi}} = \mathbf{B}(\bar{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}}) \breve{\boldsymbol{\phi}}. \tag{15.26b}$$

The above equations are already in a form where the theory of hyperbolic PDEs, as discussed in Chapter 12, can be applied. In particular, existence and Cauchy stability of Equations (15.26a) and (15.26b) are given by Theorem 12.4. The natural domains for solutions to these equations are of the form

$$\mathcal{M}_{\bar{\tau}_\bullet} \equiv [0, \bar{\tau}_\bullet] \times \mathcal{S}, \qquad \mathcal{S} \approx \mathbb{S}^3,$$

for some $\bar{\tau}_\bullet > 0$. The analogue of Propositions 15.1 and 15.2 for the conformal evolution system (15.26a) and (15.26b) is given by:

Proposition 15.3 (***existence of de Sitter-like solutions to the extended conformal evolution equations and propagation of the constraints***) *Let* $\hat{\mathbf{u}}_\star = \bar{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ *denote de Sitter-like (standard or asymptotic) initial data for the conformal field equations prescribed on a 3-manifold* $\mathcal{S} \approx \mathbb{S}^3$*. Given* $m \geq 4$*:*

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(i) There exists $\varepsilon > 0$ such that if

$$\parallel \breve{\mathbf{u}}_\star \parallel_m < \varepsilon,$$

then there exists a C^{m-2} unique solution $\breve{\mathbf{u}}$ to the conformal evolution equations (15.26a) and (15.26b) defined on $(-\frac{5}{2}, \frac{5}{2}) \times \mathbb{S}^3$ for the standard Cauchy problem and on $[0, \frac{9}{2}) \times \mathbb{S}^3$ for the asymptotic Cauchy problem.

(ii) If

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\big|_{\mathcal{S}_\star} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}\big|_{\mathcal{S}_\star} = 0, \qquad \hat{\Delta}_{\boldsymbol{abc}}\big|_{\mathcal{S}_\star} = 0, \qquad \Lambda_{\boldsymbol{abc}}\big|_{\mathcal{S}_\star} = 0,$$

and

$$\delta_{\boldsymbol{a}}\big|_{\mathcal{S}_\star} = 0, \qquad \gamma_{\boldsymbol{ab}}\big|_{\mathcal{S}_\star} = 0, \qquad \varsigma_{\boldsymbol{ab}}\big|_{\mathcal{S}_\star} = 0,$$

then the solution $\breve{\mathbf{u}}$ to the conformal evolution equations given by (i) implies, by reducing ε if necessary, a C^{m-2} solution $\hat{\mathbf{u}} = \bar{\mathbf{u}} + \breve{\mathbf{u}}$ to the extended conformal field equations on $(-\frac{5}{2}, \frac{5}{2}) \times \mathbb{S}^3$ and, respectively, on $[0, \frac{9}{2}) \times \mathbb{S}^3$.

(iii) Given a sequence of initial data $\hat{\mathbf{u}}_\star^{(n)} = \bar{\mathbf{u}}_\star^{(n)} + \breve{\mathbf{u}}_\star^{(n)}$ such that

$$\parallel \breve{\mathbf{u}}_\star^{(n)} \parallel_m < \varepsilon, \quad \text{and} \quad \parallel \breve{\mathbf{u}}_\star^{(n)} \parallel_m \to 0 \qquad \text{as } n \to \infty,$$

then for the corresponding solutions $\breve{\mathbf{u}}^{(n)} \in C^{m-2}\big((-\frac{5}{2}, \frac{5}{2}) \times \mathbb{S}^3\big)$ and, respectively, $C^{m-2}([0, \frac{9}{2}) \times \mathbb{S}^3)$, one has $\parallel \breve{\mathbf{u}}^{(n)} \parallel_m \to 0$ uniformly in $\bar{\tau} \in (-\frac{5}{2}, \frac{5}{2})$ and, respectively $\bar{\tau} \in [0, \frac{9}{2})$, as $n \to \infty$.

Proof The proof of points (i) and (iii) of the above proposition is, again, a direct application of Theorem 12.4 along the lines of Proposition 15.1. The proof of point (ii) concerning the existence of an actual solution of the extended conformal field equations follows from the homogeneity of the subsidiary evolution system as given in Proposition 13.4 together with Corollary 12.1 by an argument identical to that used in Proposition 15.2. □

Remark. By an argument similar to the one leading to Proposition 15.3, using the expression (15.14) for a separation vector in the background de Sitter spacetime, it can be shown that if ε is sufficiently small, then the separation fields for the perturbed spacetime remain non-zero in $(-\frac{5}{2}, \frac{5}{2})$ and, respectively, $[0, \frac{9}{2})$. Thus, the conformal Gaussian system used in the hyperbolic reduction remains well behaved throughout.

