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Cohomology Ring of Symplectic Quotients by Circle Actions

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Abstract. In this article we are concerned with how to compute the cohomology ring of a symplectic quotient by a circle action using the information we have about the cohomology of the original manifold and some data at the fixed point set of the action. Our method is based on the Tolman-Weitsman theorem which gives a characterization of the kernel of the Kirwan map. First we compute a generating set for the kernel of the Kirwan map for the case of product of compact connected manifolds such that the cohomology ring of each of them is generated by a degree two class. We assume the fixed point set is isolated; however the circle action only needs to be "formally Hamiltonian". By identifying the kernel, we obtain the cohomology ring of the symplectic quotient. Next we apply this result to some special cases and in particular to the case of products of two dimensional spheres. We show that the results of Kalkman and Hausmann-Knutson are special cases of our result.

1 Introduction

In this article we are concerned with the cohomology ring of symplectic reductions. We would like to answer the following question: When we consider a Hamiltonian action of a Lie group on a symplectic manifold, what would the quotient space topologically look like? The interesting point is that in fact, using only the information about the moment map at the fixed point set of the action, one can at least theoretically answer this question. The Tolman-Weitsman theorem [TW1] has now enabled us to find the answer to our question with just the information mentioned. Kalkman was the first who in [Ka] calculated the cohomology ring of the symplectic reduction of a projective space by a circle action using the localization formula. However his work was not continued further. The next attempt to understand the cohomology of these spaces was based on other means.

Hausmann and Knutson used Danilov's theorem to approach the problem. Danilov's theorem specifies the cohomology rings of all toric manifolds. In cases where the original manifold is a toric manifold, one can show that its symplectic quotient is a toric manifold too. One is then able to use Danilov's theorem to find the cohomology ring of the symplectic reduction. This is what Hausmann and Knutson did in [HK] to calculate the cohomology ring of the symplectic quotient of a product of two-dimensional spheres.

We know very little when the group acting on the manifold is a general compact Lie group or even a torus. In her Ph.D. thesis, R. Goldin [G] (also see [G2]) answered this question for the action of a torus on a coadjoint orbit of $\mathrm{SU}(n)$.

Later Tolman and Weitsman [TW2] generalized the results of Hausmann and Knutson to a compact connected symplectic manifold, but they had to assume that

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the action is semi-free and the fixed point set is isolated. They found the integer cohomology ring of $M_{\text{red}} = \mu^{-1}(0)/S^1$, whenever 0 is a regular value of the moment map. The conditions of semi-free action and the fixed point set being isolated enabled them to establish a correspondence between the fixed point set of the circle action on M and that of the product of two-dimensional spheres. However to obtain those results Tolman and Weitsman did not use their own theorem [TW1], which already opens the way to answer the problem in more general settings. Our method in this article is based on this key theorem. This theorem reduces the problem of finding a generating set for the kernel of the Kirwan map

$$\kappa\colon H_T^*(M) \to H^*\left(\mu^{-1}(0)/T\right)$$

to some specific algebraic calculations. We are then done with the task of finding the cohomology ring of the quotient space because

$$H^*\left(\mu^{-1}(0)/T\right) \cong H_T^*(M)/\ker(\kappa),$$

due to Kirwan's surjectivity theorem ([Ki1], 5.4) which asserts that κ is a surjective ring homomorphism.

We would like to state the Tolman-Weitsman theorem here for the case of circle actions on which the results of this article are based:

Theorem 1 ([TW1]) *Let S^1 act on a compact symplectic manifold M with moment map $\mu\colon M \to \mathbb{R}$. Assume that r is a regular value of μ. Let $\mathcal{F}$ denote the set of fixed points of the action. Write $M_-(r) = \mu^{-1}(-\infty, r)$ and $M_+ = \mu^{-1}(r, +\infty)$. Define $K_\pm(r) = \{\alpha \in H_{S^1}^*(M, \mathbb{C}) : \alpha|_{\mathcal{F}\cap M_\pm(r)} = 0\}$ and $K(r) = K_+(r) \oplus K_-(r)$. Then there is a short exact sequence:*

$$0 \to K(r) \to H_{S^1}^*(M, \mathbb{C}) \xrightarrow{\kappa} H^*(M_{\text{red}}, \mathbb{C}) \to 0,$$

where κ is the Kirwan map and $M_{\text{red}} = \mu^{-1}(r)/S^1$.

When $r = 0$ we write $M_\pm$, $K_\pm$ and K for $M_\pm(0)$, $K_\pm(0)$ and $K(0)$.

Remark By Remark 3.4 of [TW1], we do not need to assume that the action is Hamiltonian. The statement still holds if the action is more generally *formally Hamiltonian*. This means there is a Morse-Bott function $\mu\colon M \to \mathbf{t}^* = \operatorname{Lie}(S^1)^* \cong \mathbb{R}$ (a formal moment map) such that the critical points of μ correspond exactly to the fixed points of the action. Then as long as M is compact and 0 is a regular value of μ, the theorem is true for any formal moment map.

Besides the Tolman-Weitsman theorem that we use in this article, the residue formula ([JK1], [JK3]) is another powerful tool which may enable us to answer the question even in more general cases.

For the various definitions and properties of equivariant cohomology see for example [Au] and [BGV].

2 Notation and Preliminaries

First let us fix our notation. Consider a compact connected manifold M whose cohomology ring is generated by degree two classes $x_i \in H^2(M)$, $i = 1, 2, \ldots, m$. Assume the manifold is equipped with a circle action with isolated fixed points. We label the fixed point set by F_j, $j = 1, 2, \ldots, n$. Suppose there are moment maps for the action denoted by $\mu_i \colon M \to \mathbb{R}$ such that $i_t x_i = t\, d\mu_i$ for all $t \in \mathbb{R} \cong \mathbf{t}^* = \mathrm{Lie}(T)^*$. Here $T = S^1$. Consider the two-form $x = \sum_{i=1}^m x_i$. Corresponding to this two-form we also have the function $\mu \colon M \to \mathbb{R}$ defined by $\mu = \sum_{i=1}^m \mu_i$ so that $i_t x = t\, d\mu$.

We impose the extra condition that μ does not vanish at any of the fixed points.

