1

Basic concepts

Magnetic phenomena have been known since antiquity when a natural ore later called lodestone was discovered to attract bits of iron. The scientific study of magnetism dates from around 1600, when William Gilbert summarized experiments on the subject in his treatise *De Magnete*.[1] However, interest in the subject greatly increased after 1820, when Hans Christian Øersted reported that electrical currents could deflect magnetic needles, thereby establishing a connection between the subjects of electricity and magnetism.[2] Almost immediately, André-Marie Ampère, Jean-Baptiste Biot, and Félix Savart performed a series of seminal experiments that determined the forces acting between current loops. Experimental work and theoretical developments continued throughout the first half of the nineteenth century. A long program of experimental investigations by Michael Faraday lead him to the conception that the force between current loops occurred through the action of an intermediary field that existed in the space around the loops. Faraday's field concept was developed mathematically by William Thomson (later Lord Kelvin). This work culminated in a synthesis of knowledge about electrical and magnetic phenomena by James Clerk Maxwell in his famous treatise of 1873. Many clarifications of Maxwell's ideas and studies of their implications were carried out over the next twenty years by a small group of followers. Of particular note was the work of Oliver Heaviside who introduced the use of vector analysis and reworked the set of equations in Maxwell's treatise to the four equations we use today.[3] The resulting Maxwell equations are now accepted as the theoretical description underlying electromagnetic phenomena.

Magnetostatics is the study of the fields, forces, and energy associated with steady currents and magnetic materials. In this chapter, we will review some basic concepts underlying magnetic effects due to conductor currents in free space.

1.1 Current

Experiments have shown that there exist two kinds of electrical charge q, which are denoted as positive and negative. A current I exists when there is a net temporal flow of charge across some arbitrary plane in space.

$$I = \frac{dq}{dt}. \tag{1.1}$$

If the current is flowing through a conductor with length L and cross sectional area A, we can write the current as

$$I = \frac{\rho LA}{L/v} = \rho v\, A,$$

where ρ is the charge density and v is the velocity of the charges. The current density J along some direction n is a vector given by

$$\vec{J} = \frac{I}{A}\hat{n} = \rho\,\vec{v}, \tag{1.2}$$

where $\hat{n}$ is the unit vector perpendicular to A.

If we consider a volume of space V enclosed by a surface S, the conservation of charge requires that any change in the charge density inside V must be compensated by a flow of current through the surface or

$$-\int \frac{\partial \rho}{\partial t} dV = \int \vec{J} \cdot \hat{n}\, dS.$$

Using the Gauss divergence theorem,[1] the right-hand side can be written as

$$\int \vec{J} \cdot \hat{n}\, dS = \int \nabla \cdot \vec{J}\, dV.$$

Then, since V is arbitrary, we can remove the integrands from the volume integrals on both sides of the equation and obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0. \tag{1.3}$$

In magnetostatics, we have by definition $\partial \rho / \partial t = 0$, which leads to the relation

$$\nabla \cdot \vec{J} = 0. \tag{1.4}$$

[1] Readers unfamiliar with vector analysis should review Appendix B.

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Often we are interested in the current flow in a "central" region far from the ends of a magnet. If the current and the geometry are uniform along z in this region, we can simplify the analysis by examining problems in two dimensions. If we consider a conductor whose thickness is small compared with the distance to the observation point, we can approximate the conductor as a *current sheet*.

In addition, we frequently consider line currents or "filaments," where we ignore the transverse dimensions of the conductor altogether and use the equivalent current

$$I = \int \overrightarrow{J} \cdot \hat{n}\, dS.$$

1.2 Magnetic forces

Experiments have shown that test currents and charges in the vicinity of a current-carrying conductor experience a force. We assume that this force takes place through the actions of an intermediary magnetic field. The mathematical description of a field is a continuous function that is defined for all points in space and for all times. However, the magnetic field also has physical properties associated with it, such as stored energy. The force experiments can be explained by assuming that a current produces a vector field B, and then this field produces a force on other currents and charges. The vector field B is called the *magnetic flux density*[2] or magnetic field for short. The *magnetic flux* through some surface S is defined as

$$\Phi_B = \int \overrightarrow{B} \cdot \overrightarrow{dS}. \tag{1.5}$$

The direction of the magnetic field is often represented using Faraday's concept of *lines of induction.*[3][4] The lines of induction are defined to be tangent to the magnetic field at every point in space. It follows that corresponding components of the lines of induction and the magnetic field are always proportional to each other. If ds is a small displacement along the line of induction, we have

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} = \frac{ds}{B}.$$

In two Cartesian dimensions, the lines can be plotted, for example, by integrating the equations

[2] The vector B is also known as the magnetic induction.
[3] Historically, these curves have been referred to as lines of force.

$$dx = \frac{B_x(x,y)}{B(x,y)} ds$$
$$dy = \frac{B_y(x,y)}{B(x,y)} ds.$$

The magnitude of the magnetic field can be represented by the density of lines in a given region. The lines of induction do not have to form closed loops.[5, 6] In particular, the lines become undefined at locations where $B = 0$.

