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PROXIMAL ANALYSIS AND BOUNDARIES OF CLOSED SETS IN BANACH SPACE, PART I: THEORY

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0. Introduction. As various types of tangent cones, generalized derivatives and subgradients prove to be a useful tool in nonsmooth optimization and nonsmooth analysis, we witness a considerable interest in analysis of their properties, relations and applications.

Recently, Treiman [**18**] proved that the Clarke tangent cone at a point to a closed subset of a Banach space contains the limit inferior of the contingent cones to the set at neighbouring points. We provide a considerable strengthening of this result for reflexive spaces. Exploring the analogous inclusion in which the contingent cones are replaced by pseudocontingent cones we have observed that it does not hold any longer in a general Banach space, however it does in reflexive spaces. Among the several basic relations we have discovered is the following one: the Clarke tangent cone at a point to a closed subset of a reflexive Banach space is equal to the limit inferior of the weak (pseudo) contingent cones to the set at neighbouring points. This generalizes the results of Penot [**14**] and Cornet [**7**] for finite dimensional spaces and of Penot [**14**] for reflexive spaces. What is more we show that this equality characterizes reflexive spaces among Banach spaces, similarly as the analogous equality with the contingent cones instead of the weak contingent cones, characterizes finite dimensional spaces among Banach spaces. We also present several other related characterizations of reflexive spaces and variants of the considered inclusions for boundedly relatively weakly compact sets.

For several years it has been an open question as to how to extend Clarke's proximal normal formula [**6**] from finite dimensional spaces to infinite dimension. We give the answer to this question for reflexive Banach spaces. We present a characterization of the Clarke normal cone at a point to a closed subset of a reflexive Banach space by means of the proximal normal functionals and the other one, by means of the Frechet normals. The importance of our result is underlined by the fact that we use exact normals in contrast to the ϵ-normals used for example in [**18**].

Our main results are presented in the first part of the paper, the second part is devoted to their applications. For example, we obtain generaliza-

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tions of results given in [**12**] and of the primal Bishop-Phelps theorem proven in [**1**]. We complete the results of Borwein and O'Brien [**4**] and of Borwein [**2**] on pseudoconvexity. We mention implications of our main results for vector fields and give detailed analysis of consequences in the theory of differentiability and subdifferentiability. We show how the "lim inf" inclusions under consideration relate to the subdifferential properties of spaces and functions. In particular we show that every lower semicontinuous function in a reflexive Banach space is densely sub-Frechet and that every locally Lipschitian function in a weakly compactly generated Banach space is densely sub-Hadamard. We prove the proximal subgradient formula in a reflexive Banach space which extends one for a finite dimensional space given in [**17**].

We also provide many examples which are carefully analyzed.

1. Preliminaries. Let E be a normed space with continuous linear dual E^*. For the subset $C \subset E$ and a point $x \in \bar{C}$ we will consider the following tangent cones to C at x:

1) *the contingent cone* $K_C(x)$

and

2) *the Clarke tangent cone* $T_C(x)$, as defined for example in [**6**], where also the corresponding references may be found;

3) *the weak* (*sequential*) *contingent cone* $WK_C(x)$, which is the set of weak limits of sequences $t_n^{-1}(c_n - x)$, with $t_n > 0$, $t_n \to 0$, and $c_n \in C$;

4) *the weak* (*sequential*) *tangent cone* $WT_C(x)$, which we define to be the set of those $y \in E$, that for any sequences x_n and t_n such that $x_n \in C$, $t_n > 0$, $x_n \to x$, $t_n \to 0$ there exists a sequence c_n, $c_n \in C$ for which the sequence $t_n^{-1}(c_n - x_n)$ tends weakly to y;

5) *the pseudocontingent cone* $P_C(x) := \overline{co}K_C(x)$;

6) *the weak pseudocontingent cone* $WP_C(x) := \overline{co}WK_C(x)$;

7) *the Bony tangent cone*,

defined as the set of vectors $y \in E$ such that whenever n is a proximal normal vector to C at x, then the inequality

$$\|n - ty\| \geqq \|n\|$$

holds for every $t > 0$.

We recall the definition of a proximal normal vector.

Definition 1.1. Vector $n \in E$ is said to be a *proximal normal vector to* C at $x \in \bar{C}$ if there are $u \notin \bar{C}$ and $\lambda > 0$ such that

$$n = \lambda(u - x) \quad \text{and} \quad \|u - x\| = d_C(u),$$

where

$$d_C(u) := \inf_{c \in C} \|u - c\|.$$

As a straightforward consequence of the introduced definitions we obtain the following inclusions:

$$(1.1)\qquad \begin{array}{ccccc} WT_C(x) & \subset & WK_C(x) & \subset & WP_C(x) \\ \cup & & \cup & & \cup \\ T_C(x) & \subset & K_C(x) & \subset & P_C(x). \end{array}$$

PROPOSITION 1.1. *$B_C(x)$ is a closed cone which satisfies the inclusion*

(1.2) $K_C(x) \subset B_C(x)$.

Furthermore it is convex, whenever E is a smooth Banach space in which case also

(1.3) $P_C(x) \subset B_C(x)$.

If the norm of E is Frechet differentiable (*that is Frechet differentiable away from zero*), *then*

(1.4) $WP_C(x) \subset B_C(x)$.

Proof of the above proposition follows from **[14]**. Inclusion (1.4) is implied by the one obtained in **[14]** for another type of the weak contingent cone: its definition uses bounded nets instead of sequences. It may be observed that in a reflexive space these two definitions are equivalent.

In all the sequel, whenever we refer to a general approximating cone which is among the defined above, we will denote it $R_C(x)$.

Definition 1.2. We say that C is *R-pseudoconvex at $x \in C$* if

(1.5) $C - x \subset R_C(x)$.

If (1.5) holds for all $x \in C$, C is *R-pseudoconvex.*

Definition 1.3. We say $x \in C$ is an *R-proper* point of C if $R_C(x) \neq E$.

Inclusions (1.1)-(1.4) imply obvious relations for R-pseudoconvexity and R-properness.

We will consider also the polars of the approximating cones:

$$R_C^\circ(x) := \{ y^* \in E^* | (y^*, y) \leqq 0 \text{ for all } y \in R_C(x) \}.$$

$N_C(x) := T_C^\circ(x)$ is called *the Clarke normal cone.*

We point out that the pseudocontingent cone and the weak pseudocontingent cone are essential for dual and differentiability results.

The following theorem was first proven in **[18]**. Alternative proofs may be also found in **[15]**, **[16]** and in **[5]**.

