An Analytical Treatment of the Cam Problem.

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The cam problem may be stated as follows:

To design a curve rotating about O so as to actuate by contact a given follower curve to rotate about O' in a prescribed periodic manner.

Reference may be made to Barr, *Kinematics of Machinery*, Chapter 5, for the engineer's method of constructing cams by means of a template. A fuller analysis of the problem is aimed at in this paper.

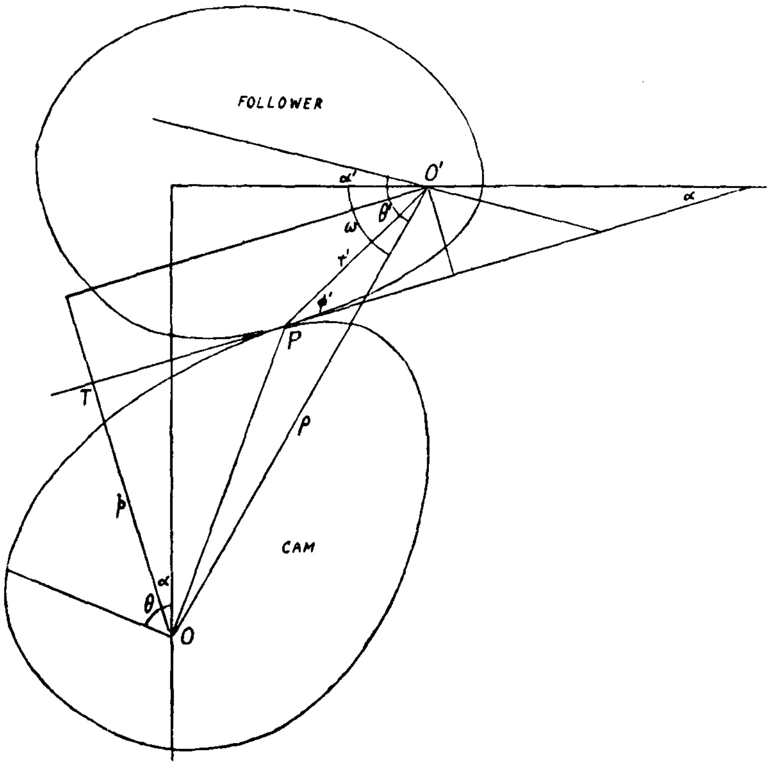

Fig. 1.

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Let the follower curve be defined by $r' = f(\theta')$ and its prescribed motion by $\alpha' = F(\theta)$. The equation of TP (Fig. 1) is

$$x \cos(\theta - \alpha) + y \sin(\theta - \alpha) = p$$

$$\therefore \qquad -x \sin(\theta - \alpha) + y \cos(\theta - \alpha) = dp/d\theta \div (1 - d\alpha/d\theta)$$

$$\therefore \qquad x^2 + y^2 = p^2 + \left(\frac{dp}{d\theta}\right)^2 \div \left(1 - \frac{d\alpha}{d\theta}\right)^2 = r^2$$

$$\sqrt{(r^2 - p^2)} = PT = \left(\frac{dp}{d\theta}\right) \div \left(1 - \frac{d\alpha}{d\theta}\right) \quad \text{.................................(1)}$$

Also $$PT = \rho \cos(\omega - \alpha) - r' \cos\phi' \quad \text{.....................(2)}$$

$$p = \rho \sin(\omega - \alpha) - r' \sin\phi' \quad \text{.....................(3)}$$

$$\therefore \qquad \frac{dp}{d\theta} = -\rho \cos(\omega - a)\frac{d\alpha}{d\theta} - \frac{dr'}{d\theta}\sin\phi' - r'\cos\phi'\frac{d\phi'}{d\theta}.$$

But $$r'\,d\theta' = -\tan\phi'\,dr' \text{ and } \theta' - \phi' = \alpha + \alpha' \quad \text{..................(4)}$$

$$\frac{dp}{d\theta} = -\rho\cos(\omega - \alpha)\frac{d\alpha}{d\theta} + r'\cos\phi'\frac{d\theta'}{d\theta} - r'\cos\phi'\frac{d\phi'}{d\theta}$$

$$= -\rho\cos(\omega - \alpha)\frac{d\alpha}{d\theta} + r'\cos\phi'\left(\frac{d\alpha}{d\theta} + \frac{d\alpha'}{d\theta}\right)$$

$$= -PT\frac{d\alpha}{d\theta} + r'\cos\phi'\frac{d\alpha'}{d\theta}$$

$$= PT\left(1 - \frac{d\alpha}{d\theta}\right) \text{ by (1)}.$$

$$\therefore \qquad PT = r'\cos\phi'\frac{d\alpha'}{d\theta} = \rho\cos(\omega - \alpha) - r'\cos\phi' \text{ by (2)}$$

$$1 + \frac{d\alpha}{d\theta} = \frac{\rho}{r'\cos\phi'}\cos(\omega - \alpha) = 2b\cos(\beta + \alpha') \quad \text{............(5)}$$

where $2br'\cos\phi' = \rho$ and $\beta = \omega - \theta' + \phi'$.