Now consider two circuits carrying currents I_a and I_b. The force exerted by circuit a on circuit b is found experimentally to be

$$\vec{F}_{ab} = \frac{\mu_0}{4\pi} I_a \, I_b \oint\!\!\oint \overrightarrow{dl_b} \times \frac{\overrightarrow{dl_a} \times \vec{R}}{R^3}, \tag{1.6}$$

where the constant $\mu_0 = 4\pi\, 10^{-7}$ is known as the *permeability of free space*,[4] dl is a displacement along the circuit in the direction of the current, and R is the distance vector from dl_a to dl_b. Note that the force is proportional to the product of the currents times a geometric factor that depends on the shape and orientations of the two circuits. It is possible to rewrite this equation in a form that manifestly obeys Newton's Third Law of motion. Using the vector triple product identity from Equation B.1 in Appendix B, we have

$$\overrightarrow{dl_b} \times (\overrightarrow{dl_a} \times \vec{R}) = \overrightarrow{dl_a}(\overrightarrow{dl_b} \cdot \vec{R}) - \vec{R}(\overrightarrow{dl_a} \cdot \overrightarrow{dl_b}).$$

The double integral of the first term on the right-hand side is then

$$\oint\!\!\oint \frac{\vec{dl}_a(\vec{dl}_b \cdot \vec{R})}{R^3} = \oint \vec{dl}_a \oint \frac{(\vec{dl}_b \cdot \vec{R})}{R^3} = \oint \vec{dl}_a \oint \frac{dR}{R^2}.$$

The last integral vanishes because the scalar integrand is taken over a closed path. Thus we can express the force as

$$\vec{F}_{ab} = -\frac{\mu_0}{4\pi} I_a I_b \oint\!\!\oint \frac{\vec{R}(\vec{dl}_a \cdot \vec{dl}_b)}{R^3}. \tag{1.7}$$

In this form, we see that Newton's law $F_{ab} = -F_{ba}$ is obeyed since R changes direction for the two cases.

Returning to Equation 1.6, we rewrite the force on circuit b in a form that explicitly depends on the current in circuit b and on an integration of the elemental

[4] We will use SI units exclusively in this book. For more details, see Appendix A.

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interactions taking place around that circuit. We collect the other factors in Equation 1.6 into a new vector B_a, which we define as the magnetic field due to circuit a. Then the force on the circuit can be written as

$$\vec{F}_{ab} = I_b \oint \vec{dl}_b \times \vec{B}_a \,. \tag{1.8}$$

The force acts at right angles to the direction of B_a. Dropping the subscripts, we see that the force on a charge q moving with velocity v can be written as

$$\vec{F} = \int \frac{dq}{dt} \vec{dl} \times \vec{B} = q\vec{v} \times \vec{B} \,. \tag{1.9}$$

Note that the force only acts on moving charges.

Now consider a rectangular current loop with length L and width w in a constant magnetic field B, as shown in Figure 1.1. The forces on each pair of opposite sides cancel, so there is no net force on the loop. However, there are moment arms between sides 1 and 3 and the axis of the loop. This creates a torque given by

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

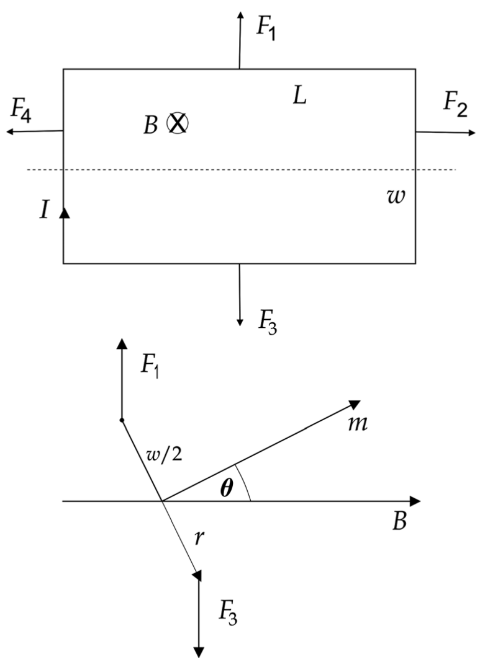

Figure 1.1 Rectangular current loop in an external field.

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For the example here,

$$\tau = 2\frac{w}{2}NILB\sin\theta,$$

where N is the number of turns in the loop. We define the *magnetic moment* $\boldsymbol{m}$ of a planar loop to lie along the normal n to the loop, so that

$$\overrightarrow{m} = NIA\,\hat{n}, \tag{1.10}$$

where A is the area of the loop. Then the torque acting on the loop can be expressed as

$$\overrightarrow{\tau} = \overrightarrow{m} \times \overrightarrow{B}. \tag{1.11}$$

1.3 The Biot-Savart law

Comparing Equations 1.6 and 1.8, we see that the force experiments require that the magnetic field can be expressed in the form

$$\overrightarrow{B} = \frac{\mu_0}{4\pi}I\oint\frac{\overrightarrow{dl}\times\overrightarrow{R}}{R^3}, \tag{1.12}$$

where we have dropped the subscripts referring to circuit a. The vector R points from the current element source to the observation (or field) point where the magnetic field is determined. This relation, known as the *Biot-Savart Law*, is an important tool for finding analytic and numerical solutions for the magnetic field produced by known current distributions. For a surface distribution of current, the total current in the Biot-Savart law can be generalized to give

$$\overrightarrow{B} = \frac{\mu_0}{4\pi}\int\frac{\overrightarrow{K}\times\overrightarrow{R}}{R^3}dS, \tag{1.13}$$

where K is the surface current density. Likewise, for a volume distribution of current, we have

$$\overrightarrow{B} = \frac{\mu_0}{4\pi}\int\frac{\overrightarrow{J}\times\overrightarrow{R}}{R^3}dV. \tag{1.14}$$

It is important to keep in mind that the Biot-Savart law and many of the other mathematical laws that we will subsequently develop ultimately depend on the validity of the experimental results on magnetic forces.