Now consider the equivariant cohomology algebra $H_T^*(M)$. As a vector space it can be written as

$$\mathcal{R} = H_T^*(M) \cong H_T^*(\text{point}) \otimes H^*(M),$$

where $H^*(M) \cong \mathbb{C}[x_1, \ldots, x_m]/\mathcal{I}$. Here $\mathcal{I}$ is the set of relations in $H^*(M)$. Also $H_T^*(\text{point}) = \mathbb{C}[t]$, the polynomial ring in the variable t.

If $\tilde{x}_i = x_i + t\mu_i$ are the equivariant extensions of the corresponding x_i's, then we see that $\tilde{x}_1, \ldots, \tilde{x}_m$ together with t generate the equivariant cohomology $H_T^*(M)$ as a ring. We also consider the equivariant extension $\tilde{x} = x + t\mu$.

The values of the moment maps at the fixed points are of great importance. We denote them as follows: $\mu_i(F_j) = \theta_{ij}$ so that the restriction of $\tilde{x}_i$ to the j-th component of the fixed point set is θ_{ij}: $\tilde{x}_i|_{F_j} = \theta_{ij} t$. Then $\mu(F_j) = \sum_{i=1}^m \theta_{ij}$ and $\tilde{x}|_{F_j} = \sum_{i=1}^m \theta_{ij} t$.

Now we would like to specify K_+ and K_- in the ring $\mathcal{R}$. According to the Tolman-Weitsman theorem,

$$K_+ = \{\alpha \in H_T^*(M) : \alpha|F_j = 0 \text{ for all } j \text{ such that } \mu(F_j) > 0\}.$$

Equivalently,

$$K_+ = \Big\{\alpha \in \mathcal{R} : \alpha(\theta_{1j}t, \ldots, \theta_{mj}t) = 0 \text{ for all } j \text{ such that } \mu(F_j) = \sum_{i=1}^m \theta_{ij} > 0\Big\}.$$

The ideal K_- is defined similarly with the difference that $>$ is replaced with $<$ in the definition of the set. We can consider K_+ and K_- as the intersection of a finite number of ideals as follows: Consider the multivariable polynomial ring

$$\bar{\mathcal{R}} = \mathbb{C}[t][\tilde{x}_1, \ldots, \tilde{x}_m]$$

in the variables $\tilde{x}_i$ with coefficients in $\mathbb{C}[t]$ (the polynomial ring in one variable t with complex coefficients). Thus $H_T^*(M)$ is the quotient of $\bar{\mathcal{R}}$ by an ideal of relations.

For $1 \le j \le n$ define the ideals

$$\mathcal{I}_j = \{\alpha \in H_T^*(M) : \alpha|_{F_j} = 0\} \cong \{\alpha \in \bar{\mathcal{R}} : \alpha(\tilde{x}_1 = \theta_{1j}t, \ldots, \tilde{x}_m = \theta_{mj}t) = 0\}$$

in $\bar{\mathcal{R}}$. Then $\bar{K}_+$ is the intersection of those $\mathcal{I}_j$'s that correspond to the j's for which the value of the moment map μ is positive:

$$\bar{K}_+ = \bigcap_{1\le j\le n:\mu(F_j)>0} \mathcal{I}_j \qquad \Big(\text{similarly } \bar{K}_- = \bigcap_{1\le j\le n:\mu(F_j)<0} \mathcal{I}_j\Big).$$

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In fact we know the generators of each $\mathcal{I}_j$. They are simply $\tilde{x}_1-\theta_{1j}t,\dots,\tilde{x}_m-\theta_{mj}t$. The problem of classifying the intersection ideal (say by specifying a generating set) is very hard and still open! We can solve this problem for a special case that is important to our geometric concerns. In the next section we will explain this special case and will show that a generating set for the intersection ideal exists such that each of its elements is a product of proper linear terms.

3 The Main Result and Its Proof

We shall consider the special case when M is a product of compact connected symplectic manifolds M_i, $i = 1, 2, \dots, m$, *i.e.*, $M = M_1 \times M_2 \times \cdots \times M_m$ such that the cohomology ring of each M_i is generated by a two-form $x_i \in H^2(M_i)$, *i.e.*, $H^*(M_i) = \langle x_i \rangle$. Consider the extensions of these forms to M by $x_i = 1 \otimes \cdots \otimes x_i \otimes \cdots \otimes 1 \in H^*(M) = \bigotimes_{i=1}^m H^*(M_i)$. Each M_i is equipped with a Hamiltonian circle action with isolated fixed points. Consider the diagonal action on M. The fixed points are labeled by m-tuples $\mathbf{F} = (F_{1j_1}, F_{2j_2}, \dots, F_{mj_m})$ for all choices of $1 \leq j_i \leq n_i$, where n_i is the number of the fixed points of M_i with distinct moment map value. Here F_{ij} denotes the union of those fixed points of M_i whose value under the moment map μ_i is θ_{ij}. Therefore $j = j' \Leftrightarrow \theta_{ij} = \theta_{ij'}$ for all i, j, j'. If the value of μ_i at F_{ij} is denoted by θ_{ij}, then $\mu(\mathbf{F}) = \mu(F_{1j_1}, F_{2j_2}, \dots, F_{mj_m}) = \sum_{i=1}^m \theta_{ij_i}$. The restrictions of each $\tilde{x}_i$ and $\tilde{x}$ to the fixed point $\mathbf{F} = (F_{1j_1}, F_{2j_2}, \dots, F_{mj_m})$ are given by $\tilde{x}_i|_{\mathbf{F}} = \theta_{ij_i}t$ and $\tilde{x}|_{\mathbf{F}} = \sum_{i=1}^m \theta_{ij_i}t$.

As usual we are concerned about the kernel of the Kirwan map: $K = K_+ \oplus K_-$. The following proposition is of fundamental importance to us:

Proposition 1 *The ideal $\bar{K}_+$ has a generating set such that each generator is a product of linear terms of the form $\tilde{x}_i - \theta_{ij_i}t$. Moreover the linear terms that appear in each generator are mutually distinct. The same as for $\bar{K}_+$ is true for $\bar{K}_-$.*

Note that here i indexes the manifolds M_i and j_i indexes the fixed point set of the i-th manifold M_i.

The proposition is an immediate consequence of the following lemma.