Constructing solutions to the Einstein field equations

The discussion of this section can be summarised in the following two technical versions of the main theorem of this chapter:

Theorem 15.2 (***global existence and stability of de Sitter-like spacetimes: conformal Gaussian systems version***) *Given $m \geq 4$, a solution $\mathbf{u}_\star = \bar{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ to the conformal constraint equations with $\lambda = -3$ on a standard Cauchy hypersurface $\mathcal{S}_\star \approx \mathbb{S}^3$ such that $\parallel \breve{\mathbf{u}}_\star \parallel_m < \varepsilon$, for $\varepsilon > 0$ suitably small, gives rise to a solution $\mathbf{u}$ to the conformal field equations on*

$$\mathcal{M} \equiv [-2, 2] \times \mathbb{S}^3.$$

This solution implies, in turn, a solution $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ to the Einstein field equations with cosmological constant $\lambda = -3$ where

$$\tilde{\mathcal{M}} \equiv (-2, 2) \times \mathbb{S}^3,$$

for which

$$\mathscr{I}^{\pm} \equiv \{\pm 2\} \times \mathbb{S}^3,$$

represent future and past conformal infinity, respectively.

In the case of asymptotic Cauchy data one obtains a similar statement:

Theorem 15.3 (***global existence and stability for the asymptotic initial value problem***) *Given $m \geq 4$, a solution $\mathbf{u}_\star = \bar{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ to the conformal constraint equations with $\lambda = -3$ on a 3-manifold $\mathcal{S} \approx \mathbb{S}^3$ representing the past component of the conformal boundary such that $\parallel \breve{\mathbf{u}}_\star \parallel_m < \varepsilon$, for $\varepsilon > 0$ suitably small, gives rise to a solution $\mathbf{u}$ to the conformal field equations on*

$$\mathcal{M} \equiv [0, 4] \times \mathbb{S}^3.$$

This solution implies, in turn, a solution $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ to the Einstein field equations with cosmological constant $\lambda = -3$ where

$$\tilde{\mathcal{M}} \equiv (0, 4) \times \mathbb{S}^3,$$

for which

$$\mathscr{I}^{-} \equiv \{0\} \times \mathbb{S}^3, \qquad \mathscr{I}^{+} \equiv \{4\} \times \mathbb{S}^3,$$

represent future and past conformal infinity, respectively.

The proofs of Theorems 15.2 and 15.3 are identical to that of Theorem 15.1. Penrose diagrams of the spacetimes thus obtained can be seen in Figure 15.5. Observe that in the gauge being considered, the Penrose diagrams for the exact de Sitter spacetime and the perturbations are identical!

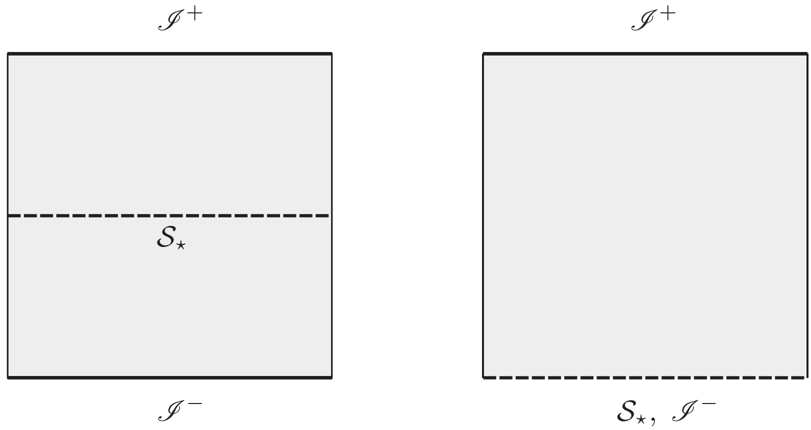

Figure 15.5 Penrose diagrams of de Sitter-like spacetimes obtained from Theorems 15.2 and 15.3: on the left is the spacetime obtained from a standard Cauchy initial value problem; on the right is the spacetime obtained from the asymptotic initial value problem.

15.4.3 Geodesic completeness and asymptotic analysis

The analysis of the existence and stability of de Sitter-like spacetimes developed in Sections 15.3 and 15.4 can be refined to include geodesic completeness. As the exact de Sitter spacetime is geodesically complete, it is to be expected that suitably small perturbations thereof will also share this property. More precisely:

Proposition 15.4 (***geodesic completeness of de Sitter-like spacetimes***) *Suitably small perturbations* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *of the de Sitter spacetime are null and timelike* $\tilde{\boldsymbol{g}}$*-geodesically complete.*

In particular, the above proposition together with the existence and stability results obtained in the previous sections show that suitably small perturbations of the de Sitter spacetime are *asymptotically simple spacetimes.*

It is convenient to divide the analysis of Proposition 15.4 into two cases.