We consider next several elementary applications of the Biot-Savart law that we will need to refer to later in this book.

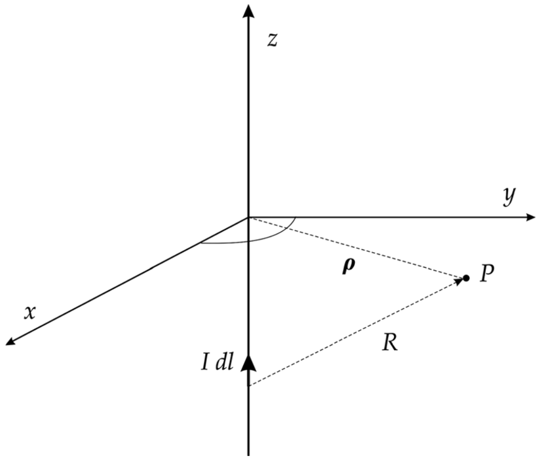

Figure 1.2 Current in a long straight wire.

Example 1.1: field from an infinitely long straight wire
Consider an infinitely long straight wire lying along the z axis, as shown in Figure 1.2. Because of the symmetry, we use cylindrical coordinates. Since the wire is infinitely long, we can chose an observation point P in the plane with $z = 0$ without loss of generality. Since

$$\begin{aligned}\overrightarrow{dl} &= dz\ \hat{z}\\ \overrightarrow{R} &= \rho\,\hat{\rho} + z\,\hat{z}\\ R &= \sqrt{\rho^2 + z^2},\end{aligned}$$

the field at point P due to the current in the wire is

$$\overrightarrow{B} = \frac{\mu_0}{4\pi}\rho\,\hat{\phi}\ 2\,\mathbb{I},$$

where[5]

$$\mathbb{I} = \int_0^\infty \frac{dz}{\{\rho^2 + z^2\}^{3/2}} = \frac{1}{\rho^2}.$$

Thus the magnetic field due to the current in the wire is

$$\overrightarrow{B} = \frac{\mu_0 I}{2\pi\,\rho}\hat{\phi}. \tag{1.15}$$

The field is directed azimuthally around the wire and falls off with distance like $1/\rho$.

[5] GR 2.271.5.

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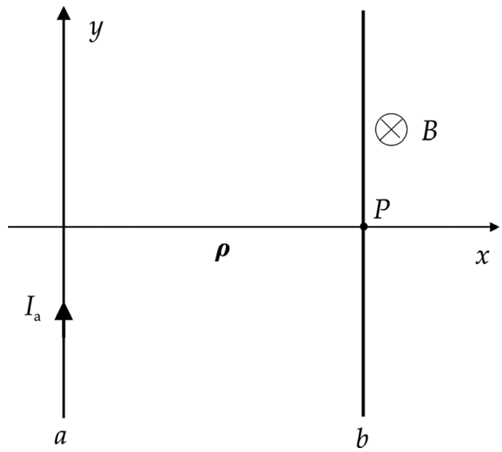

Figure 1.3 Force between two parallel wires.

Example 1.2: force between two parallel wires
Consider two infinitely long parallel wires, as shown in Figure 1.3. From Equation 1.8, the incremental force between the two wires is

$$\overrightarrow{dF}_b = I_b\,\overrightarrow{dl}_b \times \overrightarrow{B}_a$$

and from the previous example, the field at P due to the current in wire a is

$$\overrightarrow{B}_a = -\frac{\mu_0 I_a}{2\pi\rho}\hat{z}.$$

If the current direction in wire b can be either parallel or antiparallel to the current in wire a, we find that the force per unit length of the wire is

$$\frac{\overrightarrow{dF}_b}{dy} = \pm\frac{\mu_0}{2\pi\rho} I_a I_b\,\hat{x}. \tag{1.16}$$

The force between the wires is attractive when the currents are in the same direction and repulsive when they are antiparallel.

