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TWO AMALGAMS RELATED TO THE ALTERNATING GROUP ON SIX LETTERS

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A characterisation is given of some of the parabolics found in Co_3, and $SP_4(9)$ using the amalgam method.

1. INTRODUCTION

Leg G be a finite group, p a prime, $S \in \mathrm{Syl}_p(G)$ and $B = N_G(S)$. A proper subgroup of G which contains a conjugate of B is called a *parabolic subgroup* of G. The set $\mathfrak{I}$ of parabolic subgroups of G ordered by inclusion becomes a partially ordered set called *the parabolic geometry* of G. In recent years the parabolic geometry (in particular for $p = 2$) has been used to study, construct, characterise and prove uniqueness of many of the sporadic finite simple groups. The parabolic geometries (again for $p = 2$) also play an important role in the ongoing revision of the classification of the finite simple groups, in particular in the so called quasi-thin and uniqueness cases.

Parabolic subgroups have been studied most intensively for $p = 2$ but many interesting examples exist (besides the groups of Lie type) for arbitrary primes.

Recall that G is an *amalgamated product of* P_1 *and* P_2 if (G, P_1, P_2) has the following properties:

(i) P_1 and P_2 are finite subgroups of G.
(ii) $G = \langle P_1, P_2 \rangle$.
(iii) Let $S \in \mathrm{Syl}_p(P_1 \cap P_2)$ and $B = N_{P_1 \cap P_2}(S)$; then $B = N_{P_i}(S)$, $i = 1, 2$. In particular $S \in \mathrm{Syl}_p(P_i)$, $i = 1, 2$.
(iv) No nontrivial normal subgroup of G is contained in B.

To any amalgamated product (G, P_1, P_2) we can associate a graph Γ whose vertices are the cosets of P_1 and P_2 in G and two cosets are adjacent if they are distinct and have non-empty intersection We remark that if $B = N_G(S)$ then the graph Γ can be embedded into the parabolic geometry of G.

The amalgamation method has proven very successful in determining the structure of P_1 and P_2 assuming the action of P_1 and P_2 on their neighbours $\Delta(P_1)$ and $\Delta(P_2)$ respectively in the graph Γ is given.

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Let us assume for simplicity that $P_1 \cap P_2 = B$. Let $Q_i = O_p(P_i)$, $L_i = O^p(P_i) = \langle S^{P_i} \rangle$ and $P_i^{(l)}/Q_i = C_{G_i/Q_i}(L_i/Q_i)$. Then it is easy to see that $P_i^{(l)}$ is precisely the kernel of the action of P_i on $\Delta(P_i)$ and L_i acts transitively on $\Delta(P_i)$. Hence the group L_i/Q_i carries most of the information about the action of P_i and $\Delta(P_i)$ and we refer to the pair $(L_1/Q_1,\ L_2/Q_2)$ as the type of the amalgamated product $(G,\ P_1,\ P_2)$.

The main task of the amalgam method can now be described as determining $(P_1,\ P_2)$ from the type $(L_1/Q_1,\ L_2/Q_2)$. For example, in [**9**] we determined the structure of $(P_1,\ P_2)$ of type $(\Theta, \mathrm{SL}_2(3))$, where $\Theta \cong \mathrm{PSL}_2(9)$, M_{11}, M_{12} or $2 \cdot M_{12}$, for $p = 3$.

For the remainder of this paper we shall work under the following hypothesis:
(*) $(G,\ P_1,\ P_2)$ is an amalgamated product of type (Θ, Ψ) for $p = 3$ so that:

(i) $\Theta \cong \mathrm{PSL}_2(9)$, M_{11}, M_{12} or $2 \cdot M_{12}$,
(ii) $\Psi \cong \mathrm{PSL}_2(9)$ or $\mathrm{SL}_2(9)$,
(iii) $C_{P_i}(O_3(P_i)) \leqslant O_3(P_i)$ for $i = 1,\ 2$.