Lemma 1 *Consider the ring $\bar{\mathcal{R}} = \mathbb{C}[t][\tilde{x}_1, \dots, \tilde{x}_m]$, and consider the following finite set $\mathcal{F}' = \{F = (\theta_{1j_1}t, \dots, \theta_{mj_m}t) \in \mathbb{C}[t]^m : 1 \leq j_i \leq n_i$ such that $\theta_{ij} > \theta_{ij'}$ for $j < j'\}$ of points in $\mathbb{C}[t]^m$, where the real numbers θ_{ij} and positive integers n_i are given. Define $\bar{\mathcal{F}}_+ = \{(\theta_{1j_1}t, \dots, \theta_{mj_m}t) \in \mathcal{F}' : \sum_{i=1}^m \theta_{ij_i} > c\}$, where c is some fixed real number. It is a subset of $\mathcal{F}'$. Let $\mathcal{I}_+ = \{\alpha \in \bar{\mathcal{R}} : \alpha(F) = 0$ for all $F \in \bar{\mathcal{F}}_+\}$. Then the ideal $\mathcal{I}_+$ has a generating set consisting of polynomials each of which is a product of terms of the form $\tilde{x}_i - \theta_{ij_i}t$ which we will refer to as linear terms from now on. The linear terms in each generator are mutually distinct. If we replace the condition $\sum_{i=1}^m \theta_{ij_i} > c$ with $\sum_{i=1}^m \theta_{ij_i} < c$, the statement is still true.*

To prove this, we need the following algebraic lemma:

Lemma 2 *If $P(x_1, \dots, x_n)$ is a polynomial and $P(a_1, \dots, a_n) = 0$, then there are polynomials $Q_1, \dots, Q_n$ in $x_1, \dots, x_n$ such that $P = (x_1 - a_1)Q_1 + \cdots + (x_n - a_n)Q_n$.*

Proof of Lemma 2 Since $P(a_1,\dots,a_n) = 0$, the Euclidean Lemma tells us that there are polynomials Q_1 and $R_1(x_2,\dots,x_n)$ such that $P = (x_1 - a_1)Q_1 + R_1$. Then $R_1(a_2,\dots,a_n) = 0$. Thus, we can proceed by induction on k for $R_k(a_k,\dots,a_n)$, to finally obtain $P = (x_1 - a_1)Q_1 + \cdots + (x_n - a_n)Q_n + R_n$, where R_n is just a number. Then from $P(a_1,\dots,a_n) = 0$, we get $R_n = 0$. This completes the proof of the lemma. ■

Proof of Lemma 1 The proof is by induction on m. To understand how the induction works, we initially discuss both cases $m = 1$ and $m = 2$, even though mathematically we only need to check the case $m = 1$.

So assume $m = 1$. We show $\mathcal{I}_+$ is generated by one element, *i.e.* $\prod_{j:\theta_j>c}(\tilde{x} - \theta_j t)$. To see this, notice that $\alpha(\tilde{x}) \in \mathcal{I}_+$ if and only if $\alpha(F) = 0$ for every $F \in \bar{\mathcal{F}}_+$. This means $(\tilde{x} - \theta_j t)$ divides α for each j with $\theta_j > c$. So their product also divides α, which is what we wanted to prove.

Now consider the case $m = 2$ so that $\bar{\mathcal{R}} = \mathbb{C}[t][\tilde{x}_1, \tilde{x}_2]$. We arrange the points of $\bar{\mathcal{F}}_+$ in the following table:

$$
\begin{array}{cccc}
(\theta_{11}t, \theta_{21}t) & (\theta_{12}t, \theta_{21}t) & \cdots & (\theta_{1l}t, \theta_{21}t) \\
\vdots & \vdots & & \vdots \\
(\theta_{11}t, \theta_{2m_l}t) & (\theta_{12}t, \theta_{2m_l}t) & \dots & (\theta_{1l}t, \theta_{2m_l}t) \\
\vdots & \vdots & & \\
(\theta_{11}t, \theta_{2m_2}t) & (\theta_{12}t, \theta_{2m_2}t) & & \\
\vdots & & & \\
(\theta_{11}t, \theta_{2m_1}t). & & &
\end{array}
$$

Here the k-th row and i-th column is $(\theta_{1i}t, \theta_{2k}t)$ for $k \le m_i$. The integer m_i is the largest integer such that $\theta_{1i}+\theta_{2m_i} > c$ and l is the largest integer such that $\theta_{1l}+\theta_{21} > c$. Notice that $l \le n_1$.

Since $\theta_{11} > \theta_{12} > \cdots > \theta_{1n_1}$ and $\theta_{21} > \theta_{22} > \cdots > \theta_{2n_2}$, then if $\theta_{1(i+1)} + \theta_{2j} > c$, we also have $\theta_{1i} + \theta_{2j} > c$. Therefore $m_1 \ge m_2 \ge \cdots \ge m_l$ which is a crucial fact in our argument.

Fix $\alpha \in \mathcal{I}_+$. Then $\alpha(\tilde{x}_1, \tilde{x}_2)$ vanishes at all $F \in \bar{\mathcal{F}}_+$. By Lemma 2 applied to the first point of the first column, we see that there are polynomials $p(\tilde{x}_1, \tilde{x}_2)$ and $q(\tilde{x}_2)$ such that $\alpha(\tilde{x}_1, \tilde{x}_2) = (\tilde{x}_1 - \theta_{11}t)p(\tilde{x}_1, \tilde{x}_2) + (\tilde{x}_2 - \theta_{21}t)q(\tilde{x}_2)$. Since α vanishes at other points of the first column, we see that $q(\theta_{22}t) = \cdots = q(\theta_{2m_1}t) = 0$, so that $(\tilde{x}_2 - \theta_{22}t)(\tilde{x}_2 - \theta_{23}t)\cdots(\tilde{x}_2 - \theta_{2m_1}t)$ has to divide $q(\tilde{x}_2)$. Therefore, there is a polynomial $q'(\tilde{x}_2)$ such that $q(\tilde{x}_2) = (\tilde{x}_2 - \theta_{22}t)(\tilde{x}_2 - \theta_{23}t)\cdots(\tilde{x}_2 - \theta_{2m_1}t)q'(\tilde{x}_2)$. Now by considering the vanishing of α at the first point of the second column we find that there are polynomials $p_1(\tilde{x}_1, \tilde{x}_2)$ and $q_1(\tilde{x}_2)$ such that

$$p(\tilde{x}_1, \tilde{x}_2) = (\tilde{x}_1 - \theta_{12}t)p_1(\tilde{x}_1, \tilde{x}_2) + (\tilde{x}_2 - \theta_{21}t)q_1(\tilde{x}_2).$$