Example 1.3: field above an infinite current sheet
Consider an infinite current sheet with current flowing uniformly in the y direction. We calculate the magnetic field at point P, shown in Figure 1.4. We have

$$\begin{aligned}\overrightarrow{K} &= K_y\,\hat{y}\\ \overrightarrow{R} &= x\,\hat{x} + y\,\hat{y} + z_o\,\hat{z}.\end{aligned}$$

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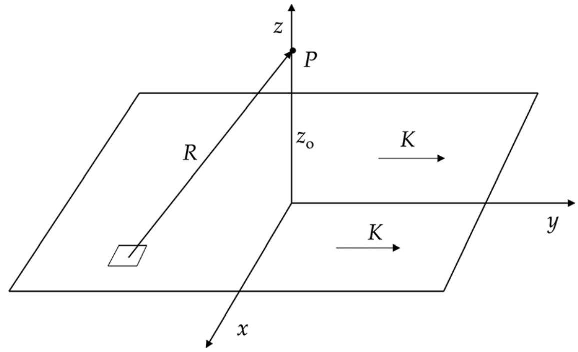

Figure 1.4 Field above an infinite current sheet.

The field is given by

$$\begin{aligned}\overrightarrow{B} &= \frac{\mu_0 K_y}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{z_o\,\hat{x} - x\,\hat{z}}{\{x^2+y^2+z_o^2\}^{3/2}}\,dx\,dy \\ &= \frac{\mu_0 K_y}{4\pi}(z_o\,\hat{x}\mathbb{I}_1 - \hat{z}\mathbb{I}_2),\end{aligned}$$

where

$$\mathbb{I}_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\{x^2+y^2+z_o^2\}^{3/2}}\,dx\,dy = \frac{2\pi}{z_o}$$

and

$$\mathbb{I}_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x}{\{x^2+y^2+z_o^2\}^{3/2}}\,dx\,dy = 0.$$

The integral $\mathbb{I}_2$ vanishes because the integrand is an odd function and the integration extends over an even interval. The magnetic field above the sheet is

$$\overrightarrow{B} = \frac{\mu_0}{2}K_y\,\hat{x}.$$

The direction of the field is parallel to the sheet and perpendicular to the current density. The magnitude of the field is constant and independent of the distance from the sheet. In the general case, the field above the sheet can be written as

$$\overrightarrow{B} = \frac{\mu_0}{2}\overrightarrow{K}\times\hat{n}, \tag{1.17}$$

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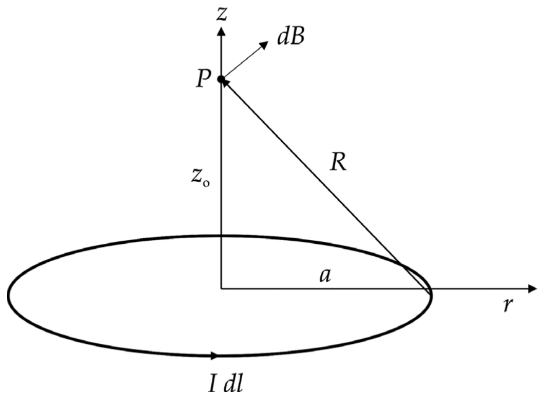

Figure 1.5 Field along the axis of a current loop.

where n is the normal to the sheet pointing to the side where B is computed. Note that the direction of B follows the right-hand rule with respect to the current filaments in the sheet.

Example 1.4: on-axis field due to a circular current loop
We look for the field at a point P that is along the axis of the loop and a distance z_o above the plane of the current loop, as shown in Figure 1.5. In cylindrical coordinates, we have

$$\begin{aligned}\overrightarrow{dl} &= a\, d\phi\, \hat{\phi}\\ \overrightarrow{R} &= -a\, \hat{r} + z_o\, \hat{z}.\end{aligned}$$

The contributions of the current elements to the field at P lie in a cone surrounding P. By symmetry, the net field must be in the z direction and

$$(\overrightarrow{dl} \times \overrightarrow{R})_z = a^2\, d\phi.$$

Thus we have

$$\begin{aligned}B_z &= \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a^2}{\{a^2 + z_o^2\}^{3/2}}\, d\phi\\ &= \frac{\mu_0 I\, a^2}{2\{a^2 + z_o^2\}^{3/2}}.\end{aligned} \tag{1.18}$$

Note that B_z is proportional to the area of the current loop and falls off at large distances like z_o^{-3}. The field is largest at the center of the loop where the value is

$$B_{z0} = \frac{\mu_0 I}{2a}.$$

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1.4 Divergence of the magnetic field

Starting from the Biot-Savart law,

$$\overrightarrow{B} = \frac{\mu_0}{4\pi}\int \frac{\overrightarrow{J} \times \overrightarrow{R}}{R^3} dV$$

and taking the divergence of both sides, we have

$$\nabla \cdot \overrightarrow{B} = \frac{\mu_0}{4\pi}\int \frac{\nabla \cdot (\overrightarrow{J'} \times \overrightarrow{R})}{R^3} dV',$$

where we use primes to indicate the use of source coordinates. The operator ∇ is defined in terms of field coordinates, while the distance R is a function of both source and field coordinates. Using the vector identity B.4, we can write

$$\nabla \cdot (\overrightarrow{J'} \times \overrightarrow{R}) = \overrightarrow{R} \cdot (\nabla \times \overrightarrow{J'}) - \overrightarrow{J'} \cdot (\nabla \times \overrightarrow{R}).$$

The first term on the right-hand side vanishes because $\nabla \times \overrightarrow{J'} = 0$. When the second term is written in terms of a determinant, we obtain

$$\nabla \times \overrightarrow{R} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ x - x' & y - y' & z - z' \end{vmatrix} = 0.$$

Thus we find that the divergence of the magnetic field vanishes.

$$\nabla \cdot \overrightarrow{B} = 0. \tag{1.19}$$

This vector relation is one of the fundamental properties of magnetic fields. Its validity depends on the fact that isolated magnetic charges (monopoles) do not appear to exist.