THEOREM 1.1 (Treiman): *If C is a closed subset of a Banach space, $x \in C$, then*

$$\liminf_{\substack{x' \to x \\ C}} K_C(x') \subset T_C(x).$$

THEOREM 1.2 (Penot). *If E is reflexive $C \subset E$, $x \in C$ then*

$$T_C(x) \subset \liminf_{\substack{x' \to x \\ C}} WK_C(x').$$

The proof of Theorem 1.2 can be found in **[14]**.

2. Tangent cone separation principle and stars of closed sets in Banach spaces.

Definition 2.1.

$$\text{star } C := \{x \in C \mid [x, y] \subset C \text{ for all } y \in C\},$$

where $[x, y] := \text{co}\{x, y\}$.

We say C is *starshaped* when star C is non-empty.

PROPOSITION 2.1. *For any subset C of a (locally convex vector) space E the following inclusion holds*

$$(2.1) \qquad \text{star } C \subset \bigcap_{x \in C} T_C(x) + x.$$

Proof. Let $\bar{x} \in$ star C and $x \in C$. Let V be any neighbourhood of 0. Then for any $x' \in (x + V) \cap C$ and any t, $0 < t \leqq 1$, we have

$$t^{-1}[(t(\bar{x} - x') + x') - x'] + V$$
$$\subset t^{-1}(C - x') + V,$$

which shows that $\bar{x} - x \in T_C(x)$, hence (2.1) follows.

In particular Proposition 2.1 implies that convex sets are T-pseudoconvex in any locally convex vector space. For closed sets in Banach spaces the inclusion in (2.1) may be replaced with equality. This is a consequence of the following theorem.

THEOREM 2.1. *Let C be a closed subset of a Banach space E. Suppose that $\bar{x}$ in C and $\bar{y}$ in E are such that*

$$(2.2) \qquad [\bar{x}, \bar{y}] \not\subset C.$$

Then for any $r > 0$ one can find x_r in C with

$$\text{(i) } \bar{y} \notin K_C(x_r) + x_r$$

and

$$\text{(ii) } \|x_r - \bar{x}\| \leqq \|\bar{y} - \bar{x}\| + r.$$

Proof. (2.2) implies that there is some $\bar{z}$ in $[\bar{x}, \bar{y}]$ and some $s > 0$ such that

$$(\bar{z} + sB) \cap C = \emptyset,$$

where B is a closed unit ball in E. Take $r \in (0, s)$. Applying Ekeland's variational principle as given in **[9]**, to the function

$$z \mapsto \|z - \bar{y}\|$$

on the set

$$D := C \cap \bigcup_{z \in [\bar{x}, \bar{z}]} (z + rB),$$

yields an x_r in D such that for all x in D

$$\|x - \bar{y}\| \geqq \|x_r - \bar{y}\| - q\|x - x_r\|, \tag{2.3}$$

for some $q \in (0, 1)$. Furthermore there exist $w > 0$ and $p > 0$ such that

$$x_r + (0, w)(\bar{y} - x_r + pB) \subset \bigcup_{z \in [\bar{x}, \bar{z}]} (z + rB). \tag{2.4}$$

But (2.3) shows that for some $\bar{p} > 0$ the "drop" (**[16]**)

$$x_r + (0, w)(\bar{y} - x_r + \bar{p}B)$$

does not intersect D, which together with (2.4) and the definition of D implies that it also does not intersect C. Thus

$$\bar{y} - x_r \notin K_C(x_r),$$

which proves (i), and (ii) follows from the choice of D.

COROLLARY 2.1. *If C is a closed subset of a Banach space then*

$$\text{star } C = \bigcap_{x \in C} T_C(x) + x = \bigcap_{x \in C} K_C(x) + x, \tag{2.5}$$

and when C is starshaped

$$\text{rec(star } C) = \bigcap_{x \in C} T_C(x) = \bigcap_{x \in C} K_C(x), \tag{2.6}$$

where rec(star C) *is the recession cone of star C.*

Proof. By Theorem 2.1

$$\bigcap_{x \in C} K_C(x) + x \subset \text{star } C.$$

With Proposition 2.1 this gives (2.5). Since $T_C(x)$ is convex for all x we have

$$\Big[\bigcap_{x \in C} T_C(x)\Big] + [T_C(x) + x] \subset T_C(x) + x,$$

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and therefore

$$\bigcap_{x \in C} T_C(x) \subset \operatorname{rec}(\operatorname{star} C).$$

Conversely, suppose that h is in rec(star C), that is

$$\operatorname{star} C + \mathbf{R}^+ h \subset \operatorname{star} C.$$

Then for y in star C and $x \in C$ we have by (2.1)

$$y + nh \in T_C(x) + x \quad \text{for all } n \in \mathbf{N}\backslash\{0\},$$

or

$$h \in n^{-1}T_C(x) + n^{-1}(x - y),$$

which since $T_C(x)$ is a closed cone shows that $h \in T_C(x)$. Thus

$$\operatorname{rec}(\operatorname{star} C) \subset \bigcap_{x \in C} T_C(x),$$

and by Theorem 1.1,

$$\bigcap_{x \in C} T_C(x) = \bigcap_{x \in C} K_C(x),$$

which finishes the proof.

In the following sections we will present a strengthening of Corollary 2.1 for boundedly relatively weakly compact sets and we will show its applications.

3. "Lim inf" inclusions and proximal normal formulas in a reflexive Banach space. Let C be a closed subset of a Banach space E, $x \in C$.

Definition 3.1. A functional $n^* \in E^*$ is said to be a *proximal normal functional to C at x* if there are $u \notin C$, $\lambda > 0$, and $y^* \in E^*$ such that

$$(3.1) \qquad n^* = \lambda y^*, \quad \|u - x\| = d_C(u),$$

and

$$(3.2) \qquad (y^*, u - x) = \|u - x\|, \quad \|y^*\| = 1,$$

that is y^* supports unit ball at $\|u - x\|^{-1}(u - x)$, which is the proximal normal vector to C at x. The set of all proximal normal functionals to C at x will be denoted $\mathrm{PN}_C(x)$.

Definition 3.2. A functional $f^* \in E^*$ is said to be a *Frechet normal to C at x* if for any $\epsilon > 0$ there exists a neighbourhood X_ϵ of x such that the inequality

$$(3.3) \qquad (f^*, x' - x) - \epsilon\|x' - x\| \leqq 0$$

holds for all $x' \in C \cap X_\epsilon$.