Since $\alpha' = F(\theta)$ the left member of equation (5) is a function of θ and may be regarded as a function of α'; b and β are functions of θ'. Hence this equation determines α' in terms of θ'. α is given by $\theta' - \phi' - \alpha'$ and p and PT by (2) and (3).

Practical Procedure.

(1) Plot the given displacement in polar form taking θ as radius vector against α'. (Fig. 2).

(2) Derive $\frac{d\alpha'}{d\theta}$ graphically, or otherwise, and plot $1 + \frac{d\alpha'}{d\theta}$ as radius vector against α' on the same diagram (dotted curve, Fig. 2).

(3) On the follower curve in the zero position mark points 1, 2, 3, etc., within the probable range of contact, draw tangents at these points and read off $r' \cos \phi'$, $r' \sin \phi'$, β, and calculate b. (Fig. 2.)

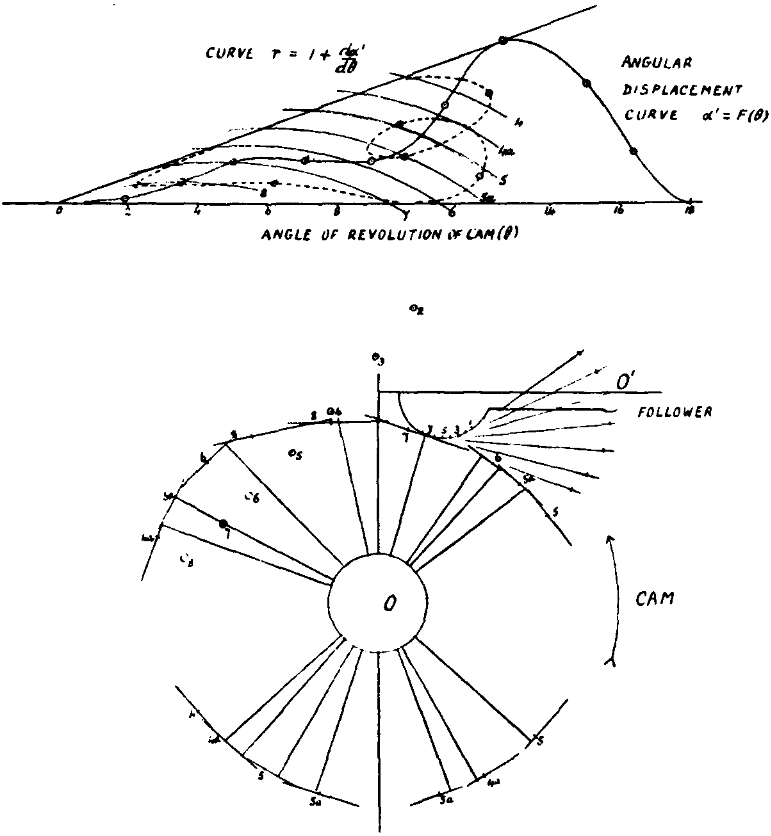


Fig. 2.

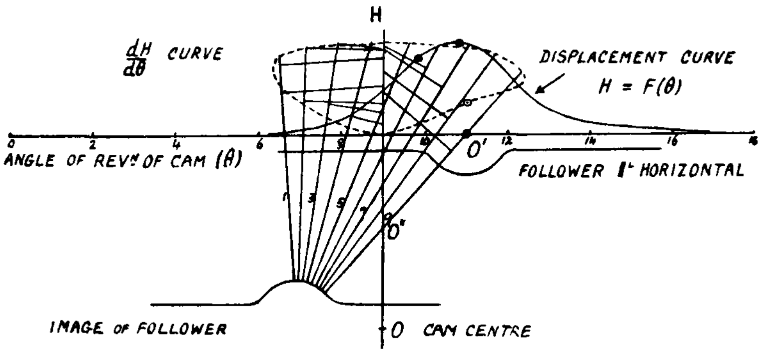
(4) Plot the points $(-\beta, b)$ on the displacement curve diagram (Fig. 2) marking them 1, 2, 3, etc., to correspond with the selected points of contact and with centres $(-\beta, b)$ and radii b describe arcs cutting the $1 + \dfrac{d\alpha'}{d\theta}$ curve. The limiting circles giving intersection define the limiting points of contact on the follower curve.

(5) Read off α' and θ corresponding to these intersections, calculate $\alpha = \omega - \alpha' - \beta$ and $\omega - \alpha$.

(6) Calculate PT by $\rho \cos(\omega - \alpha) - r' \cos \phi'$ and p by

$$\rho \sin(\omega - \alpha) - r' \sin \phi'.$$

(7) Plot the cam $\theta - \alpha$, p, PT.