Considering the rest of the points of the second column in the same way as what we concluded for $q(\tilde{x}_2)$, we see that $q_1(\tilde{x}_2) = (\tilde{x}_2 - \theta_{22}t)(\tilde{x}_2 - \theta_{23}t)\cdots(\tilde{x}_2 - \theta_{2m_2}t)q_1'(\tilde{x}_2)$ for some polynomial $q_1'(\tilde{x}_2)$. One can now write α as

$$\begin{aligned}\alpha(\tilde{x}_1, \tilde{x}_2) = (\tilde{x}_1 - \theta_{11}t)(\tilde{x}_1 - \theta_{12}t)p_1(\tilde{x}_1, \tilde{x}_2)&\\ + (\tilde{x}_1 - \theta_{11}t)(\tilde{x}_2 - \theta_{21}t)\cdots(\tilde{x}_2 - \theta_{2m_2}t)q_1'(\tilde{x}_2)&\\ + (\tilde{x}_2 - \theta_{21}t)\cdots(\tilde{x}_2 - \theta_{2m_1}t)q'(\tilde{x}_2).&\end{aligned}$$

Proceeding by induction we write α as

$$\begin{aligned}\alpha(\tilde{x}_1, \tilde{x}_2) &= (\tilde{x}_1 - \theta_{11}t)(\tilde{x}_1 - \theta_{12}t)\cdots(\tilde{x}_1 - \theta_{1l}t)q_l\\ &\quad + (\tilde{x}_1 - \theta_{11}t)(\tilde{x}_1 - \theta_{12}t)\cdots(\tilde{x}_1 - \theta_{1(l-1)}t)(\tilde{x}_2 - \theta_{21}t)\cdots(\tilde{x}_2 - \theta_{2m_l}t)q_{l-1}'\\ &\quad + \cdots + (\tilde{x}_1 - \theta_{11}t)(\tilde{x}_2 - \theta_{21}t)\cdots(\tilde{x}_2 - \theta_{2m_2}t)q_1'\\ &\quad + (\tilde{x}_2 - \theta_{21}t)\cdots(\tilde{x}_2 - \theta_{2m_1}t)q'.\end{aligned}$$

This not only completes the proof for the case $m = 2$, but also gives a specific list of the generators in the form that was claimed.

Inductively assume the lemma is true for any polynomial in $m-1$ variables and for any value of c, so that any two linear terms in each of the contributing products are distinct. We then show it also holds for any polynomial in m variables and any value of c, so that any two linear terms in each of the contributing products are distinct.

Assume $\alpha \in \mathcal{I}_+$, so that it vanishes at the given points of $\mathbb{C}[t]^m$. As before we arrange the points at which α vanishes in the following way: the first column consists of the points $(\theta_{11}t, \theta_{2j_2}t, \ldots, \theta_{mj_m}t)$ for $(j_2, \ldots, j_m) \in A_1 \subset \{1, \ldots, n_2\} \times \cdots \times \{1, \ldots, n_m\}$, the second column is $(\theta_{12}t, \theta_{2j_2}t, \ldots, \theta_{mj_m}t)$ for $(j_2, \ldots, j_m) \in A_2$, and the last column is $(\theta_{1l}t, \theta_{2j_2}t, \ldots, \theta_{mj_m}t)$ for $(j_2, \ldots, j_m) \in A_l$, where A_i are specified by the definition of the set $\bar{\mathcal{F}}_+$ so that

$$A_i = \{(j_2, \ldots, j_m) : j_i \leq n_i \text{ and } \theta_{1i} + \sum_{k=2}^{m} \theta_{kj_k} > c\}.$$

Here l is the largest integer such that there is some point in $\bar{\mathcal{F}}_+$ whose first coordinate is $\theta_{1l}t$.

Because $\theta_{11} > \theta_{12} > \cdots > \theta_{1l}$, we see that if we have $\theta_{1(i+1)} + \sum_{k=2}^{m} \theta_{kj_k} > c$, then we also have $\theta_{1i} + \sum_{k=2}^{m} \theta_{kj_k} > c$. Therefore $A_l \subset A_{l-1} \subset \cdots \subset A_2 \subset A_1$.

Considering the first point of the first column we see that by the division algorithm $\alpha(\tilde{x}_1, \ldots, \tilde{x}_m) = (\tilde{x}_1 - \theta_{11}t)p(\tilde{x}_1, \ldots, \tilde{x}_m) + q(\tilde{x}_2, \ldots, \tilde{x}_m)$ for some polynomials p and q. Considering the rest of the points of the first column we find that q has to satisfy $q(\theta_{2j_2}t, \ldots, \theta_{mj_m}t) = 0$ for all $(j_2, \ldots, j_m) \in A_1$. Consider the points in $\mathbb{C}[t]^{m-1}$ corresponding to A_1. Then $\theta_{2j_2} + \cdots + \theta_{mj_m} > -\theta_{11} + c$, so that we can apply the induction hypothesis to q and $c' = c - \theta_{11}$ and conclude that q can be written as a linear combination of products of linear terms of the form $(\tilde{x}_2 - \theta_{2j_2}t)$, $(\tilde{x}_3 - \theta_{3j_3}t), \ldots$ and $(\tilde{x}_2 - \theta_{mj_m}t)$, where $j_2, \ldots, j_m$ are specified by A_1 and no linear term appears twice in each resulting product.