PROPOSITION 3.1. *The set of all Frechet normals to C at x is included in $WK_C^\circ(x)$ and it coincides with $WK_C^\circ(x)$ whenever E is reflexive.*

Proof. Let f^* be a Frechet normal to C at x and $y \in WK_C(x)$. Then there exists a sequence $t_n^{-1}(c_n - x)$ converging weakly to y with $t_n > 0$, $t_n \to 0$ and $c_n \in C$. Take any $\epsilon > 0$. Since weak convergent sequences are norm bounded

$$t_n^{-1}\|c_n - x\| \leqq M \quad \text{for } n \in \mathbf{N} \text{ and some } M > 0.$$

Furthermore for sufficiently large n all c_n lie in X_ϵ from Definition 3.2. Hence by (3.3)

$$(f^*, y) = \lim_{n\to\infty} t_n^{-1}(f^*, c_n - x) \leqq \epsilon M,$$

which, since ϵ can be arbitrarily small, implies $(f^*, y) \leqq 0$. Therefore $f^* \in WK_C^\circ(x)$. Suppose now that E is reflexive, $f^* \in WK_C^\circ(x)$ but (3.3) does not hold. Then there exists $\epsilon > 0$ and a sequence c_n, $c_n \in C$, $c_n \to x$ for which

$$(3.4) \qquad (f^*, c_n - x) > \epsilon\|c_n - x\|.$$

Let y be a weak limit point of $\|c_n - x\|^{-1}(c_n - x)$. Then $y \in WK_C(x)$ and (3.4) shows $(f^*, y) \geqq \epsilon$, which is a contradiction. Hence we conclude that f^* is a Frechet normal and the proof is finished.

PROPOSITION 3.2. *If E is a smooth space then*

$$(3.5) \qquad B_C(x) = \mathrm{PN}_C^\circ(x).$$

Proof. Observe that whenever E is smooth, then $n^* \in \mathrm{PN}_C(x)$, $\|n^*\| = 1$ implies that n^* is the Gateaux derivative of the norm at some proximal normal vector to C at x. Furthermore any Gateaux derivative of the norm at the proximal normal vector to C at x is in $\mathrm{PN}_C(x)$.

Propositions 1.1, 3.1, and 3.2 imply the following.

COROLLARY 3.1. *If E is reflexive and the norm of E is Frechet differentiable, then for any closed set $C \subset E$, $x \in C$, any proximal normal functional to C at x is a Frechet normal to C at x.*

Equipped with the definitions and basic relations of normal functionals we take on the task of extending the *proximal normal formula*

$$(3.6) \qquad N_C(x) = \overline{\mathrm{co}}\{y^* \in E^* | y^* = \lim y_n^*, x_n \to x, y_n^* \text{ is a proximal normal functional to } C \text{ at } x_n\}$$

from the Euclidean space to a reflexive space.

The following two propositions concerning the properties of reflexive spaces will be helpful on our way.

PROPOSITION 3.3. *Every reflexive Banach space can be given an equivalent Frechet differentiable and locally uniformly convex (hence Kadec) norm.*

The proof is given in [**8**], where also the corresponding definitions may be found. Recall that a Kadec norm is one for which the weak and norm topologies agree on the boundary of the unit sphere.

PROPOSITION 3.4. *If E is a reflexive Banach space with a Kadec norm and C is a closed subset of E, then the set of those points which have a nearest point in C is dense in E.*

The above proposition follows from Lau's nearest point result proven in [**13**].

LEMMA 1. *Let E be a Banach space with a closed unit ball B. For $\alpha > 0$ and $y \in E$ define*

(3.7) $\quad W_\alpha := \mathrm{co}\{y + \alpha B, -y + \alpha B, B\}.$

Let $\|\ \|$ and $\|\ \|_\alpha$ denote the norms on E associated with B and W_α, respectively. Then

(i) *if $\alpha > 0$ and E is reflexive, or if $\alpha \geqq 1$ then the ball W_α is closed, the norms $\|\ \|$ and $\|\ \|_\alpha$ are equivalent and the norm $\|\ \|_\alpha$ is smooth whenever the norm $\|\ \|$ is smooth*;

(ii) if *$\alpha > 0$ and E is reflexive or if $\alpha \geqq 1$, then the norm $\|\ \|_\alpha$ is Kadec whenever the norm $\|\ \|$ is Kadec*;

(iii) *if $\alpha \geqq 1$, or if $\alpha > 0$ and E is reflexive and the norm $\|\ \|$ is Kadec, then the norm $\|\ \|_\alpha$ is Frechet differentiable whenever the norm $\|\ \|$ is Frechet differentiable.*

Proof. Let us observe that W_α of (3.7) may be written as

(3.8) $\quad W_\alpha = \mathrm{co}([-1, 1]y + \alpha B, B),$

and

(3.9) $\quad W_\alpha = [-1, 1]y + \alpha B$ if $\alpha \geqq 1$.

It is easy to see that under given assumptions W_α is closed and equivalence of the norms holds.

To finish proof of (i) assume that the norm $\|\ \|$ is smooth and let $\bar{w} \in E$ be given with $\|\bar{w}\|_\alpha = 1$. Suppose that $x^* \in E^*$ and

(3.10) $\quad (x^*, \bar{w}) = 1, (x^*, w) \leqq \|w\|_\alpha$ for all $w \in W_\alpha$.

Then by (3.8) there exist $0 \leqq \lambda \leqq 1$, $-1 \leqq \beta \leqq 1$ and $b^1, b^2 \in B$ such that

(3.11) $\quad \bar{w} = \lambda b^1 + (1 - \lambda)(\beta y + \alpha b^2).$

Since $\|\bar{w}\|_\alpha = 1$ only the following cases are possible:

(a) $\lambda = 1$ and $||b^1|| = 1$,
(b) $\lambda = 0$ and $||b^2|| = 1$,
(c) $0 < \lambda < 1$ and $||b^1|| = ||b^2|| = 1$.

Let $\bar{b}$ denote b^1 if (a) holds and let $\bar{b}$ denote b^2 if (b) or (c) hold. Then it follows from (3.10) that if

$$y^* := (x^*, \bar{b})^{-1}x^*$$

then

$$(y^*, \bar{b}) \geqq (y^*, b) \quad \text{for all } b \in B \text{ and } (y^*, \bar{b}) = 1.$$

However, since the norm $|| \ ||$ is smooth there is the unique functional in E^* which satisfies the above conditions. This in turn implies that the conditions in (3.10) are satisfied by the unique functional x^*, hence the norm $|| \ ||_\alpha$ is smooth.

Assume now that the norm $|| \ ||$ is Kadec. Let a sequence w_n, $n \in \mathbf{N}\backslash\{0\}$ and w_0 in E be given such that

(3.12) $\quad ||w_n||_\alpha = 1 \quad$ for $n \in \mathbf{N}$ and $w_n \to w_0$ weakly.

Then by (3.8)

(3.13) $\quad w_n = \lambda_n b_n^1 + (1 - \lambda_n)(\beta_n y + \alpha b_n^2)$

for some λ_n, b_n, b_n^1, b_n^2 such that $0 \leqq \lambda_n \leqq 1$, $-1 \leqq \beta_n \leqq 1$, $b_n^1, b_n^2 \in B$, $n \in \mathbf{N}$.