Null geodesics

The key observation required to prove null geodesic completeness is the following: *given the conformal representation* $(\mathbb{R} \times \mathbb{S}^3, \bar{\boldsymbol{g}}_{\mathscr{E}})$ *any null* $\bar{\boldsymbol{g}}_{\mathscr{E}}$*-geodesic starting within the unphysical spacetime reaches the conformal boundary for a finite value of its affine parameter.*

In what follows, let $(\bar{\mathcal{M}}, \bar{\boldsymbol{g}})$ be one of the de Sitter-like spacetimes obtained from, say, a standard Cauchy initial value problem with data prescribed on a hypersurface $\mathcal{S}_\star$. Making use of a perturbative argument similar to the ones employed in Propositions 15.1 and 15.3 and by reducing ε, if necessary, it can be shown that given a point $p \in \mathcal{S}_\star$ and a fixed $\delta > 0$, for all points $q \in \mathcal{S}_\star$ lying

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in an $\boldsymbol{h}$-metric ball of radius δ centred at p, the future directed null $\bar{\boldsymbol{g}}$-geodesics starting at q will reach $\mathscr{I}^+$ in a finite value of their affine parameter. As $\mathcal{S}_\star$ is a compact hypersurface, it can be covered with a finite number of such $\boldsymbol{h}$-metric balls of radius δ, and, accordingly, there exist non-trivial perturbations of the de Sitter spacetime for which all null $\bar{\boldsymbol{g}}$-geodesics starting on $\mathcal{S}_\star$ reach $\mathscr{I}^+$ in a finite value of their affine parameter. Now, every $\bar{\boldsymbol{g}}$-null geodesic is (up to a reparametrisation) also a $\tilde{\boldsymbol{g}}$-null geodesic. Moreover, making use of the discussion in Chapters 7 and 10, one can find an affine parameter s of the $\tilde{\boldsymbol{g}}$-null geodesic such that $\mathrm{s} \to \infty$ as $\Theta \to 0$. Hence, one concludes null $\tilde{\boldsymbol{g}}$-completeness.

Timelike geodesics

In the case of timelike geodesics, following Lemma 5.2 every timelike $\tilde{\boldsymbol{g}}$-geodesic is, up to a reparametrisation, a timelike $\bar{\boldsymbol{g}}_{\mathscr{E}}$-conformal geodesic. It can be explicitly checked – starting, for example, from the general solution to the conformal geodesic equations in the Minkowski spacetime as discussed in Section 6.2.3 – that every $\bar{\boldsymbol{g}}_{\mathscr{E}}$-conformal geodesic starting inside the region of the Einstein cylinder associated to the conformal de Sitter spacetime reaches the conformal boundary of the spacetime in a finite amount of its unphysical proper time $\bar{\tau}$. Using, as in the case of the null geodesics, a perturbative argument, this property is seen to be preserved for suitably small perturbations of the de Sitter spacetime. Of course, not every $\bar{\boldsymbol{g}}$-conformal geodesic can be reparametrised to a $\tilde{\boldsymbol{g}}$-geodesic. This is the case only for those curves reaching the conformally boundary orthogonally – as can be checked using Lemma 5.3. Finally, using the properties of conformal geodesics in Einstein spaces as discussed in Section 5.5.6, the physical proper time of $\tilde{\boldsymbol{g}}$ satisfies $\tilde{\tau} \to \infty$ as $\Theta \to 0$. This implies geodesic completeness.

15.5 Extensions

The results of this chapter can be extended to the case of the Einstein equations coupled with suitable trace-free matter; see Chapter 9.

For simplicity, the subsequent discussion will be restricted to the standard conformal field equations. One of the main difficulties when attempting a direct extension to include matter is the presence of the rescaled Cotton tensor T_{cdb} in the Cotton and Bianchi equations. As discussed in Chapter 9 this tensor involves derivatives of the matter fields. As the Cotton and the Bianchi equations are interpreted as differential conditions on the components of the Schouten tensor and the rescaled Weyl tensor, the inclusion of matter in the analysis brings further terms into the principal part of the conformal evolution equations which, in principle, destroy the symmetric hyperbolicity. In general, these derivatives cannot be eliminated using the field equations for the matter model. Thus, one introduces the derivatives of the matter variables as new unknowns into the problem. Equations for these ***auxiliary variables*** can be obtained by applying a covariant derivative to the matter equation and commuting derivatives. This procedure has been described, for the Maxwell field, in Section 9.2. For suitable matter models – such as the Maxwell field, the conformally invariant scalar field and radiation fluids – the resulting field equations for the auxiliary field admit a symmetric hyperbolic reduction without the need of introducing further gauge source functions.