Introduce now the following notation: $G \sim 3^{d_1 + \dots + d_n} H$ means that there exists a normal series

$$1 = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_n \leqslant G,$$

so that for $i = 1,\ 2,\ \ldots,\ n$, H_i/H_{i-1} is an elementary Abelian minimal normal subgroup of G/H_{i-1} with $|H_i/H_{i-1}| = 3^{d_i}$ and $G/H_n \cong H$.

Also, by $G \sim 2 \cdot H$ we mean that $G/Z(G) \cong H$, $|Z(G)| = 2$ and $Z(H) \leqslant H'$.

We are now able to state our main result.

Theorem. *Under hypothesis (*) the possible pairs* $(L_1,\ L_2)$ *are:*

(i) $(3^5 M_{11},\ 3^{1+4}\, \mathrm{SL}_2(9))$,
(ii) $(3^6\, \mathrm{PSL}_2(9),\ 3^{1+1+4}\, \mathrm{SL}_2(9))$.

Note that the above examples can be found in $G \cong \mathrm{Co}_3$ *and* $\mathrm{PSp}_4(9)$ *respectively.*

2. Properties of Θ, Ψ and their modules – the graph Γ

A *Steiner system* $S(l,\ m,\ n)$ is a pair $(\Omega,\ \mathcal{B})$, where Ω is a set of size n, $\mathcal{B}$ is a set of subsets of size m called blocks and such that every subset of size l in Ω lies in a unique member of $\mathcal{B}$.

By [**10**], there exists, up to isomorphism, a unique Steiner system of type $S(5,\ 6,\ 12)$. Let $\mathcal{S} = S(5,\ 6,\ 12)$. Define then the *Mathieu group on 12 points* to be the group $M_{12} = \mathrm{Aut}(\mathcal{S}) = \{\pi \in \mathrm{Sym}(12) \mid B^\pi \text{ is a block for all blocks } B\}$.

Define M_{11} to be the stabiliser of a point in M_{12}. Then M_{11} is 4-transitive on eleven points and its corresponding Steiner system is $S(4,\ 5,\ 11)$.

LEMMA 2.1.

(a) $|M_{12}| = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 2^6 \cdot 3^3 \cdot 5 \cdot 11$.

(b) *M_{12} has two classes of involutions, say D_1 and D_2. Moreover if $D_1 = \{x \mid x$ fixes a point$\}$, then $x \in D_1$,*

if and only if x fixes a point

if and only if x fixes four points

if and only if x belongs to a normaliser of a Sylow 3-subgroups of M_{12}

if and only if x lifts to an involution in $2 \cdot M_{12}$.

PROOF: See [1] and [6]. □

NOTATION 2.2. To avoid repetitions we shall use the following notation throughout:

$$X \cong (2)H \text{ means that either } X \cong 2 \cdot H \text{ or } X \cong H.$$

DEFINTION 2.3: Let X be a finite group. Slightly abusing the standard defintion we shall say that X is 3-stable provided that the following condition holds: If V is an irreducible $GF(3)X$-module and $A \leqslant X$ is such that $[V, A, A] = 1$ then $[V, A] = 1$.

LEMMA 2.4. *Let Y be a finite group. Then:*

(a) *The following statement is equivalent to Y being 3-stable: let V be any $GF(3)$-module and $A \leqslant Y$ with $[V, A, A] = 1$. Then $AC_Y(V)/C_Y(V) \leqslant O_3(Y/C_Y(V))$.*

(b) *Y is 3-stable if and only if $Y/O_3(Y)$ is 3-stable.*

(c) *If every element of order 3 in Y lies in a perfect simple 3-stable subgroup of Y then Y is 3-stable.*

PROOF: See [9].

DEFINITION 2.5: A $\mathrm{GF}(3)X$-module V is called an FF-module for X if $C_X(V) = 1$ and if there exists a non-identity 3-subgroup A of X such that $|V| / |C_V(A)| \leqslant |A|$.

LEMMA 2.6. *Θ is 3-stable; in particular Θ does not have an FF-module.*

PROOF: The proof that Θ is 3-stable can be found in [9]; Thompson's Replacement Theorem [5, 8.2.5] implies that a group with an irreducible FF-module is not 3-stable and hence Θ does not have an FF-module. □

DEFINITION 2.7: Let $X \cong \mathrm{SL}_2(9)$ and let W be a faithful $GF(3)X$-module. Then W is called a natural $\mathrm{SL}_2(9)$-module for X if W carries the structure of a 2-dimensional vector space over $GF(9)$ invariant under the action of X.