Note that if $\alpha \geqq 1$, we may assume by (3.9) $\lambda_n = 0$, $b_n^1 = 0$ for all $n \in \mathbf{N}$.

The weak (sequential) compactness of the ball B if E is reflexive or the compactness of the set $[-1, 1]y$ when $\alpha \geqq 1$ imply, that there exist λ_0 and β_0, limit points of the sequences λ_n and β_n and there exist b_0^1, b_0^2, weak limit points of the sequences b_n^1, b_n^2. Hence

(3.14) $\quad w_0 = \lambda_0 b_0^1 + (1 - \lambda_0)(\beta_0 y + \alpha b_0^2)$,

and without loss of generality we assume that all considered sequences converge instead of taking their subsequences.

Again only the three cases are possible. If $\lambda_0 = 0$, then for n sufficiently large $0 \leqq \lambda_n < 1$, which together with $||w_0||_\alpha = ||w_n||_\alpha = 1$ implies

$$||b_0^2|| = ||b_n^2|| = 1.$$

Since B is Kadec we obtain $b_n^2 \to b_0^2$. Hence also $w_n \to w_0$, because $\lambda_n \to 0$.

If $\lambda_0 = 1$, the analogous argument shows that $b_n^1 \to b_0^1$, hence also $w_n \to w_0$, because $(1 - \lambda_n) \to 0$.

If $0 < \lambda_0 < 1$, then we may prove that

$$||b_0^1|| = ||b_0^2|| = ||b_n^1|| = ||b_n^2|| = 1$$

for $n \in \mathbf{N}$ sufficiently large and

$$b_n^1 \to b_0^1, \quad b_n^2 \to b_0^2$$

on using Kadec property of the ball B. Hence also $w_n \to w$ and the ball W_α is Kadec.

To prove (ii), suppose that sequences $w_n \in E$, $x_n^* \in E^*$, $n \in \mathbf{N}\backslash\{0\}$ and $w_0 \in E$, $x_0^* \in E^*$ are given such that

(3.15) $\quad w_n \to w_0, \quad \|w_0\|_\alpha = \|w_n\|_\alpha = 1,$

(3.16) $\quad (x_n^*, w_n) = 1, \quad (x_n^*, w) \leqq 1$ for all $w \in W_\alpha$, $n \in \mathbf{N}$.

To prove that the norm $\| \ \|_\alpha$ is Frechet differentiable it is enough to show that

(3.17) $\quad x_n^* \to x_0^*$ in norm,

while (i) implies that

(3.18) $\quad x_n^* \to x_0^*$ weak star.

(See for example [**8**] for justification.) We may assume that w_n and w_0 are as in (3.13) and (3.14). Let $\bar{b}_n$, $\bar{b}_0$ denote b_n^2, b_0^2, respectively if $\lambda_0 = 0$ and b_n^1, b_0^1 if $0 < \lambda_0 \leqq 1$. Then whenever n is sufficiently large, (3.15) when $\alpha \geqq 1$ and (3.15) together with weak sequential compactness and Kadec property of the ball B when $\alpha > 0$ imply

(3.19) $\quad \|\bar{b}_0\| = \|\bar{b}_n\| = 1$ and $\bar{b}_n \to \bar{b}_0,$

while (3.16) implies

$$(x_n^*, \bar{b}_n) \geqq (x_n^*, b) \quad \text{for all } b \in B,$$

(where again without loss of generality we use the sequences instead of the subsequences). Define

$$x_{\bar{b}_n}^* := \{x_n^*, \bar{b}_n)^{-1} x_n^*, \quad n \in \mathbf{N}.$$

Then

(3.20) $\quad (x_{\bar{b}_n}^*, b_n) = 1, \quad (x_{\bar{b}_n}^*, b) \leqq 1$ for all $b \in B$, $n \in \mathbf{N}$.

Now (3.19) and (3.20) together with the assumption that the norm $\| \ \|$ is Frechet differentiable imply that

$$x_{\bar{b}_n}^* \to x_{\bar{b}_0}^* \text{ in norm.}$$

Using this and (3.18) we get (3.17) which finishes the proof.

LEMMA 2. *Let E be a smooth Banach space and let B, W_α, $\| \ \|$, $\| \ \|_\alpha$ be as in Lemma* 1. *Suppose C is a closed subset of E and* $0 \notin C$.

Let $\bar{w} \in C$ be such that

(3.21) $\quad \|\bar{w}\|_\alpha = \inf\{ \|z\|_\alpha | z \in C\},$

and denote the derivative of the norm $\| \, \|_\alpha$ *at* $\bar{w}$ *by* x^*. *Then there exists* $\bar{y} \in E, \bar{y} \notin C$ *such that*

(3.22) $\|\bar{w} - \bar{y}\| = \inf\{ \|z - \bar{y}\| \,|\, z \in C\}$

and $x^*_{\bar{w}-\bar{y}}$, *the derivative of the norm* $\| \, \|$ *at* $\bar{w} - \bar{y}$, *is some positive multiple of* x^*.

Proof. Assume without loss of generality that $\|\bar{w}\|_\alpha = 1$ and suppose that $\bar{w}$ is given by (3.11) and $\bar{b}$ is defined as in the proof of (i) in Lemma 1. Let $\bar{y} := 0$ if $\lambda = 1$ and let

$$\bar{y} := \lambda b^1 + (1 - \lambda)\beta y$$

otherwise. Then if $\lambda = 1$, we have for any $z \in C$

$$\|\bar{w} - \bar{y}\| = \|b^1\| = 1 = \|\bar{w}\|_\alpha \leqq \|z\|_\alpha \leqq \|z\| = \|z - \bar{y}\|,$$

hence (3.22) holds. Also similarly as in Lemma 1 we get

$$x^*_{\bar{w}-\bar{y}} = x^*_{\bar{b}} = (x^*, \bar{b})^{-1}x^*.$$

So assume now that $0 \leqq \lambda < 1$ and suppose to the contrary that

$$\|\bar{c} - \bar{y}\| < \|\bar{w} - \bar{y}\| = (1 - \lambda)\alpha$$

for some $\bar{c}$ in C. Then there exists $\theta > 0$, such that

(3.23) $\bar{c} \in \bar{y} + \|\bar{c} - \bar{y}\|B \subset \lambda b^1 + (1 - \lambda)\beta y + (1 - \lambda - \theta)\alpha B.$

Assume $0 \leqq |\beta| < 1$, then for $\theta < (1 - \lambda)(1 - |\beta|)$, (3.23) shows

$$\bar{c} \in (1 - \theta)W_\alpha$$

which contradicts (3.21) as $\|\bar{w}\|_\alpha = 1$. Similar contradiction follows also when $|\beta| = 1$. This proves (3.22). Furthermore

$$x^*_{\bar{w}-\bar{y}} = x^*_{\bar{w}-\bar{y}} = x^*_{\bar{b}} = (x^*, \bar{b})^{-1}x^*,$$

and the proof is finished.