Next considering the second column, we find polynomials p_1 and q_1 such that $p(\tilde{x}_1, \dots, \tilde{x}_m) = (\tilde{x}_1 - \theta_{12}t)p_1(\tilde{x}_1, \dots, \tilde{x}_m) + q_1(\tilde{x}_2, \dots, \tilde{x}_m)$. Because $A_2 \subset A_1$, we see that $q_1(\theta_{2j_2}t, \dots, \theta_{mj_m}t) = 0$ for all $(j_2, \dots, j_m) \in A_2$. So q_1 is a combination of products of linear terms by the induction hypothesis so that no linear term appears twice in each resulting product. One can now write α as

$$\alpha = (\tilde{x}_1 - \theta_{11})(\tilde{x}_1 - \theta_{12})p_1 + (\tilde{x}_1 - \theta_{11})q_1 + q.$$

Note that the term $(\tilde{x}_1 - \theta_{11})$ does not appear anywhere in $q_1(\tilde{x}_2, \dots, \tilde{x}_m)$, so that after multiplying by it in each term of q_1, the linear terms that appear in each resulting product are still mutually distinct.

Proceeding inductively on the columns we obtain polynomials $q, q_1, q_2, \dots$, all of which are combinations of products of linear terms, so that eventually α can also be written in this way with the same property that the linear terms in each resulting product are mutually distinct.

If the condition in the definition of $\bar{\mathcal{F}}_+$ is $\sum_{i=1}^m \theta_{ij_i} < c$, we simply need to write the table of the points of $\bar{\mathcal{F}}_-$ in reverse order so that the points corresponding to the largest indices appear at the top of the table. Then because $\theta_{11} > \theta_{12} > \cdots > \theta_{1l}$, we see that for each i if $\theta_{1i} + \sum_{k=2}^m \theta_{kj_k} < c$, then also $\theta_{1(i+1)} + \sum_{k=2}^m \theta_{kj_k} < c$. Therefore $A_1 \subset A_2 \subset \cdots \subset A_l$. We then need to start the argument from the index l proceeding down to 1. The rest of the proof is the same. This finishes the proof of the lemma. ■

Let us return to geometry and the case of the product of manifolds. We shall give a specific representation of some generating sets of $\bar{K}_+$ and $\bar{K}_-$ which are of the specific form described in Lemma 1.

For simplicity and convenience we relabel the fixed point set in the following way:

Consider $\mathcal{A} := N_1 \times N_2 \times \cdots \times N_m$, where $N_i = \{1, \dots, n_i\}$. Then we have a one-to-one correspondence between the components of the fixed point set on which the value of the moment map is the same and the elements of $\mathcal{A}$:

$$\mathbf{F} = \mathbf{F}(J) = (F_{1j_1}, F_{2j_2}, \dots, F_{mj_m}) \sim J = (j_1, j_2, \dots, j_m) \in \mathcal{A}.$$

Definition 1 We define the *long elements* of $\mathcal{A}$ as members of the set

$$\mathcal{L} = \left\{ J \in \mathcal{A} \mid \mu\big(\mathbf{F}(J)\big) > 0 \right\},$$

and *short elements* as members of the set

$$\mathcal{S} = \left\{ J \in \mathcal{A} \mid \mu\big(\mathbf{F}(J)\big) < 0 \right\}.$$

Consider the projections $\begin{cases} P_i \colon \mathcal{A} \to N_i \\ P_i(j_1, \dots, j_i, \dots, j_m) = j_i \end{cases} \quad 1 \leq i \leq m.$

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Definition 2 We call a collection $\{\mathcal{A}_i\}_{1\leq i\leq m}$ where $\mathcal{A}_i \subset N_i$ a *covering* of $\mathcal{L}$ (respectively, $\mathcal{S}$), if

$$\mathcal{L} \subset \bigcup_{i=1}^{m} P_i^{-1}(\mathcal{A}_i) \quad \Bigl(\text{respectively } \mathcal{S} \subset \bigcup_{i=1}^{m} P_i^{-1}(\mathcal{A}_i)\Bigr).$$

We call it a *minimal covering* if, whenever we drop just one element from one of the $\mathcal{A}_i$'s, it will no longer be a covering of $\mathcal{L}$ (respectively, $\mathcal{S}$).

Notice that some of the $\mathcal{A}_i$'s may be empty sets and $P_i^{-1}(\mathcal{A}_i) = N_1 \times \cdots \times N_{i-1} \times \mathcal{A}_i \times N_{i+1} \times \cdots \times N_m$.

Consider the composition of the map

$$\mathbb{C}[t, \tilde{x}_1, \dots, \tilde{x}_m] \xrightarrow{\eta} H_T^*(M, \mathbb{C})$$

with

$$H_T^*(M, \mathbb{C}) \xrightarrow{\kappa} H^*(M_{\mathrm{red}}, \mathbb{C}).$$

Let $\bar{K}_+$ and $\bar{K}_-$ denote the preimages under η of K_+ and K_- defined in Theorem 1. Now we are ready to state our main result:

Theorem 2 *Consider the case of products of compact connected manifolds such that the cohomology of each of them is generated by a degree-two form.*

(i) *The following family of classes of equivariant forms belongs to and generates $\bar{K}_+$:*

$$\Bigl\{ \prod_{1\leq i\leq m} \prod_{j_i\in\mathcal{A}_i} (\tilde{x}_i - \theta_{i j_i} t) : \mathcal{A}_i \textit{ is a minimal covering of } \mathcal{L} \Bigr\}. \tag{1}$$

(ii) *The following family of classes of equivariant forms belongs to and generates $\bar{K}_-$:*

$$\Bigl\{ \prod_{1\leq i\leq m} \prod_{j_i\in\mathcal{A}_i} (\tilde{x}_i - \theta_{i j_i} t) : \mathcal{A}_i \textit{ is a minimal covering of } \mathcal{S} \Bigr\}. \tag{2}$$

Remark The minimality condition was added to avoid some extra terms which do not contribute to generating $\bar{K}_+$ or $\bar{K}_-$.

Proof By Lemma 1, $\bar{K}_+$ has a set of generators that are products of distinct linear terms. Moreover the lemma precisely specifies these linear terms: $\tilde{x}_i - \theta_{i j_i} t$, where the $\theta_{i j_i} t$ are the components of the points in $\mathbb{C}[t]^m$ at which the elements of $\bar{K}_+$ vanish. There are a finite number of polynomials that can be written in this form. Considering all possible choices there are a total of $2^{n_1 n_2 \cdots n_m}$ polynomials made out of these linear terms so that no linear term appears more than once. So we have the task of separating all those that belong to $\bar{K}_+$ and giving an adequate set of generators for it.