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It is worth mentioning at this point that

$$A_6 \cong \mathrm{PSL}_2(9) \text{ and } 2 \cdot A_6 \cong \mathrm{SL}_2(9).$$

REMARK 2.8.

(i) $\mathrm{PSL}_2(9)$ has four irreducible $GF(3)$-modules; their dimensions are: 1, 4, 6 and 9.

(ii) Let $X = \mathrm{SL}_2(9)$ and let V be an FF-module. Then

$$V = [V, Z(X)] \oplus C_V(X)$$

and $[V, Z(X)]$ is a natural $\mathrm{SL}_2(9)$-module. [**8**, p.469 and 470].

(iii) M_{11} has two non-trivial irreducible $GF(3)$-modules of dimension less than or equal to 8; moreover, both have dimension five and they are dual to each other [**7**].

DEFINITION 2.9: Let $\Gamma = \{P_i x \mid x \in G,\ i = 1, 2\}$. From now on, small Greek letters will always denote elements of Γ. Make Γ into a graph by defining α to be adjacent to β if and only if $\alpha \neq \beta$ and $\alpha \cap \beta \neq \emptyset$. Then G operates on Γ by right multiplication.

For $\delta \in \Gamma$, let $G_\delta = \mathrm{Stab}_G(\delta)$, let $G_\delta^{(n)}$ equal the largest normal sugroup of G_δ fixing all vertices of distance at most n from δ and let $\Delta(\delta)$ be the set of all vertices adjacent to δ.

LEMMA 2.10. *Let X be any of our groups Θ or Ψ, $S_1 \in \mathrm{Syl}_3(X)$ and $B_1 = N_X(S_1)$. Then B_1 is irreducible on $Z(S_1)$; in particular B_1 is irreducible on S_1 for $X \cong (P)\mathrm{SL}_2(9)$ or M_{11} and $S = Q_\alpha Q_\beta$.*

PROOF: See [**9**]. ▯

LEMMA 2.11. *The normaliser of a Sylow 3-subgroup is maximal in* $\mathrm{SL}_2(9)$.

PROOF: See [**3**, 8.3.2 and 11.3.2]. ▯

LEMMA 2.12. *Let $i = 1, 2$. Then:*

(a) $G_{P_i x} = P_i^x$,

(b) *The edge-stabilisers in G are conjugate to B,*

(c) *Let $\delta_i = P_i$. Then $\Delta(\delta_i) \cong P_i/B$ as a G_{δ_i}-set; in particular, G_{δ_i} is transitive on $\Delta(\delta_i)$,*

(d) *Let (δ, λ) be an edge of λ; then $G = \langle G_\delta, G_\lambda\rangle$,*

(e) *G acts faithfully on Γ,*

(f) *Γ is connected.*

PROOF: See [**4**, 2.1, 2.2 and 3.1] ▯

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NOTATION 2.13. Let $d(,)$ denote the usual distance on the graph Γ. For $\delta \in \Gamma$ and $i \geqslant 1$,

$$\Delta^{(i)}(\delta) = \{\lambda \in \Gamma \mid d(\delta, \lambda) \leqslant i\},$$
$$Q_\delta = O_3(G_\delta),$$
$$Z_\delta = \langle \Omega_1 Z(T) \mid T \in \mathrm{Syl}_3(G_\delta)\rangle,$$
$$V_\delta = \langle Z_\lambda \mid \lambda \in \Delta(\delta)\rangle,$$
$$b_\delta = \min_{\delta' \in \Gamma}\{d(\delta, \delta') \mid Z_\delta \not\leqslant G_{\delta'}^{(l)}\},$$
$$b = \min_{\delta' \in \Gamma}\{b_{\delta'}\},$$
$$G_{\delta\lambda} = G_\delta \cap G_\lambda \text{ and } Q_{\delta\lambda} = Q_\delta \cap Q_\lambda \text{ if } \delta \in \Delta(\lambda).$$

A pair of vertices (δ, δ') such that $Z_\delta \not\leqslant G_{\delta'}^{(l)}$ and $d(\delta, \delta') = b$ is called a critical pair.