Let E be a Banach space and let A be a multifunction

$$A:E \mapsto 2^E.$$

Then the following inclusions are straightforward,

(3.24) $$d \liminf_{x' \to x} A(x') \subset (\text{norm})\liminf_{x' \to x} A(x') \subset (w^* \limsup_{x' \to x} A^\circ(x'))^\circ,$$

where

$$w^*(\text{sequential})\limsup_{x' \to x} A^\circ(x')$$

is the set of weak star limits of bounded sequences y_n^* with $y_n^* \in A^\circ(x_n)$ and $x_n \to x$, while

$$d \liminf_{x' \to x} A(x')$$

(the discrete limit inferior) is defined as follows

$$\text{(3.25)} \quad d \liminf_{x' \to x} A(x') := \bigcup_{\delta > 0} \bigcap_{x' \in x + \delta B} A(x'),$$

where B is the unit ball in E. Recall that

$$(\text{norm})\liminf_{x' \to x} A(x') := \bigcap_{\epsilon > 0} \bigcup_{\delta > 0} \bigcap_{x' \in x + \delta B} A(x') + \epsilon B.$$

THEOREM 3.1. *If C is a closed subset of a reflexive Banach space E with Kadec and Frechet differentiable norm, $x \in C$, then*

$$\text{(3.26)} \quad T_C(x) = \liminf_{\substack{x' \to x \\ C}} WK_C(x')$$

$$= \liminf_{\substack{x' \to x \\ C}} WP_C(x') = \liminf_{\substack{x' \to x \\ C}} B_C(x')$$

and the Frechet normal formula and the proximal normal formula

$$\text{(3.27)} \quad N_C(x) = \overline{\text{co}}\ w^* \limsup_{\substack{x' \to x \\ C}} WK_C^\circ(x'),$$

$$N_C(x) = \overline{\text{co}}\ w^* \limsup_{\substack{x' \to x \\ C}} \text{PN}_C(x')$$

hold.

Proof. Using Theorem 1.2, Proposition 1.1, Proposition 3.2 and the last inclusion in (3.24) we get

$$\text{(3.28)} \quad T_C(x) \subset \liminf_{\substack{x' \to x \\ C}} WK_C(x') \subset \liminf_{\substack{x' \to x \\ C}} WP_C(x')$$

$$\subset \liminf_{\substack{x' \to x \\ C}} B_C(x') \subset (w^* \limsup_{\substack{x' \to x \\ C}} \text{PN}_C(x'))^\circ.$$

From (3.24) and Corollary 3.1 it follows that

$$\text{(3.29)} \quad \liminf_{\substack{x' \to x \\ C}} WK_C(x') \subset (w^* \limsup_{\substack{x' \to x \\ C}} WK_C^\circ(x'))^\circ$$

$$\subset (w^* \limsup \text{PN}_C(x'))^\circ.$$

We prove that

(3.30) $\quad (w^* \limsup_{\substack{x' \to x \\ C}} \mathrm{PN}_C(x'))^\circ \subset T_C(x).$

Let B be the closed unit ball in E and let $\| \; \|$ denote the norm of E, which is by our assumption Kadec and Frechet differentiable.

Suppose $y \notin T_C(x)$ and $\|y\| = 1$. Using the characterization of $T_C(x)$ given in [**18**] or Penot [**15**], [**16**], there exists $\epsilon > 0$ such that for each $n \in \mathbf{N}$ we can find $\lambda_n > 0$ and $x_n \in C \cap (x + 2^{-n}B)$ with

(3.31) $\quad C \cap (x_n + (0, \lambda_n](y + 2\epsilon B)) = \emptyset.$

Without loss of generality we suppose $\epsilon < 1/4$. Let us choose $\rho_n > 0$ with

$$\rho_n < \min(2^{-n}, \lambda_n/4, \epsilon/4)$$

and $\alpha_n > 0$ such that

$$\alpha_n < \min(\epsilon\rho_n/6, \lambda_n/6).$$

Let us set for $n \in \mathbf{N}$

$$\begin{aligned} W_n &:= \mathrm{co}(\epsilon B, (1 + \alpha_n)^{-1}(y + \alpha_n\epsilon B), \\ &\qquad\qquad (1 + \alpha_n)^{-1}(-y + \alpha_n\epsilon B)) \\ &= \mathrm{co}((1 + \alpha_n)^{-1}([-1, 1]y + \alpha_n\epsilon B), \epsilon B). \end{aligned}$$

Then for all $n \in \mathbf{N}$ $\;\epsilon B \subset W_n \subset B$. By Lemma 1 the norm $\| \; \|_n$ associated with the ball W_n is equivalent with the norm $\| \; \|$ and it is Kadec and smooth. Select, using Lau's nearest point result (Proposition 3.4), some y_n in E with

(3.32) $\quad y_n - y \in \alpha_n W_n$ and

$$\|y_n - \bar{x}_n\|_n = \inf\{ \|y_n - z\|_n | z \in C_n\},$$

for some $\bar{x}_n \in C_n := (C - x_n) \cap \rho_n B$.

We have $y \in (1 + \alpha_n)W_n$, hence

$$\|y_n - \bar{x}_n\|_n \leqq \|y_n - 0\|_n \leqq \|y_n - y\|_n + \|y\|_n \leqq 1 + 2\alpha_n,$$

so that

(3.33) $$\begin{aligned} \bar{x}_n &= \bar{x}_n - y_n + y_n - y + y \\ &\in (1 + 2\alpha_n)W_n + \alpha_n W_n + y = y + W_n + 3\alpha_n W_n. \end{aligned}$$