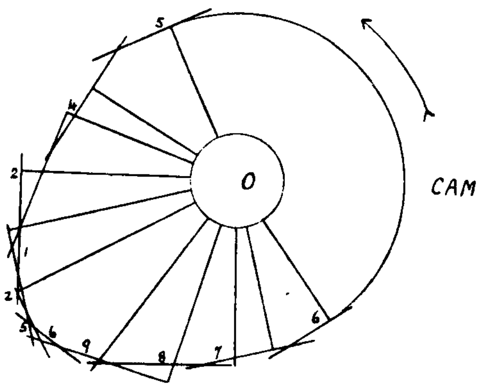


Fig. 3.

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The reason for using polar curves will now be apparent. The problem is essentially reduced to the solution of a system of transcendental equations of the form

$$f_1(\alpha') = f_2(\theta') \cos \{f_3(\theta') + \alpha'\}$$

which is a restatement of the equation (5) in which θ' is parameter of the system. The solution is effected by the intersections of the single curve $r = f_1(\alpha') \equiv 1 + \dfrac{d\alpha'}{d\theta}$ by the family of circles

$$r = f_2(\theta') \cos \{f_3(\theta') + \alpha'\}$$

which are plotted very easily as they all pass through the origin.

The curve $r = f_1(\alpha')$ is necessarily closed owing to periodicity and may have one or more loops (see Fig. 2), one corresponding to each rise and fall of the follower curve. Hence each circle gives $2n$ intersections which means that the point of contact belonging to that circle is engaged $2n$ times by the cam in each revolution. When the rise or fall is at a uniform rate $1 + \dfrac{d\alpha'}{d\theta}$ is constant, and the relative portion of the curve $r = f_1(\alpha')$ is an arc of a circle with centre at the origin. The case of a uniform rise followed by a uniform fall provides an interesting study in discontinuity, the dotted curve in this case becomes two concentric arcs unclosed, and only one intersection with a member of the circle family may appear. In practice it will usually suffice to select 3 or 4 contact points which will ensure intersection of the circles with the $r = f_1(\alpha')$ curve, calculate and plot the circle centres carefully, and interpolate others as the diagram indicates. The accuracy of the construction is essentially that of graphical differentiation.

Parallel Motion.

This is the case where O' the centre of rotation of the follower curve moves off to infinity along the horizontal. If H be the rise corresponding to θ and (x, y) the point of contact, we have

$$H = \rho \cos \omega \tan \alpha'$$
$$k = \rho \sin \omega$$
$$x = \rho \cos \omega - r' \cos (\theta' - \phi')$$
$$y = H + r' \sin (\theta' - \phi').$$

Hence

$$
\begin{aligned}
p &= \rho \sin \omega \cos \alpha - \rho \cos \omega \sin \alpha - r' \sin (\theta' - \alpha' - \alpha) \\
&= k \cos \alpha - \rho \cos \omega \sin \alpha - \cos \alpha \, (y - H) + \sin \alpha \, (\rho \cos \omega - x) \\
&= (k - y + H) \cos \alpha - x \sin \alpha \\
PT &= \rho \cos \omega \cos \alpha + \rho \sin \omega \sin \alpha - r' \cos (\theta' - \alpha' - \alpha) \\
&= \rho \cos \omega \cos \alpha + k \sin \alpha - \{\cos \alpha \, (\rho \cos \omega - x) + \sin \alpha \, (y - H)\} \\
&= (k - y + H) \sin \alpha + x \cos \alpha.
\end{aligned}
$$

By differentiating H with respect to θ and proceeding to the limit when $\alpha' = 0$ and $\rho = \infty$, we obtain the analogue to equation (5) for parallel motion in the form

$$\frac{dH}{d\theta} = H \tan \alpha + x + (k - y) \tan \alpha$$

Construction for Parallel Motion.

(1) Draw the rectangular displacement curve $H = f(\theta)$. (Fig. 3.)

(2) Derive $\frac{dH}{d\theta}$ graphically, or otherwise, and plot on the same diagram, using the negative side of the x-axis for the $\frac{dH}{d\theta}$ axis.

(3) Draw the image of the given follower curve with respect to the point O''.

(4) From selected contact points on this image curve draw normals to the image curve cutting the $\frac{dH}{d\theta}$ curve each in $2n$ points.

(5) Project these intersections on the H axis and re-project these latter on the normals. PT and p are then given by these projections.

The foregoing construction embodies the two diagrams for the previous case in one, and the circle system is replaced by a straight line system, the normals to the image curve, with α as parameter. Another simplification is that it is wholly geometrical and is easily carried out with a little practice. In the case of uniform rise or fall $\frac{dH}{d\theta}$ is constant, and is represented by a straight line parallel to the H-axis.