A NOTE ON p-GROUPS WITH POWER AUTOMORPHISMS

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1. Introduction. Let G be a group. The *norm*, or *Kern* of G is the subgroup of elements of G which normalize every subgroup of the group. This idea was introduced in 1935 by Baer [**1**, **2**], who delineated the basic properties of the norm. A related concept is the subgroup introduced by Wielandt [**10**] in 1958, and now named for him. The *Wielandt subgroup* of a group G is the subgroup of elements normalizing every subnormal subgroup of G. In the case of finite nilpotent groups these two concepts coincide, of course, since all subgroups of a finite nilpotent group are subnormal. Of late the Wielandt subgroup has been widely studied, and the name tends to be the more used, even in the finite nilpotent context when, perhaps, norm would be more natural. We denote the Wielandt subgroup of a group G by $\omega(G)$. The Wielandt series of subgroups $\omega_i(G)$ is defined by: $\omega_1(G) = \omega(G)$ and for $i \geq 1$, $\omega_{i+1}(G)/\omega_i(G) = \omega(G/\omega_i(G))$. The subgroups of the upper central series we denote by $\zeta_i(G)$.

In both cases, whether norm or Wielandt subgroup, interesting questions concern the structure of the subgroup, and the nature of its embedding in its parent group. Thus the norm is a Dedekind group, so its structure is completely known [**4**]; and it is contained in the second centre of the group: see Schenkman [**9**]. The Wielandt subgroup is a T-group (i.e. its subnormal subgroups are all normal), so much is known about its structure, everything if the group is finite and soluble. We know no easily stateable results concerning the embedding of the Wielandt subgroup. In both cases non-abelianness, of norm or Wielandt subgroup, seems to impose constraints on the structure of the group. Thus we may cite Baer's result [**2**], that in a 2-group non-abelian norm can occur only if the group is Hamiltonian (i.e. non-abelian and Dedekind). In [**3**] non-abelianness of the Wielandt subgroup in a finite soluble group is used to obtain bounds on derived length.

The aim of the present note is to investigate the constraints imposed on the structure of certain p-groups consequent upon their having non-central norms. The corollary below shows that for metabelian groups of exponent dividing p^2 and of sufficiently large class, the Wielandt series and the upper central series coincide.

THEOREM. *Let G be a metabelian group of exponent dividing p^2 in which the norm is not central. Then the nilpotency class of G is at most $2p-2$, and this bound is best possible.*

COROLLARY. *Let G be metabelian of exponent dividing p^2, where p is an odd prime, and of class at least $2p$. Then, for $i \geq 1$, $\omega_i(G) = \zeta_i(G)$.*

Every element in the norm induces, by conjugation, a power automorphism in the group: every element of the group is mapped to a power of itself. On the other hand if p is an odd prime and P is a non-abelian metabelian group of exponent p^2 with a power automorphism α, it is easy to check that the semidirect product $P\langle\alpha\rangle$ is a metabelian group of exponent p^2 and the same class as P. That $P\langle\alpha\rangle$ has a non-central norm is a special case of the Lemma in Section 2. Thus our result can be regarded, for odd primes,

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as an improvement on Theorem 3.8 of Meixner [8], which shows that a metabelian group of exponent p^2 with a non-trivial power automorphism has nilpotency class at most $2p-1$. Almost all the ideas in the proof of our theorem and the construction of our examples are to be found in Meixner. Meixner also shows that a metabelian group of exponent 4 with $\omega(G)$ non-central has class at most 2 and that this result is best possible.

2. Proofs. We begin by proving the following result.

LEMMA. *If P is a p-group, p an odd prime, and α a nontrivial power automorphism of P of order p then, in the natural semidirect product $H = P\langle\alpha\rangle$, $\alpha \in \omega(H)\backslash\zeta_1(H)$. Moreover H and P have the same exponent and they have the same nilpotency class and derived length unless P is abelian in which case both class and derived length of H are two.*

Proof. Let $g \in P$ have order $p^{r+1}(r \geq 1)$. Then since α has order p we may, by replacing it by some power of itself, suppose that

$$g^\alpha = g^{p^r+1}.$$

Then for $i \in \{1, 2, \ldots, p-1\}$

$$(\alpha^i g)^{p^r+1} = \alpha^{i(p^r+1)} g^N,$$

where

$$\begin{aligned} N &= \sum_{l=0}^{p^r} (p^r+1)^{il} \\ &= \sum_{l=0}^{p^r} (ilp^r + 1) \qquad (\operatorname{mod} p^{r+1}) \\ &= 1 + p^r + \frac{ip^{2r}(p^r+1)}{2} \\ &= p^r + 1 \qquad (\operatorname{mod} p^{r+1}) \end{aligned}$$

since $p^r + 1$ is even. Thus

$$(\alpha^i g)^{p^r+1} = \alpha^i g^{p^r+1} = \alpha^{-1}(\alpha^i g)\alpha$$

and so α is in $\omega(H)$; and clearly α is not central.