If we integrate Equation 1.19 over some volume of space V that is enclosed by a surface S, we get

$$\int \nabla \cdot \overrightarrow{B}\, dV = 0.$$

Then using the divergence theorem, we find *Gauss's Law for magnetism*

$$\int \overrightarrow{B} \cdot \hat{n}\, dS = 0. \tag{1.20}$$

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1.5 Circulation of the magnetic field

Returning again to the Biot-Savart law, consider the integral

$$\begin{aligned}\mathbb{I} &= \int \frac{\vec{J'} \times \vec{R}}{R^3} dV' \\ &= \int \left(\nabla \left(\frac{1}{R} \right) \times \vec{J'} \right) dV'.\end{aligned}$$

Using the vector identity B.6, we can write this as

$$\mathbb{I} = \int \nabla \times \frac{\vec{J'}}{R} dV' - \int \frac{1}{R} \nabla \times \vec{J'} dV'.$$

The quantity $\nabla \times \vec{J'} = 0$ in the second integral. We can bring the ∇ operator in the first term outside the integral sign because it operates on the observation point coordinates, while the integral is over the source point coordinates. Thus

$$\mathbb{I} = \nabla \times \int \frac{\vec{J'}}{R} dV'$$

and an alternate expression for the magnetic field is

$$\vec{B} = \frac{\mu_0}{4\pi} \int \nabla \times \frac{\vec{J'}}{R} dV'. \tag{1.21}$$

Taking the curl of both sides of this equation, we find

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \nabla \times \nabla \times \int \frac{\vec{J'}}{R} dV'.$$

Using the vector relation B.7, we can write this in the form

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \left[\nabla \int \nabla \cdot \left(\frac{\vec{J'}}{R} \right) dV' - \int \nabla^2 \left(\frac{\vec{J'}}{R} \right) dV' \right].$$

Then since ∇ does not operate on J', we have

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \left[\nabla \int \vec{J'} \cdot \nabla \left(\frac{1}{R} \right) dV' - \int \vec{J'} \nabla^2 \left(\frac{1}{R} \right) dV' \right]. \tag{1.22}$$

Consider for the moment the relation

$$\nabla \left(\frac{1}{R} \right) = -\nabla' \left(\frac{1}{R} \right).$$

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In the second integral in Equation 1.22 when $R \neq 0$, we have

$$\nabla^2\left(\frac{1}{R}\right) = \frac{1}{r^2}\partial_r\left[r^2\partial_r\left(\frac{1}{R}\right)\right]$$

in spherical coordinates. Then since $R = |\overrightarrow{r} - \overrightarrow{r'}|$, we find that $\nabla^2\left(\frac{1}{R}\right) = 0$. Although $\nabla^2\left(\frac{1}{R}\right)$ is undetermined when $R = 0$, the integral of this expression is still defined. Performing the integral on a small sphere surrounding $R = 0$, we find

$$\int \nabla^2\left(\frac{1}{R}\right) dV = \int \nabla\cdot\nabla\left(\frac{1}{R}\right) dV = \int \nabla\left(\frac{1}{R}\right)\cdot\overrightarrow{dS}$$

using the divergence theorem. Evaluating the last integral on the surface of the small sphere, we find that

$$\int \nabla\left(\frac{1}{R}\right)\cdot\overrightarrow{dS} = -\frac{1}{R^2}4\pi R^2 = -4\pi.$$

We can summarize these results by writing the expression in terms of the Dirac delta function δ.

$$\nabla^2\left(\frac{1}{R}\right) = -4\pi\delta(R). \tag{1.23}$$

Now we can do the first integral in Equation 1.22 using Equation B.3 to give

$$\int \overrightarrow{J'}\cdot\nabla'\left(\frac{1}{R}\right) dV' = \int \nabla'\cdot\frac{\overrightarrow{J'}}{R} dV' - \int \frac{1}{R}\nabla\cdot\overrightarrow{J'} dV'.$$

The first term on the right-hand side can be converted to a surface integral using the divergence theorem. It vanishes if the surface enclosing the volume in the integrals is sufficiently large. The second integral also vanishes because $\nabla'\cdot\overrightarrow{J'} = 0$ for magnetostatics. Thus we are only left with the second integral in Equation 1.22, which because of the delta function from Equation 1.23, gives

$$\nabla \times \overrightarrow{B} = \mu_0\overrightarrow{J}. \tag{1.24}$$

Thus we have shown that a steady current creates a magnetic field that circulates around the current. This is a second fundamental vector relation for magnetic fields.[6]

[6] We will find in Chapter 10 that this relation requires an additional term if the current varies with time.