Now

(3.34) $$\begin{aligned} y + W_n \subset \mathrm{co}(y + \epsilon B, (1 + \alpha_n)^{-1}(2 + \alpha_n)(y + \epsilon B), \\ (1 + \alpha_n)^{-1}\alpha_n(y + \epsilon B)) \subset (0, +\infty)(y + \epsilon B) \end{aligned}$$

and

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$$\begin{aligned}
&((0, +\infty)(y + \epsilon B) + 3\alpha_n W_n) \cap C_n \\
&= \{((0, +\infty)(y + \epsilon B) + 3\alpha_n W_n) \cap \rho_n B\} \cap C_n \\
&\subset \{(0, (1 - \epsilon)^{-1}(\rho_n + 3\alpha_n)](y + \epsilon B) + 3\alpha_n W_n\} \cap C_n \\
&\subset \{(0, \lambda_n](y + \epsilon B) + 3\alpha_n W_n\} \cap C_n \\
&\subset \{(0, 3\alpha_n \epsilon^{-1})(y + \epsilon B) + 3\alpha_n W_n\} \cap C_n \\
&\cup \{3\alpha_n \epsilon^{-1}, \lambda_n](y + \epsilon B) + 3\alpha_n W_n\} \cap C_n \\
&\subset \{(0, 3\alpha_n \epsilon^{-1})(y + \epsilon B) + 3\alpha_n W_n\} \cap C_n \\
&\cup \{[3\alpha_n \epsilon^{-1}, \lambda_n](y + 2\epsilon B)\} \cap C_n \\
&= \{(0, 3\alpha_n \epsilon^{-1})(y + \epsilon B) + 3\alpha_n W_n\} \cap C_n,
\end{aligned}$$

where the last equality follows from (3.31). Hence using (3.33) and (3.34) we get

$$\bar{x}_n \in (0, 3\alpha_n \epsilon^{-1})(y + \epsilon B) + 3\alpha_n W_n.$$

Therefore

$$\begin{aligned}
\|\bar{x}_n\| &< 3\alpha_n \epsilon^{-1}(1 + \epsilon) + 3\alpha_n = 3\alpha_n \epsilon^{-1}(1 + 2\epsilon) \\
&\leqq 6\alpha_n \epsilon^{-1} < \rho_n,
\end{aligned}$$

and we conclude that there exists $r_n > 0$ such that

$$(3.35) \quad \|\bar{x}_n - y_n\|_n = \inf\{\|z - y_n\|_n | z \in (C - x_n) \cap (\bar{x}_n + r_n B)\}.$$

Let $y_n^* \in E^*$ be the derivative of the norm $\| \|_n$ at $y_n - \bar{x}_n$. Using (3.35), by Lemma 2, we choose $\bar{y}_n \in E$ such that

$$(3.36) \quad \|\bar{x}_n - \bar{y}_n\| = \inf\{\|z - \bar{y}_n\| \,|\, z \in (C - x_n) \cap (\bar{x}_n + r_n B)\}$$

and $\lambda_n > 0$ such that

$$(3.37) \quad y_n^* = \lambda_n y^*_{\bar{y}_n - \bar{x}_n},$$

where $y^*_{\bar{y}_n - \bar{x}_n}$ is the derivative of the norm $\| \|$ at $\bar{y}_n - \bar{x}_n$. Now it is not difficult to see that (3.36) and (3.37) imply that y_n^*, $n \in \mathbf{N}$ is the proximal normal functional to C at $x_n + \bar{x}_n$ as defined in Definition 3.1, that is

$$(3.38) \quad y_n^* \in \mathrm{PN}_C(x_n + \bar{x}_n), \quad n \in \mathbf{N}.$$

As E is reflexive the closed unit ball of E^* is sequentially weak* compact, hence y_n^* being bounded contains a weak star convergent subsequence. Without loss of generality assume that y_n^* converges to some $y^* \in E^*$. Then as $x_n + \bar{x}_n \to x$, we have by (3.38)

$$(3.39) \quad y^* \in w^* \limsup_{\substack{x' \to x \\ C}} \mathrm{PN}_C(x').$$

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But as

$$(y + W_n) \cap C_n \subset (0, +\infty)(y + \epsilon B) \cap C_n$$
$$\subset (0, \lambda_n)(y + \epsilon B) \cap C_n = \emptyset,$$

as shown above, we have

$$\|\bar{x}_n - y_n\|_n > 1 - \alpha_n.$$

Thus

$$\begin{aligned}(y_n^*, y) &= (y_n^*, y_n - \bar{x}_n) + (y_n^*, y - y_n + \bar{x}_n)\\ &\geqq \|y_n - \bar{x}_n\|_n - \|y - y_n\|_n - \|\bar{x}_n\|_n\\ &\geqq 1 - \alpha_n - \alpha_n - \rho_n\epsilon^{-1} > 1/2,\end{aligned}$$

which shows that $(y^*, y) > 0$. This by (3.39) gives

$$y \notin (w^* \limsup_{\substack{x' \to x \\ C}} \mathrm{PN}_C(x'))^\circ,$$

which finishes the proof of (3.30).

(3.28), (3.29), and (3.30) imply (3.26). Taking the polars of the sets in (3.28), (3.29), (3.30) and using the fact that if E is reflexive then the weak star closed convex sets in E^* are norm closed, we get (3.27) and the proof is finished.

COROLLARY 3.2. *If C is a closed subset of a reflexive Banach space, $x \in C$ then*

$$(3.40) \quad T_C(x) = \liminf_{\substack{x' \to x \\ C}} WK_C(x') = \liminf_{\substack{x' \to x \\ C}} WP_C(x')$$

and

$$(3.41) \quad N_C(x) = \overline{\mathrm{co}}\, w^* \limsup_{\substack{x' \to x \\ C}} WK_C^\circ(x').$$

Proof. Combine Proposition 3.3 and Theorem 3.1.

The first equality in (3.40) was proven in [**14**] under stronger assumptions on the set C.

Note that proximal normal formula (3.6) and the corresponding equalities for finite dimensional spaces obtained in [**14**], [**7**], [**11**], follow from Theorem 3.1.

We complete Theorem 1.2 and Theorem 3.1 with the following important observation.

THEOREM 3.2. *Let E be a Banach space. Then the following are equivalent.*

$$1)\quad \liminf_{\substack{x' \to x \\ C}} WP_C(x') = T_C(x)$$

for all closed sets $C \subset E$, $x \in C$,

$$2)\quad \liminf_{\substack{x' \to x \\ C}} WP_C(x') \supset T_C(x)$$

for all closed sets $C \subset E$, $x \in C$,

$$3)\quad \liminf_{\substack{x' \to x \\ C}} WK_C(x') \supset T_C(x)$$

for all closed sets $C \subset E$, $x \in C$,

4) *E is reflexive.*

Proof. Clearly 1) implies 2) and 3) implies 2). Theorem 1.2 shows that 3) is implied by 4) and Theorem 3.1 shows that 1) is implied by 4). Hence it suffices to show that 2) implies 4).