The argument above is easily modified to show that for every $g \in P$ and $i \in \{0, 1, \ldots, p-1\}$, $\alpha^i g$ has order no bigger than that of g. Hence H and P have the same exponent.

As to the class and derived length of H: every commutator with entries from the set $P \cup \{\alpha\}$, including α, and of weight greater than or equal to three, is trivial, since α is second central in H. Hence P and H have the same class and derived length except when P is abelian, in which case all commutators in H of weight three or greater are trivial so H has class and derived length two.

Next we give a proof of our Theorem. The case $p = 2$ is covered in [8] so from now on we suppose that p is odd.

Let w be a non-central element of the norm in a metabelian group G of exponent p^2. We write $C = C_G(w)$. Of course $C \neq G$. As a first step we prove

$$G/G' \quad \text{has exponent} \quad p. \tag{2.1}$$

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Consider the subgroup $N = [G, w]$. It is normal in G and has the property that, for all $x \in G \backslash C$, $x^p \in N$. We note that

$$N \leq G' \leq C.$$

Every element of G/N outside C/N has order p. Therefore every element of G/G' outside C/G' has order p. If G/G' is not of exponent p it is generated by elements of order p^2 since it is abelian. But all elements of order p^2 in G/G' are in C/G'. Therefore $G/G' = C/G'$ and hence $G = C$. This is a contradiction to the non-centrality of w. Hence (2.1) is confirmed.

We now write M for the subgroup of G generated by the pth powers of the elements of G. Note that, by (2.1), $M \leq G'$, so it is abelian and of exponent p. Next we prove that

$$\text{for all} \quad m \in M, \quad \text{and for all} \quad x \in G, [m, (p-1)x] = 1. \tag{2.2}$$

As a first step choose $x \in G \backslash C$, and $m \in M$. Consider the elements $x_j = xm^j$ $(0 \leq j \leq p-1)$. None of these is in C. Hence for some integer k_j satisfying $1 \leq k_j \leq p-1$,

$$x_j^{pk_j} = [x_j, w] = [x, w].$$

Since there are p elements x_j and only $p-1$ different k_j, we must have that two of the latter are equal. It follows that for some i, j with $0 \leq i < j \leq p-1$

$$x_i^p = x_j^p$$

or, in other words,

$$x_i^p = (x_i m^{j-i})^p.$$

Write $m_0 = m^{j-i}$ and $x_0 = x_i$. Then we have from the last line

$$\begin{aligned} x_0^p &= (x_0 m_0)^p \\ &= x_0^p m_0^{1+x_0+x_0^2+\dots+x_0^{p-1}}, \end{aligned}$$

whence

$$m_0^{1+x_0+x_0^2+\dots+x_0^{p-1}} = 1.$$

In this equation we can replace m_0 by m since $j-i$ is prime to p. Also we may replace x_0 by x since M is abelian. Therefore, regarding M as a module, and the elements of G as endomorphisms of M, we have shown that, for all $x \in G \backslash C$

$$1 + x + x^2 + \dots + x^{p-1} = 0. \tag{2.3}$$

The next step is to show that (2.3) holds for all x in G. To see this let c be an arbitrary element of C. Then

$$\sum_{i=0}^{p-1} c^i = \sum_{j=0}^{p-1} \sum_{i=0}^{p-1} (cx^j)^i$$

since the new summands added are all zero, by (2.3). Therefore

$$\begin{aligned} \sum_{i=0}^{p-1} c^i &= \sum_{i=0}^{p-1} c^i \sum_{j=0}^{p-1} x^{ij} \\ &= 0, \end{aligned}$$

by (2.3) again and the fact that M has exponent p. Hence (2.3) does hold for all $x \in G$.