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1.6 The Ampère law

If we integrate both sides of Equation 1.24 over an arbitrary surface S, we find

$$\int (\nabla \times \overrightarrow{B}) \cdot \overrightarrow{dS} = \mu_0 \int \overrightarrow{J} \cdot \overrightarrow{dS}.$$

On the right side, the current density integrated over the surface gives the total current I. On the left side, we can use Stokes's theorem from Appendix B to give

$$\int (\nabla \times \overrightarrow{B}) \cdot \overrightarrow{dS} = \oint \overrightarrow{B} \cdot \overrightarrow{dl},$$

where the contour on the right-hand side extends along the perimeter of the surface S. Thus we have the result

$$\oint \overrightarrow{B} \cdot \overrightarrow{dl} = \mu_0 I. \tag{1.25}$$

This equation is known as the Ampère law.[7] It can be most usefully applied in highly symmetric cases where, for example, the magnitude of the field is constant along the integration path.

Again let us consider several elementary examples of using the Ampère law to derive results that we use later in the book.

Example 1.5: a long cylindrical conductor

Consider a long cylindrical conductor with constant current density J inside the radius a, as shown in Figure 1.6. Since by symmetry the magnitude of the field must be independent of ϕ, we choose a circular path of integration. When the path is outside the conductor, all of the current is enclosed by the path and the Ampère law gives $B_\phi\, 2\pi\rho = \mu_0 I$. Thus the field outside the conductor is

$$B_\phi = \frac{\mu_0 I}{2\pi\rho}, \tag{1.26}$$

which falls off like $1/\rho$. Since this result is independent of the radius of the conductor, it also applies to the field from a current filament, which we previously derived using the Biot-Savart equation.

When the path of integration is inside the conductor, only part of current is enclosed by the path and the Ampère law gives

$$B_\phi 2\pi\rho = \mu_0 \frac{\pi\rho^2}{\pi a^2} I.$$

[7] According to O. Darrigol,[2] this equation was first given by Maxwell. In that case, we agree with him that it's really not appropriate to call it Ampère's law.

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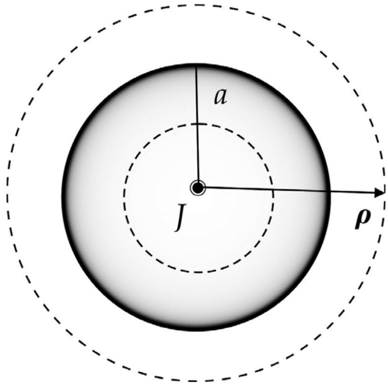

Figure 1.6 Cylindrical conductor of radius a. The two integration paths are shown with dotted lines.

Thus the field inside the conductor is

$$B_\phi = \frac{\mu_0 I \rho}{2\pi a^2} = \frac{\mu_0 J}{2}\rho, \tag{1.27}$$

which increases linearly with ρ.

Example 1.6: ideal solenoid

We define an ideal solenoid as an infinitely long system of parallel circular current loops with radius a, as shown in Figure 1.7. This is an approximation to a real solenoid when the observation points are far from the ends of the solenoid and the conductor is tightly wound, such that we can ignore any gaps or the helical nature of the windings. First consider a cylinder containing the points *achj*. From Gauss's law, Equation 1.20, we know that the flux passing through the surface must be 0. From symmetry, the flux through the top and through the bottom faces of the cylinder have to cancel. The contribution through the side of the cylinder then gives

$$2\pi\,\rho\,L\,B_\rho = 0.$$

Since the same argument applies for a cylinder of any radius, we must have $B_\rho = 0$ everywhere for the ideal solenoid.

Now consider the Ampère law applied to the path *bdgi*. The contributions to the integral vanish along *bd* and *gi* since $B_\rho = 0$. Then we have

$$B_z\,(0)L - B_z\,(r_1)L = \mu_0\,n\,I\,L,$$

where n is the number of conductor turns per unit length, or that

$$B_z\,(r_1) = B_z\,(0) - \mu_0\,n\,I.$$

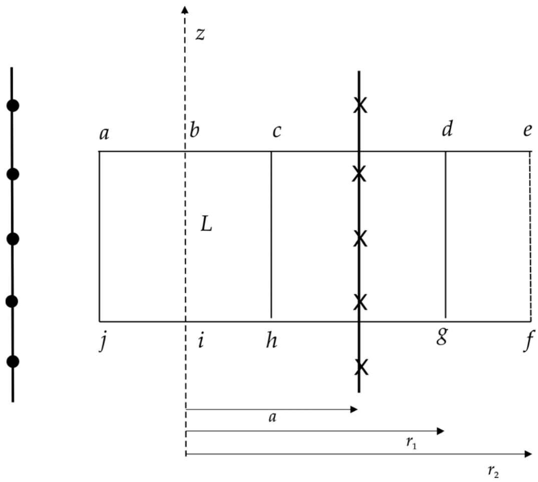

Figure 1.7 Cross-section of an ideal solenoid. The dashed line is the axis of the solenoid. The dots and crosses refer to the direction of the current.

If we apply the same argument over the path *befi*, we find that

$$B_z\,(r_2) = B_z\,(0) - \mu_0\, n\, I.$$

Both of these equations for B_z outside the solenoid have the same right-hand side. Thus the value of B_z outside the solenoid is constant, independent of radius. However, if we consider the total flux outside the solenoid, we find

$$\Phi_B = \int_0^{2\pi} \int_a^{\infty} B_z^{out}\; r\, dr\, d\,\phi.$$

The total flux would be infinite if B_z outside the solenoid is any constant other than 0. This is clearly non-physical, so we must have $B_z = 0$ everywhere outside the solenoid.