Let E be irreflexive. Choose $z \in E$ and $z^* \in E^*$ with

$$(3.42)\quad (z^*, z) = \|z\| = 1 = \|z^*\|.$$

Let $y \in E$ be such that

$$(3.43)\quad (z^*, y) = 0 \text{ and } \|y\| = 1.$$

Choose $y^* \in E^*$ with

$$(3.44)\quad (y^*, y) = 1 = \|y^*\|.$$

Then

$$\widetilde{E} := \{x \in E |\, (z^*, x) = 0, (y^*, x) = 0\}$$

is obviously an irreflexive subspace of E. Let us observe that the unit sphere in $\widetilde{E}$

$$S := \{x \in \widetilde{E} |\, \|x\| = 1\}$$

is not countably relatively weakly compact. Indeed, otherwise by Eberlein's theorem (see for example [**10**]) it is relatively weakly compact, which implies that

$$\overline{\text{co}}\, S := \{x \in \widetilde{E} |\, \|x\| \leqq 1\}$$

is weakly compact. However this is impossible since $\widetilde{E}$ is irreflexive. Thus there exists a sequence x_n in $\widetilde{E}$, $x_n \in S$ for $n \in \mathbf{N}$, with no weak cluster point.

Define

$$C := \bigcup_{m=1}^{\infty} A_m;\ A_m := m^{-1}z + \bigcup_{n=1}^{\infty} [n^{-1}, \infty)(m^{-2}x_n + y).$$

We will verify that

(3.45) $WP_C(m^{-1}z) = 0$ for all $m \in \mathbf{N}$

and

(3.46) $y \in T_C(0)$.

As $m^{-1}z \in \mathrm{cl}\, C$ and since both the Clarke tangent cone and the weak contingent cones always are the same for a set and its closure, this will violate 2).

To prove (3.45) suppose $h \in WK_C(m^{-1}z)$, $h \neq 0$. Then there exists $M > 0$ and a sequence h_n with the weak limit h such that

$$t_nh_n + m^{-1}z \in C$$

for some $t_n > 0$, $t_n \to 0$ and $\|h_n\| < M$. Thus there also exist sequences m_n, k_n, with $m_n, k_n \in \mathbf{N}$, $m_n, k_n > 0$ and λ_n with $\lambda_n \geqq 1$ for which

$$m^{-1}z + t_nh_n = m_n^{-1}z + \lambda_nk_n^{-1}(m_n^{-2}x_{k_n} + y).$$

As $(z^*, x_{k_n}) = 0$ for $n \in \mathbf{N}$, this together with (3.42) and (3.43) imply

$$m_n^{-1} - m^{-1} = t_n(x^*, h_n) \to 0.$$

Hence without loss of generality we assume $m_n = m$ for all $n \in \mathbf{N}$. Then

$$t_nh_n = \lambda_nk_n^{-1}(m_n^{-2}x_{k_n} + y).$$

As $(y^*, x_{k_n}) = 0$ for $n \in \mathbf{N}$ this and (3.44) show

$$t_n(y^*, h_n) = \lambda_nk_n^{-1} \geqq k_n^{-1}.$$

This in turn implies that $k_n \to \infty$. Let

$$\beta_n := t_nk_n\lambda_n^{-1}.$$

Then

$$\beta_nh_n - y = m^{-2}x_{k_n}.$$

Select $h^* \in E^*$ with $(h^*, h) = \|h\|$, $\|h^*\| = 1$. Then for n sufficiently large $(h^*, h_n) \geqq 2^{-1}\|h\|$, and as

$$\beta_n(h^*, h_n) \leqq \|h\|\,\|y + m^{-2}x_{k_n}\| \leqq 2,$$

β_n remains bounded. It follows now that $m^{-2}x_{k_n}$ has a weak cluster point. Since k_n tends to ∞, this means x_n has a cluster point. This contradiction shows

$$WP_C(m^{-1}z) = WK_C(m^{-1}z) = 0.$$

To prove (3.46), let sequences t_n, z_n be given such that,

$$t_n \to 0,\ t_n > 0,\ z_n \to 0,\ z_n \in C, \text{ then}$$

$$z_n = m_n^{-1}z + \lambda_n k_n^{-1}(m_n^{-2}x_{k_n} + y),$$

for some sequences λ_m, m_n, k_n as above. As $(z^*, z) = 1$,

$$(z^*, x_{k_n}) = (z^*, y) = 0,$$

we may argue that $m_n \to \infty$. Setting

$$c_n := m_n^{-1}z + (\lambda_n k_n^{-1} + t_n)(m_n^{-2}x_{k_n} + y),$$

we have

$$c_n \in A_{m_n} \subset C, \quad t_n^{-1}(c_n - z_n) \to y$$

and $y \in T_C(0)$. (3.46) is proven and the proof is finished.

If E is an infinite dimensional Banach space, then replacing the weak topology with the norm topology in the construction of the set C above, we recover the example from [**18**]. As a consequence we get the following.

THEOREM 3.3. *Let E be a Banach space. Then the following are equivalent.*

$$1) \quad \liminf_{\substack{x' \to x \\ C}} P_C(x') = T_C(x)$$

for all closed sets $C \subset E$, $x \in C$,

$$2) \quad \liminf_{\substack{x' \to x \\ C}} P_C(x') \supset T_C(x)$$

for all closed sets $C \subset E$, $x \in C$,

$$3) \quad \liminf K_C(x') \supset T_C(x)$$

for all closed sets $C \subset E$, $x \in C$,

4) *E is finite dimensional.*

The analysis of the proof of Theorem 3.1, Proposition 1.1 and Lemmas 1 and 2 shows that similar methods may be used to prove the following theorem.

THEOREM 3.4. *If C is a weakly compact subset of a Banach space E, $x \in C$ then*

$$\liminf_{\substack{x' \to x \\ C}} P_C(x') \subset T_C(x),$$

whenever E may be given an equivalent smooth norm, and

$$\liminf_{\substack{x' \to x \\ C}} WP_C(x') \subset T_C(x),$$

whenever E may be given a Frechet differentiable norm.

4. "Discrete lim inf" inclusions and stars of boundedly relatively weak compact sets. It is an easy consequence of the corresponding definitions that if

$$\liminf_{\substack{x' \to x \\ C}} R_C(x') \subset T_C(x)$$

then

$$d \liminf_{\substack{x' \to x \\ C}} R_C(x') + x' \subset \liminf_{\substack{x' \to x \\ C}} R_C(x') + x' \subset T_C(x) + x.$$

Corollary 2.1 implies that if C is a closed subset of a Banach space, then

$$\bigcap_{x' \in C} R_C(x') + x' \subset \text{star } C,$$

whenever for all $x \in C$

$$d \liminf_{\substack{x' \to x \\ C}} R_C(x') + x' \subset T_C(x) + x.$$

This motivates our following considerations for the boundedly relatively weakly compact sets. First we recall Borwein's nearest point result proven in [**3**].