The bounding of the parameter b which we just introduced, will allow us to deduce a considerable amount of information about P_1 and P_2.

LEMMA 2.14.

(a) *G acts edge- but not vertex-transitively on Γ,*
(b) *G_δ is finite,*
(c) *$C_{G_\delta}(Q_\delta) \subseteq Q_\delta$,*
(d) *if α is adjacent to β they $\mathrm{Syl}_3(G_\alpha \cap G_\beta) \subseteq \mathrm{Syl}_3(G_\alpha) \cap \mathrm{Syl}_3(G_\beta)$.*

PROOF: See [**4**, p.73]. ∎

REMARK 2.15. Notice that as G acts edge-transitively, $b = \min\{b_\alpha, b_\beta\}$ for any pair of adjacent vertices α, β. Thus, we are allowed to choose α, β such that $b_\alpha = b \leqslant b_\beta$ and $\{G_\alpha, G_\beta\} = \{P_1, P_2\}$. In particular, $G_\alpha \cap G_\beta = B$ and $S \in \mathrm{Syl}_3(G_\alpha) \cap \mathrm{Syl}_3(G_\beta)$.

Let $\alpha \in \Gamma$ be such that $d(\alpha, \alpha) = b$ and $Z_\alpha \not\leqslant G_{\alpha'}^{(l)}$. Let p be a path of length b from α to α. We label the vertices of p by

$$p = (\alpha, \alpha+1, \dots, \alpha+b) = (\alpha' - b, \dots, \alpha' - 1, \alpha'),$$

that is, $\alpha + i$ (respectively $\alpha - i$) is the unique vertex in p with $d(\alpha, \alpha+i) = i$ (respectively $d(\alpha - i) = i$). furthermore, from 2.12 (c) we may assume that

$$\beta = \alpha + 1 \text{ if } b \geqslant 1.$$

Note also that if $Q_\delta = Q_\lambda$ for some $\delta \in \Delta(\lambda)$ then $Q_\delta \trianglelefteq \langle G_\delta, G_\lambda\rangle = G$, a contradiction. Hence

$$Q_\delta \neq Q_\lambda \ \forall \delta \in \Delta(\lambda).$$

LEMMA 2.16. *Let (δ, λ) be an edge and let N be a subgroup of $G_{\delta,\lambda}$ such that $N_{G_\mu}(N)$ acts transitively on $\Delta(\mu)$ for $\mu \in \{\delta, \lambda\}$. Then $N = 1$.*

PROOF: See [**4**, (3.2)]. ∎

Lemma 2.17. *For $\delta \in \Gamma$,*

(a) $Q_\delta \leqslant G_\delta^{(1)}$,
(b) $Z_\delta \leqslant Z(Q_\delta) \cap V_\delta$; *in particular,* $b \geqslant 1$ *and* $Z_\alpha \nleqslant Q_{\alpha'}$,
(c) $Z_{\alpha'} \leqslant G_\alpha$ *and* $[Z_\alpha, Z_{\alpha'}] \leqslant Z_\alpha \cap Z_{\alpha'}$
(d) $Z_\alpha \neq \Omega_1 Z(T)$, $T \in \mathrm{Syl}_3(G_\alpha)$,
(e) *If* $S \in \mathrm{Syl}_3(B)$ *and* $\Omega_1(Z(S))$ *is centralised by a sugbroup* R *of* G_β *which acts trasitively on* $\Delta(\beta)$ *then* $Z(L_\alpha) = 1$.

Proof: See [**9**]. ∎

Remark 2.18.

(i) $Z_\delta \leqslant G_\gamma$ $\forall \gamma \in \Delta^{(1)}(\delta)$, $B = G_{\alpha\beta}$, $Z_\alpha \leqslant B$, $Z_\beta \leqslant B$.
(ii) Also $\mathrm{Syl}_3(B) \subseteq \mathrm{Syl}_3(P_1) \cap \mathrm{Syl}_3(P_2)$.
(iii) A Frattini argument gives that $L_\delta S = L_\delta$ and for $\mu \in \Delta(\delta)$, $G_\delta = L_\delta G_{\delta\mu}$.