Clearly every element of (5) vanishes at $\mathbf{F}(J)$ for all $J \in \mathcal{L}$. Assume $\alpha \in \bar{K}_+$ is a product of the linear terms specified. We show that α is a multiple of some polynomial in the class (5). This means the class of polynomials (5) form a generating set for $\bar{K}_+$.

To show this for each i, define the sets $\mathcal{B}_i$ as $\mathcal{B}_i = \{j_i : (\tilde{x}_i - \theta_{i j_i} t) \text{ divides } \alpha\}$. Then $\{\mathcal{B}_i\}_{i \leq m}$ is a covering of $\mathcal{L}$, since $\alpha \in \bar{K}_+$, hence α should vanish at $\mathbf{F}(J)$ for all $J \in \mathcal{L}$. Then $\alpha = \prod_{1 \leq i \leq m} \prod_{j_i \in \mathcal{B}_i} (\tilde{x}_i - \theta_{i j_i} t)$. This covering does not have to be a minimal one. However it is clear that every covering has a minimal sub-covering, *i.e.*, a minimal covering $\{\mathcal{A}_i\}_{i \leq m}$ such that $\mathcal{A}_i \subset \mathcal{B}_i$ for each i. Then the polynomial in (i) corresponding to this minimal covering is a divisor of α so that the classes (5) corresponding to minimal coverings suffice to form a generating set for $\bar{K}_+$. This finishes the proof of (i).

The proof of (ii) is similar. ∎

4 Examples

Example 1 Consider the projective space $M = \mathbb{C}P^n$ equipped with a circle action with weights $m_1, \ldots, m_n$ so that $g \cdot [z_0 : \cdots : z_n] = [g^{m_0} z_0 : \cdots : g^{m_n} z_n]$ for $g \in S^1$ and $[z_0 : \cdots : z_n] \in \mathbb{C}P^n$. This action is Hamiltonian with the moment map $\mu \colon \mathbb{C}P^n \to R$; $[z_0 : \cdots : z_n] \mapsto \frac{\sum_i m_i z_i \bar{z}_i}{\sum_i z_i \bar{z}_i}$. The fixed points of this action are $F_i = [0 : \cdots : 1 : \cdots : 0]$ where 1 is in the i-th position.

Kalkman [Ka] used the localization formula to find the cohomology ring of the symplectic quotient $\mu^{-1}(0)/S^1$. As we show this is a special case of Theorem 2:

The cohomology ring of $\mathbb{C}P^n$ is generated by the degree-two class of the symplectic form $x \in H^2(\mathbb{C}P^n)$. Define $\tilde{x}$ as before. Also $\mathcal{L} = \{i \mid \mu(F_i) > 0\}$ and $\mathcal{S} = \{i \mid \mu(F_i) < 0\}$. By Theorem 2, the polynomials $P = \prod_{i \in \mathcal{L}} \bigl(\tilde{x} - \mu(F_i) t\bigr)$ and $Q = \prod_{i \in \mathcal{S}} \bigl(\tilde{x} - \mu(F_i) t\bigr)$ (families (1) and (2) of Theorem 2) generate $\bar{K}_+$ and $\bar{K}_-$ respectively. They correspond to the minimal coverings $\{\mathcal{L}\}$ and $\{\mathcal{S}\}$ of $\mathcal{L}$ and $\mathcal{S}$ respectively. This result is the content of Theorem 5.2 in [Ka]. ∎

Example 2 Now consider the product of two projective spaces $M = \mathbb{C}P^k \times \mathbb{C}P^l$ with symplectic forms x_1 and x_2 and a circle acting on both with weights $m_0, \ldots, m_k$ and $n_0, \ldots, n_l$ and moment maps μ_1 and μ_2. Suppose we have ordered the weights so that $m_0 > m_1 > \cdots > m_k$ and $n_0 > n_1 > \cdots > n_l$. Then $N_1 = \{0, \ldots, k\}$ and $N_2 = \{0, \ldots, l\}$. Consider the diagonal circle action on M and assume 0 is a regular value of the moment map $\mu = \mu_1 + \mu_2$ on M so that $m_i + n_j \neq 0$ for all $0 \leq i \leq m_k$ and $0 \leq j \leq n_l$, since this is the value of μ on the fixed point with 1 in the i-th place in $\mathbb{C}P^k$ and in the j-th place in $\mathbb{C}P^l$ and 0 everywhere else. Note that in the notation of Theorem 2, $\theta_{1i} = m_i$ and $\theta_{2j} = n_j$. Following the explanation in the proof of Lemma 1 (the case $m = 2$ in the notation of that lemma) we see that there are integers q and $l_0, \ldots, l_q$ (specified by the weights of the actions on $\mathbb{C}P^k$ and $\mathbb{C}P^l$) such that $q \leq k$, $l \geq l_0 \geq l_1 \geq \cdots \geq l_k$, and for $0 \leq i \leq q$, $m_i + n_j > 0$ for $0 \leq j \leq l_i$.

By Theorem 2 and in the notation of that theorem, we obtain the following classes

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that generate $\bar{K}_+$:

$$(\tilde{x}_1 - m_0 t)(\tilde{x}_1 - m_1 t)\cdots(\tilde{x}_1 - m_q t),$$
$$(\tilde{x}_1 - m_0 t)(\tilde{x}_1 - m_1 t)\cdots(\tilde{x}_1 - m_{q-1} t)(\tilde{x}_2 - n_0 t)\cdots(\tilde{x}_2 - n_{l_q} t),$$
$$\vdots$$
$$(\tilde{x}_1 - m_0 t)(\tilde{x}_2 - n_0 t)\cdots(\tilde{x}_2 - n_{l_1} t),$$
$$(\tilde{x}_2 - n_0 t)\cdots(\tilde{x}_2 - n_{l_0} t).$$

Likewise we obtain classes of the above form which generate $\bar{K}_-$, with the only difference that now, we have them for $q' \leq i \leq k$, $m_i + n_j < 0$ and for $l'_i \leq j \leq l$, for some q' and $l'_{q'}, \dots, l'_l$ that are specified by the weights. Then $H^*(M_{\mathrm{red}}) \cong \mathbb{C}[t, \tilde{x}_1, \tilde{x}_2]/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by the two families of classes introduced in the example. ∎

Example 3 As the next example we would like to consider the case of the product of m spheres of radii $r_1, \dots, r_m$ and the diagonal circle action. The result for this case was first obtained by Hausmann and Knutson [HK]. They however had a different approach.