The construction of suitable symmetric hyperbolic evolution equations for the auxiliary fields needs to be supplemented with their associated subsidiary evolution equations and a further subsidiary equation for the definition of the auxiliary variable. This procedure is similar in spirit to the construction of subsidiary equations for the geometric fields described in Sections 13.3 and 13.4.5.

The procedure briefly described in the previous paragraph has been implemented by Friedrich (1991) for the Einstein-Yang-Mills system, using the standard conformal field equations and a hyperbolic reduction involving gauge source functions to obtain a generalisation of the existence and stability result given in the main theorem of this chapter. The same basic ideas can be used to obtain a future global existence and stability result for perturbations of radiation perfect fluid Friedman-Robertson-Walker cosmological models; see Lübbe and Valiente Kroon (2013b).

Matter and the extended conformal field equations

The implementation of the ideas discussed in the previous paragraphs to the extended conformal field equations requires further consideration. The matter field equations are usually expressed in terms of the Levi-Civita connection $\boldsymbol{\nabla}$ of the unphysical metric $\boldsymbol{g}$. However, the conformal field equations provide direct access only to the Riemann tensor of the Weyl connection $\hat{\boldsymbol{\nabla}}$. Equation (5.30a), relating the Riemann tensors of the connections $\boldsymbol{\nabla}$ and $\hat{\boldsymbol{\nabla}}$, involves the covariant derivatives of the covector $f_{\boldsymbol{a}}$. Thus, further derivatives of the conformal fields enter the principal part of the evolution system in a way which destroys the symmetric hyperbolicity. The antisymmetric part of the derivative $\hat{\nabla}_{[\boldsymbol{a}} f_{\boldsymbol{b}]}$ can be replaced by terms not containing derivatives using the equation

$$\hat{\nabla}_{\boldsymbol{a}} f_{\boldsymbol{b}} - \hat{\nabla}_{\boldsymbol{b}} f_{\boldsymbol{a}} = \hat{L}_{\boldsymbol{ab}} - \hat{L}_{\boldsymbol{ba}};$$

compare Equation (8.45). However, a similar substitution is not possible for the symmetric part $\hat{\nabla}_{(\boldsymbol{a}} f_{\boldsymbol{b})}$. In order to obtain a suitable symmetric hyperbolic system one needs to introduce $\hat{\nabla}_{(\boldsymbol{a}} f_{\boldsymbol{b})}$ as an unknown of the system – or, alternatively, the components $L_{\boldsymbol{ab}}$ of the Schouten tensor of the unphysical Levi-Civita connection $\boldsymbol{\nabla}$. In the case of the Einstein-Maxwell equations it is possible to find suitable evolution equations for the auxiliary field $\psi_{\boldsymbol{AA'BC}} \equiv \nabla_{\boldsymbol{AA'}} \phi_{\boldsymbol{BC}}$ which do not contain the symmetrised derivative $\hat{\nabla}_{(\boldsymbol{a}} f_{\boldsymbol{b})}$; see Lübbe and Valiente Kroon (2012). This, however, is an exceptional situation.

15.6 Further reading

The results discussed in this chapter were first obtained in Friedrich (1986b). Similar results starting from asymptotic Cauchy data were first discussed in Friedrich (1986a). These results have been extended to the case of the Einstein equations coupled to a Yang-Mills field in Friedrich (1991). Alternative proofs, which make use of the extended conformal field equations and conformal gauge systems, in the vacuum and Einstein-Maxwell case, have been given in Lübbe and Valiente Kroon (2009, 2012).

A different way of generalising the global existence and stability results discussed in this chapter is to consider higher dimensions. In this case one cannot make use of the conformal Einstein field equations of Chapter 8, which are valid only for four-dimensional spacetimes. Alternative field equations are required. Global existence and stability results for de Sitter-like vacuum spacetimes of arbitrary dimension have been given in Anderson (2005a) and Anderson and Chruściel (2005) using the *conformal equations* implied by the Fefferman-Graham obstruction tensor.

The methods of this chapter can be adapted to analyse perturbations of cosmological models with radiation perfect fluids and an asymptotic structure similar to that of de Sitter spacetime. An example of this type of analysis can be found in Lübbe and Valiente Kroon (2013b).