Since the field vanishes outside the solenoid, applying the Ampère law to the path *bdgi* gives

$$B_z\,(0) = \mu_0 n I.$$

Similarly, on the path *cdgh* we find

$$B_z\,(r) = \mu_0 n I, \tag{1.28}$$

where we write r for the length *bc*. Thus the field of the ideal solenoid is constant and along the axis of the solenoid on the inside and it vanishes outside.

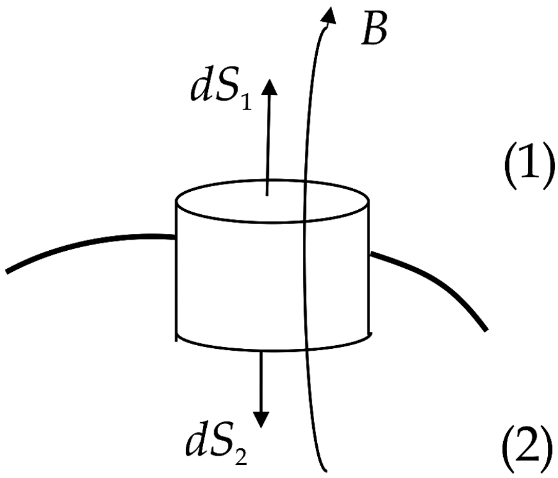

Figure 1.8 Gaussian pillbox across a current sheet.

1.7 Boundary conditions at a current sheet

We now consider how the magnetic field is influenced by the presence of a current sheet. Assume we have a current sheet, as shown in Figure 1.8. We construct a cylindrical pillbox across the sheet with an infinitesimal height along the normal to the surface. Then applying Equation 1.20, we find that

$$B_{1n}\, S - B_{2n}\, S = 0.$$

Since the surface S is arbitrary, it follows that $B_{1n} = B_{2n}$ or that

$$(\overrightarrow{B_2} - \overrightarrow{B_1})\cdot\hat{n} = 0. \tag{1.29}$$

Thus the normal component of B must be continuous across a current sheet.

Next construct a closed path across the current sheet, as shown in Figure 1.9. Assume the path length perpendicular to the surface is infinitesimally small. The path encloses any current present in the sheet. Applying the Ampère law, we find

$$-B_{1t}\, L + B_{2t}\, L = \mu_0\, K\, L.$$

Thus the change in the field across the sheet is

$$B_{2t} - B_{1t} = \mu_0\, K \tag{1.30}$$

or in general

$$(\overrightarrow{B_2} - \overrightarrow{B_1}) \times \hat{n} = \mu_0\, \overrightarrow{K}. \tag{1.31}$$

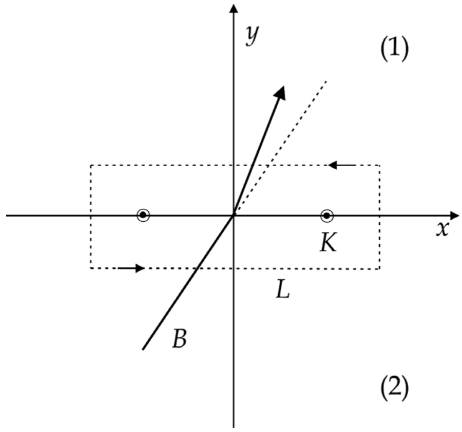

Figure 1.9 Fields near a current sheet.

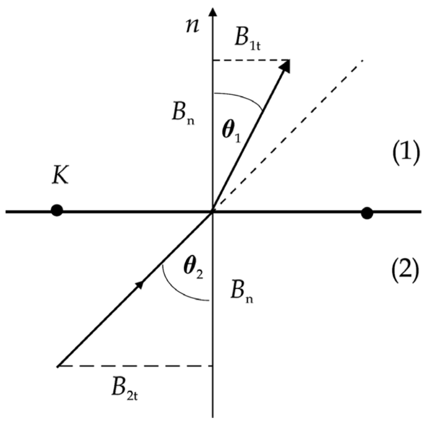

Figure 1.10 Refraction of the magnetic field crossing a current sheet.

The tangential component of B changes by an amount proportional to the current density when crossing the sheet.

Finally, let us consider the angles between the magnetic field vectors and the normal to the current sheet, as shown in Figure 1.10. In region (2), the magnetic field vector makes an angle with the normal to the current sheet given by

$$\tan\theta_2 = \frac{B_{2t}}{B_n}.$$

The corresponding angle with the normal in region (1) is given by

$$\begin{aligned}\tan\,\theta_1 &= \frac{B_{1t}}{B_n} \\ &= \frac{B_{2t} - \mu_0\,K}{B_n} \\ &= \tan\,\theta_2 - \frac{\mu_0\,K}{B_n}.\end{aligned}$$

We see that in crossing the current sheet, the vector B is refracted in the direction of the field from the current sheet, i.e., toward the normal for the positive current density K shown here.