So $M = S^2_{r_1} \times \cdots \times S^2_{r_m}$ and x_j is the symplectic form of the j-th sphere. The group G is $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$ acting diagonally on M and $T = U(1)$ is its maximal torus acting by rotation around a fixed axis, say the z-axis on each sphere. The fixed point set of the circle action on M is then $\mathcal{F} = \{(i_1 r_1 \hat{k}, \dots, i_m r_m \hat{k}) \mid i_j = \pm 1, 1 \leq j \leq m\}$, where $\hat{k}$ is the unit vector in the z-axis direction. The moment map of the j-th sphere is $\mu_j \colon S^2_{r_j} \to \mathbb{R}$; $\mu_j(x_j, y_j, z_j) = z_j$.

We label the fixed point set in the following way: Let $\mathcal{A} = \{1, \dots, m\}$ and $J \subset \mathcal{A}$ an arbitrary subset and consider the following fixed point associated to J, $F_J = (i_1 r_1 \hat{k}, \dots, i_m r_m \hat{k})$, where $i_j = 1$ if $j \in J$ and $i_j = -1$ if $j \notin J$.

The restriction of each $\tilde{x}_j$ to F_J is given by

$$\tilde{x}_j|_{F_J} = \begin{cases} r_j t & \text{if } j \in J \\ -r_j t & \text{if } j \notin J, \end{cases}$$

and the value of the moment map μ at F_J is $\mu(F_J) = \sum_{j \in J} r_j - \sum_{j \notin J} r_j$. We assume 0 is a regular value of the moment map so that $\mu(F_J) \neq 0$ for all fixed points F_J.

Definition 3 The set $J \subset \mathcal{A}$ is called *long* if $\mu(F_J) > 0$, otherwise it is called *short*. The set of all long subsets of $\mathcal{A}$ is denoted by $\mathcal{L}$, and that of short subsets is denoted by $\mathcal{S}$.

Therefore $J \in \mathcal{A}$ is long if and only if $\sum_{j \in J} r_j > \sum_{j \notin J} r_j$.

For every subset J of $\mathcal{A}$ define $P_J = \prod_{j \in J} (\tilde{x}_j - r_j t)$ and $Q_J = \prod_{j \in J} (\tilde{x}_j + r_j t)$ in the equivariant cohomology ring of M. These are polynomials in the variables $\tilde{x}_j$.

Consider the following families of classes of polynomials in $H^*_{S^1}(M)$:

$$\begin{array}{llll} & \text{(i)} & (\tilde{x}_j - r_j t)(\tilde{x}_j + r_j t), & j \in \mathcal{A} \\ (3) & \text{(ii)} & P_J, & J \subset \mathcal{A} \text{ long} \\ & \text{(iii)} & Q_J, & J \subset \mathcal{A} \text{ long} \end{array}$$

Theorem 3 *Let $M = \prod_i S^2_{r_i}$ and $\bar{K}_+$ and $\bar{K}_-$ be the preimages under η in $\mathbb{C}[t, \tilde{x}_1, \dots, \tilde{x}_m]$ defined above. Then*

(a) *The families* (i) *and* (ii) *together form a set of generators of $\bar{K}_+$.*
(b) *The families* (i) *and* (iii) *together form a set of generators of $\bar{K}_-$.*

Corollary 1 *The cohomology ring of M_{red} can be written as*

$$H^*(M_{\text{red}}) \cong \mathbb{C}[t, \tilde{x}_1, \dots, \tilde{x}_m]/\mathcal{I},$$

where $\mathcal{I}$ is the ideal generated by the families (i), (ii) *and* (iii) *in* (2).

Proof of Theorem 3 Let $N_i = \{1, 2\}$, $\theta_{i1} = -r_i$, $\theta_{i2} = r_i$ and let the long/short subsets defined in Definition 3 correspond to the long/short elements defined in Definition 1. Fix $1 \le j \le m$ and define $\mathcal{A}_j = N_j = \{1, 2\}$ and $\mathcal{A}_i = \varnothing$ if $i \ne j$. Then $\mathcal{A} = N_1 \times \cdots \times N_m \subset P_j^{-1}(\mathcal{A}_j)$, hence $\{\mathcal{A}_i\}_{1 \le i \le m}$ is a covering of both $\mathcal{L}$ and $\mathcal{S}$, clearly a minimal one in the notation of Theorem 2. The classes (1) and (2) in Theorem 2 corresponding to this minimal covering are both $(\tilde{x}_j - r_j t)(\tilde{x}_j + r_j t)$ which is (i) in the collection (3).

Next suppose L is a long element of $\mathcal{A}$, define $\mathcal{A}_i = \{2\}$ if $P_i(L) = 2$ and $\varnothing$ otherwise. To proceed, we need to show that any two long subsets have nonempty intersection. In fact if J and L are long and $J \cap L = \varnothing$, then $\sum_{j \in J} r_j > \sum_{j \notin J} r_j$ and $\sum_{j \in L} r_j > \sum_{j \notin L} r_j$ and therefore

$$\sum_{j \in L} r_j > \sum_{j \notin L} r_j = \sum_{j \in J} r_j + \sum_{j \notin J \cup L} r_j \ge \sum_{j \in J} r_j > \sum_{j \notin J} r_j = \sum_{j \in L} r_j + \sum_{j \notin J \cup L} r_j > \sum_{j \in L} r_j,$$

which is a contradiction. Consequently $\{\mathcal{A}_i\}_{1 \le i \le m}$ is a covering of $\mathcal{L}$. If J is another long element, there is some i such that $P_i(J) = P_i(L) = \{2\}$, hence $J \in P_i^{-1}(\mathcal{A}_i)$. The corresponding class in (1) is then the class P_L in the collection (3) where here L denotes the long subset corresponding to the long element being considered. If S is a short element, using its long counterpart L (in terms of the subsets $L = \mathcal{A} - S$), we obtain the class Q_L in (3)(iii). ∎