PROPOSITION 4.1 (Borwein). *If C is a norm-closed relatively weakly compact subset of a Banach space E, which has a Kadec norm and is such that E^* has an equivalent strictly convex dual norm, then the set of those points which have a nearest point in C is dense in E.*

THEOREM 4.1. *Let E be a Banach space and let C be a norm-closed boundedly relatively weakly compact subset of E, $x \in C$, then*

(i) *the following inclusion holds*

$$d \liminf_{\substack{x' \to x \\ C}} P_C(x') + x' \subset T_C(x) + x; \tag{4.1}$$

(ii) *if E can be given an equivalent Frechet differentiable and Kadec norm then also*

$$d \liminf_{\substack{x' \to x \\ C}} WP_C(x') + x' \subset T_C(x) + x. \tag{4.2}$$

Proof. Let us observe first that as C is boundedly relatively weakly compact, then $\widetilde{E} := \overline{\text{span}}\, C$ is a weakly compactly generated subspace of E (see [**2**] for the proof) and as such it possesses an equivalent smooth and Kadec norm. ([**8**]). Let $\|\ \|$ denote some equivalent norm on $\widetilde{E}$ which is smooth and Kadec in the case (i) and which is Frechet and Kadec in the case (ii). Let B be a unit ball associated with the norm $\|\ \|$. Suppose that

$$y \notin T_C(x) + x \quad \text{and} \quad \|y - x\| = 1.$$

Then as in the proof of Theorem 3.1 we can find $\epsilon > 0$, $\lambda_n > 0$ and $x_n \in C \cap (x + 2^{-n}B)$ for $n \in \mathbf{N}$ with

(4.3) $$C \cap (x_n + (0, \lambda_n](y - x + 2\epsilon B)) = \emptyset.$$

Without loss of generality suppose that $\epsilon < 1/4$ and choose $\rho_n > 0$ with

$$\rho_n < \min(2^{-n}, \lambda_n/4, \epsilon/4)$$

and $\alpha_n > 0$ such that

$$\alpha_n < \min(\epsilon\rho_n/12, \lambda_n/6).$$

Let us set for $n \in \mathbf{N}$

$$W_n := (1 + \alpha_n)^{-1}([-1, 1](y - x_n) + \alpha_n(\epsilon/2)B).$$

By Lemma 1, the norm $\| \|_n$ associated with the ball W_n is equivalent norm on $\widetilde{E}$, which is smooth and Kadec in case (i) and Frechet differentiable and Kadec in case (ii). Note also that there exists an equivalent strictly convex dual norm on $\widetilde{E}^*$, because $\widetilde{E}$ is weakly compactly generated (**[8]**). Select, using Borwein's nearest point result (Proposition 4.1), some y_n in $\widetilde{E}$ with

$$y_n - (y - x_n) \in \alpha_n W_n \quad \text{and}$$

$$\|y_n - \bar{x}_n\|_n = \inf\{ \|y_n - z\|_n | z \in C_n\},$$

for some $\bar{x}_n \in C_n := (C - x_n) \cap \rho_n B$. Following the proof of Theorem 3.1 we get

$$\|y_n - \bar{x}_n\|_n \leqq 1 + 2\alpha_n, \quad W_n \subset 2B,$$

and

(4.5) $$\bar{x}_n \in y - x_n + W_n + 3\alpha_n W_n.$$

Also

(4.6) $$\begin{aligned} y - x_n + W_n &\subset (0, \infty)(y - x_n + (\epsilon/2)B) \\ &= (0, +\infty)(y - x + (x - x_n) + (\epsilon/2)B) \\ &\subset (0, \infty)(y - x + \epsilon B), \end{aligned}$$

whenever $2^{-n} \leqq \epsilon/2$. Let us restrict our considerations to those n for which this inequality is satisfied. Furthermore we get

$$\begin{aligned} &\{ (0, \infty)(y - x + \epsilon B) + 3\alpha_n W_n\} \cap C_n \\ &\subset \{ (0, \lambda_n](y - x + \epsilon B) + 3\alpha_n W_n\} \cap C_n \\ &= \{ (0, 6\alpha_n\epsilon^{-1})(y - x + \epsilon B) + 3\alpha_n W_n\} \cap C_n, \end{aligned}$$

where the last equality follows from (4.3). Hence using (4.5) and (4.6) we have

$$\|\bar{x}_n\| \leqq 6\alpha_n\epsilon^{-1}(1 + \epsilon) + 6\alpha_n \leqq 12\alpha_n\epsilon^{-1} < \rho_n,$$

which in turn implies that $\bar{x}_n$ satisfies (3.35) for some $r_n > 0$. Thus $y_n - \bar{x}_n$ is the proximal normal vector to C at $x_n + \bar{x}_n$ in the norm $\| \ \|_n$. However as

$$(y - x_n + W_n) \cap C_n = \emptyset$$

we get

$$\|\bar{x}_n - y_n\|_n \geqq 1 - \alpha_n.$$

Thus

$$\begin{aligned} (4.7) \qquad & \| y_n - \bar{x}_n - (y - (x_n + \bar{x}_n)) \|_n \\ & \leqq \alpha_n < 1 - \alpha_n \leqq \|\bar{x}_n - y_n\|_n. \end{aligned}$$

Now (4.7) implies in the case (i), via (1.3), that

$$y - (x_n + \bar{x}_n) \notin P_C(x_n + \bar{x}_n),$$

which proves (4.1) and in the case (ii), via (1.4), that

$$y - (x_n + \bar{x}_n) \notin WP_C(x_n + \bar{x}_n),$$

which proves (4.2).

COROLLARY 4.1. *If C is a norm-closed boundedly relatively weakly compact subset of a Banach space E, then*

$$(4.8) \qquad \bigcap_{x \in C} P_C(x) + x = \text{star } C, \quad \textit{and}$$

$$\bigcap_{x \in C} P_C(x) = \text{rec(star } C)$$

whenever C is starshaped.

If E can be given an equivalent Frechet differentiable and Kadec norm then

$$(4.9) \qquad \bigcap_{x \in C} WP_C(x) + x = \text{star } C, \quad \textit{and}$$

$$\bigcap_{x \in C} WP_C(x) = \text{rec(star } C)$$

whenever C is starshaped.

In particular (4.9) *holds for closed subsets in reflexive spaces.*

Proof. Combine Theorem 4.1 and Corollary 2.1. Note that always

$$\bigcap_{x \in C} P_C(x) \subset \text{rec}(\bigcap_{x \in C} P_C(x) + x) \quad \text{and}$$

$$\bigcap_{x \in C} WP_C(x) \subset \operatorname{rec}(\bigcap_{x \in C} WP_C(x) + x),$$

which follows from the convexity of the corresponding cones.

This completes Part I. In Part II we continue on applications of this theory as indicated at the introduction.

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