1.8 Inductance

Consider a coil with N turns. We define the *flux linkage* to be the product of the magnetic flux going through the coil multiplied by the number of turns. The flux linkage is proportional to the current flowing through the coils. We define the coefficient of proportionality to be the *self-inductance* L of the coil. Thus we have

$$L = \frac{N\Phi_B}{I}. \tag{1.32}$$

Example 1.7: self-inductance of an ideal solenoid
For an ideal solenoid with N turns in a length d and radius R, the field from Equation 1.28 is

$$B_z \simeq \frac{\mu_0\,N\,I}{d}.$$

The flux in the solenoid is

$$\Phi_B = \frac{\mu_0\,N\,I}{d}\pi R^2,$$

so the self-inductance is

$$L = \frac{\mu_0 N^2}{d}\pi R^2. \tag{1.33}$$

This result ignores any end effects present in a real solenoid.

If we now consider two coils, the *mutual inductance* M is defined as the flux linkage in the second coil due to the current in the first coil. Thus we have

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$$M = \frac{N_2 \Phi_{2,1}}{I_1}, \tag{1.34}$$

where $\Phi_{2,1}$ is the flux in coil 2 due to the current in coil 1. In general, M can be defined using the Neumann equation [7]

$$M = \frac{\mu_0}{4\pi} \iint \frac{\overrightarrow{dl_1} \cdot \overrightarrow{dl_2}}{r_{12}}, \tag{1.35}$$

where r_{12} is the distance from the current element in the first coil to the current element in the second coil. Note that this shows that M is a constant times a geometric factor. The symmetry of this equation between the two coils shows that M of coil 2 due to current in coil 1 is the same as M for coil 1 due to current in coil 2.

Example 1.8: mutual inductance of two coaxial solenoids
Assume we have two coaxial solenoids. The first solenoid has length d_1 and both solenoids have approximately the same radius R. Then

$$\Phi_{2,1} = \frac{\mu_0 \, N_1 \, I_1}{d_1} \pi R^2$$

and the mutual inductance is

$$M = \frac{\mu_0 N_1 N_2}{d_1} \pi R^2. \tag{1.36}$$

The force between two coaxial coils can be expressed in terms of the derivative of their mutual inductance.[8]

$$F_z = I_1 \, I_2 \frac{\partial M}{\partial z}. \tag{1.37}$$

1.9 Energy stored in the magnetic field

We can obtain a rough estimate for the energy stored in a magnetic field[8] by considering a simple LR circuit, as shown in Figure 1.11. From Kirchhoff's circuit laws,[9] we have

$$IV = I^2 \, R + LI \frac{dI}{dt}.$$

[8] We will reexamine this question more rigorously in Chapter 10.

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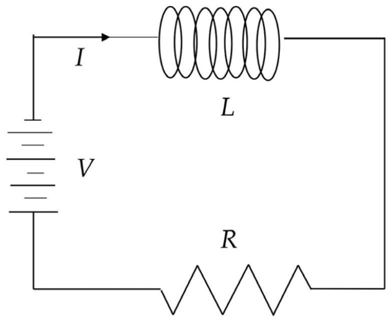

Figure 1.11 LR circuit.

The power $P = IV$ generated by the battery is distributed between the power lost to resistive heating in R and the power in the magnetic field associated with the inductor L. The energy in the magnetic field is then

$$\begin{aligned} W_B &= \int P\,dt = \int LI\frac{dI}{dt}\,dt = \int LI\,dI \\ &= \frac{1}{2}LI^2. \end{aligned} \tag{1.38}$$

If we consider the inductor to be a long solenoid, then using Equations 1.28 and 1.33,

$$W_B = \frac{1}{2}\,\frac{\mu_0 N^2}{d}\,\pi R^2\,\frac{B^2 d^2}{\mu_0^2 N^2} = \frac{1}{2}\,\frac{B^2}{\mu_0}\,\pi R^2 d$$

and the energy density in the magnetic field is

$$w_B = \frac{B^2}{2\,\mu_0}. \tag{1.39}$$

References

[1] F. Bonaudi, Magnets in particle physics, in S. Turner (ed.), *CERN Accelerator School: Magnetic Measurements and Alignment*, Montreux, 1992, p. 2–5.
[2] O. Darrigol, *Electrodynamics from Ampère to Einstein*, Oxford University Press, 2000, p. 142.
[3] B. Hunt, *The Maxwellians*, Cornell University Press, 1991, p. 70–71, 245–247.
[4] D. Halliday & R. Resnick, *Physics for Students of Science and Engineering*, 2nd ed., Wiley, 1962, p. 759–760.
[5] K.L. McDonald, Topology of steady current magnetic fields, *Am. J. Phys.* 22:586, 1954.
[6] M. Lieberherr, The magnetic field lines of a helical coil are not simple loops, *Am. J. Phys.* 78:1117, 2010.

[7] G. Harnwell, *Principles of Electricity and Magnetism*, 2nd ed., McGraw-Hill, 1949, p. 322.
[8] P. Lorrain & D. Corson, *Electromagnetic Fields and Waves*, 2nd ed., Freeman, 1970, p. 360.
[9] D. Tomboulian, *Electric and Magnetic Fields*, Harcourt, Brace & World, 1965, p. 267–269.