The elements of the family (iii) in (3) look different from those of the third family introduced in Theorem 6.4 in [HK]. They are the same when the coefficient ring is $\mathbb{C}$. To see this start from the families (3) in Theorem 3 and write $u_j = \tilde{x}_j / r_j$. The families (i) and (ii) can be written in terms of u_j:

$$(\tilde{x}_j + r_j t)(\tilde{x}_j - r_j t) = r_j^2 (u_j + t)(u_j - t)$$

and

$$P_L = \prod_{j\in L}(\tilde{x}_j - r_j t) = \Big(\prod_{j\in L} r_j\Big) \prod_{j\in L}(\tilde{x}_j/r_j - t) = \lambda_L \prod_{j\in L}(u_j - t)$$

where $\lambda_L = (\prod_{j\in L} r_j)$. Every Q_L for $L \in \mathcal{L}$ can be rewritten as

$$Q_L = \prod_{j\in L}(\tilde{x}_j + r_j t) = \lambda_L \prod_{j\in L}(u_j + t) = \lambda_L \prod_{j\in L}(u_j - t + 2t) = \lambda_L \sum_{J\subset L} \prod_{j\in J}(u_j - t)(2t)^{|L-J|}.$$

But the long subsets of L have already been included in the second family (ii), hence we can drop the terms corresponding to $J \subset L$, $J \in \mathcal{L}$ in the last expression to obtain the classes $\lambda_L \sum_{S\subset L, S\in \mathcal{S}} \prod_{j\in S}(u_j - t)(2t)^{|L-S|}$. After dropping the scalar multiples r_j^2 and λ_L the new families still generate $\bar{K}_+$ and $\bar{K}_-$.

The families introduced in Theorem 3 still do not perfectly match with those in Theorem 6.4 in [HK] which in fact are finer than ours.

We need to extend our notation: let $\mathcal{A}_m = \{1, \dots, m\}$. Consider $r = r_m$ in Theorem 1. Define $\mathcal{L}(r_m) = \{L \subset \mathcal{A}_{m-1} : \sum_L r_j - \sum_{\mathcal{A}_{m-1}-L} r_j > r_m\}$, and $\mathcal{S}(r_m) = \{S \subset \mathcal{A}_{m-1} : \sum_S r_j - \sum_{\mathcal{A}_{m-1}-S} r_j < r_m\}$. Consider the following families:

(i)′ $(u_j - t)(u_j + t)$, $j \in \mathcal{A}_{m-1}$
(ii)′ $P'_J = \prod_{j\in L}(u_j - t)$, $J \in \mathcal{L}(r_m)$
(iii)′ $Q'_J = \sum_{S\subset L, S\in\mathcal{S}} \prod_{j\in S}(u_j - t)(2t)^{|L-S|}$, $J \in \mathcal{L}(r_m)$,

where $u_j = \tilde{x}_j/r_j$ and $\lambda_L = \prod_{j\in L} r_j$ as in Remark (1). Then r_m is a regular value of the moment map for the abelian polygon space which is defined as

$$M = \prod_{i=1}^{m-1} S^2_{r_i} /\!\!/_{r_m} \mathrm{SO}_2 .$$

Corollary 2 *Consider $\bar{K}_+(r_m)$ and $\bar{K}_-(r_m)$ for the abelian polygon space. Then*

(a) *The families* (i)′ *and* (ii)′ *together form a set of generators of $\bar{K}_+(r_m)$.*
(b) *The families* (i)′ *and* (iii)′ *together form a set of generators of $\bar{K}_-(r_m)$.*

The proof is the same as that of Theorem 3 adding the comments that we gave after the proof of that theorem. ■

The following corollary is Theorem 6.4 in [HK] when the coefficient ring is $\mathbb{C}$:

Corollary 3 *For the abelian polygon space, $\bar{K}(r_m)$ is generated by the families* (i)″, (ii)″ *and* (iii)″ *which are defined as follows:*

(i)″ $(u_j - t)(u_j + t)$ $j \in \mathcal{A}_{m-1}$
(ii)″ $P''_L = \prod_{j\in L}(u_j - t)$ $L \in \mathcal{L}_m$
(iii)″ $Q''_L = \sum_{S\subset L, S\in\mathcal{S}_m} \prod_{j\in S}(u_j - t)(2t)^{|L-S|}$ $L \in \mathcal{P}(\mathcal{A}_{m-1}) \cap \mathcal{L}$.

Here $\mathcal{P}(\mathcal{A}_{m-1})$ is the set of all subsets of $\mathcal{A}_{m-1}$, and $\mathcal{L}_m$ and $\mathcal{S}_m$ are defined as $\mathcal{L}_m = \{L \subset \mathcal{A}_{m-1} : L \cup \{m\} \in \mathcal{L}\}$, and $\mathcal{S}_m = \{S \subset \mathcal{A}_{m-1} : S \cup \{m\} \in \mathcal{S}\}$.

Proof The argument used after the proof of Theorem 3 applies here too to show that the classes Q_J in (iii)$'$ can be replaced by (iii)$'''$ $\sum_{S\subset L, S\in \mathcal{S}(r_m)} \prod_{j\in S}(u_j - t)(2t)^{|L-S|}$ for $J \in \mathcal{L}(r_m)$, which together with (i)$'$ and (ii)$'$ still generate $\bar{K}(r_m)$.

Then, notice that $\mathcal{L}(r_m) \subset \mathcal{L} \cap \mathcal{P}(\mathcal{A}_{m-1}) \subset \mathcal{L}_m$, hence the families (ii)$''$ and (iii)$''$ are larger than the families (ii)$'$ and (iii)$'''$ respectively. Furthermore this allows us to remove some of the terms in the elements in (iii)$'''$ to obtain the elements in (iii)$''$. In fact consider a term in a class in (iii)$'''$ corresponding to some $S \in \mathcal{S}(r_m) - \mathcal{S}_m$. This means $S \cup \{m\} \in \mathcal{L}$, hence $S \in \mathcal{L}_m$. Thus the term corresponding to this S has already been considered in (ii)$''$. This establishes that the new families suffice to generate $\bar{K}(r_m)$. ■

Remark The classes V_j and R in Theorem 6.4 in [HK] correspond to our classes $u_j - t$ and $2t$ respectively.

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