Now

$$\binom{p}{i} = \binom{p-1}{i} + \binom{p-1}{i-1}, \qquad i \geq 1$$

so that

$$\binom{p-1}{i} \equiv (-1)^i \pmod{p}, \quad 0 \leq i \leq p-1.$$

It follows that, for all $x \in G$,

$$\begin{aligned}(x-1)^{p-1} &= \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} x^i \\ &= \sum_{i=0}^{p-1} x^i \\ &= 0.\end{aligned}$$

This completes the proof of (2.2).

To complete the proof of the first statement in the theorem observe that G/M is metabelian of exponent p. Hence, if x, y are arbitrary elements of G, $[x, (p-1)y] \in M$: see [**7**]. Combining this with (2.3) yields that

$$[x, (p-1)y, (p-1)z] = 1$$

is a law in G. We now invoke the result of Gupta and Newman [**5**] to conclude that G has class at most $2p-2$.

We take up the claim that this bound is best possible in the next section. This section is complete by proving the Corollary.

First note that in a metabelian group of exponent dividing p^2, and of class at least $2p$, $\omega_1(G) = \zeta_1(G)$. This follows immediately from the Theorem. For the same reason $\omega_2(G) = \zeta_2(G)$ which we see by considering $G/\omega(G)$. We now prove by induction on r that $\omega_r(G) = \zeta_r(G)$ for $r \geq 2$. To this end let $w \in \omega_{r+1}(G)$, where $r \geq 2$. For an arbitrary $g \in G$, $[w, g] = g^l$ modulo $\omega_r(G)$ for some integer l. It follows that, on the assumption that $\omega_r(G) = \zeta_r(G)$,

$$[w, g, g] = [g^{-l}[w, g], g] \in \zeta_{r-1}(G). \tag{2.4}$$

For all $g, h \in G$, $[w, gh, gh] \in \zeta_{r-1}(G)$. Routine commutator calculations show that then

$$[w, g, h][w, h, g] \in \zeta_{r-1}(G).$$

We use here that $\omega_{r+1}(G) \leq \zeta_{r+2}(G)$ under the hypothesis $\omega_r(G) = \zeta_r(G)$. Then, because G is metabelian, we obtain by commuting with h,

$$[w, g, h, h][w, h, h, g] \in \zeta_{r-2}(G),$$

whence

$$[w, g, h, h] \in \zeta_{r-2}(G)$$

by (2.4). Finally, from $[w, g, hk, hk] \in \zeta_{r-2}(G)$ for all $h, k \in G$, we deduce by standard commutator calculations that $[w, g, h, k]^2 \in \zeta_{r-2}(G)$, whence

$$[w, g, h, k] \in \zeta_{r-2}(G)$$

since p is odd. From this it follows that $w \in \zeta_{r+1}(G)$. Since $w \in \omega_{r+1}(G)$ was chosen arbitrarily, $\omega_{r+1}(G) \leq \zeta_{r+1}(G)$. However, $\zeta_{r+1}(G) \leq \omega_{r+1}(G)$ so $\omega_{r+1}(G) = \zeta_{r+1}(G)$. This completes the induction and the proof of the Corollary.

3. A construction. Let $A = \langle a \rangle$, $B = \langle b \rangle$ be cyclic groups of order p^r $(r \geq 2)$. Write Z for the subgroup of B of order p. Also write $W = A \; twr_Z \; B$ where Z acts trivially on A. There is an automorphism α of W which satisfies

$$a^\alpha = a, \qquad b^\alpha = b^{q+1}$$

where $q = p^{r-1}$.

Let X be the base group of W and write

$$Y = \langle a a^b a^{b^2} \dots a^{b^{q-1}} \rangle.$$

Then $Y \leq \zeta_1(W)$ and it admits α. Let $W_1 = W/Y$. We claim that the automorphism α_1 induced in W_1 by α, is a power automorphism. First note that α acts trivially on X/Y since $b^q \in Z$. Therefore, for all $c \in B$, and all $f \in X$,

$$\begin{aligned}(cf)^\alpha &= c^{q+1} f \\ &= c^{q+1} f^{1+c+c^2+\dots+c^q} \qquad (\text{mod } Y) \\ &= (cf)^{q+1} \qquad (\text{mod } Y).\end{aligned}$$

Hence α_1 is a power automorphism on W_1. The order of α_1 is, of course, p.

We are now in a position to complete the proof of the Theorem. Let $r = 2$ and let H be the semi-direct product $W_1 \langle \alpha_1 \rangle$. By the Lemma this is metabelian and of exponent p^2, because W_1 is; and its class is the same as that of W_1. By Liebeck [6] this class is $2p - 2$.

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