A list of properties follows, the proofs of which can be found in [**9**].

Lemma 2.19.

(i) $[Z_\alpha, Z_{\alpha'}, Z_{\alpha'}] = 1$.
(ii) $V_\delta \trianglelefteq G_\delta$ $\forall \delta \in \Gamma$.
(iii) Z_α *normalises* V_α.
(iv) *If* $\beta > 2$ *then* V_β *is Abelian.*
(v) *If* $Z_\delta \leqslant Z(L_\delta)$ *then* $Z_\delta \leqslant Z_\lambda$ $\forall \lambda \in \Delta(\delta)$.
(vi) $Z_\alpha \nleqslant Z(L_\alpha)$.
(vii) *If* $Z_{\alpha'} \leqslant Z(L_{\alpha'})$ *then* α *is not conjugate to* α'.
(viii) $Z_\alpha \cap Q_{\alpha'} \neq C_{Z_\alpha}(Z_{\alpha'})$ *if and only if* $Z_{\alpha'} \leqslant Z(L'_\alpha)$.
(ix) *Let* $\delta \in \{\alpha, \beta\}$ *and* A *be a 3-subgroup of* G_δ *with* $A \nleqslant Q_\delta$. *Then*

$$O^3(L_\delta) \leqslant \langle A^{L_\delta} \rangle \text{ and } L_\delta = \langle A^{L_\delta} \rangle Q_\delta.$$

Remark 2.20.

(a) By 2.19 (vi), $Z_\alpha \nleqslant Z(L_\alpha)$ and so by 2.19 (ix) $C_{G_\alpha}(Z_\alpha)/Q_\alpha$ is a 3-group.
(b) If $Z_{\alpha'} \leqslant Q_\alpha$ then $[Z_\alpha, Z_{\alpha'}] = 1$ and so $Z_\alpha = C_{Z_\alpha}(Z_{\alpha'})$. Hence $Z_\alpha \cap Q_{\alpha'} \neq z_\alpha$ and $Z_\alpha \cap Q_{\alpha'} \neq C_{Z_\alpha}(Z_{\alpha'})$.
(c) If $Z_{\alpha'} \nleqslant Q_\alpha$ then by (a) $C_{Z_\alpha}(Z_{\alpha'}) = Z_\alpha \cap Q_{\alpha'}$ and since we have a complete symmetry between α and α' in this case, we get that

$$C_{Z'_\alpha}(Z_\alpha) = Z_{\alpha'} \cap Q_\alpha.$$

DEFINITION 2.21:

(a) $\overline{L_\delta} = L_\delta/O_3(L_\delta)$.

(b) Let K be a complement for S in B and

$$K_\alpha = K \cap L_\alpha \text{ and } K_\beta = K \cap L_\beta.$$

(c) Let $\delta \in \{a, \beta\}$. Let t_δ be an element of order 2 in K_δ with $t_\delta Q_\delta/Q_\delta \in Z(L_\delta/Q_\delta)$ if L_δ/Q_δ is isomorphic to one of the groups $\mathrm{SL}_2(9)$, $2 \cdot M_{12}$; otherwise let $t_\delta = 1$.

The following corollary will be useful for the proof of 3.6.

COROLLARY 2.22. *If $\delta \in \Delta(\lambda)$, $t_\delta \neq 1$ and $L_\lambda/Q_\lambda \cong (P)\mathrm{SL}_2(9)$, M_{11} or $(2)M_{12}$ then t_δ does not centralise S/Q_λ.*

PROOF: See [**9**].

3. THE CASE $[Z_\alpha Z_{\alpha'}] \neq 1$

In this section we work under the hypothesis $Z_{\alpha'} \not\leqslant Q_\alpha$. Notice that under this hypothesis, we have a complete symmetry between α and α', so $Z_{\alpha'} \not\leqslant Z(L_{\alpha'})$.

PROPOSITION 3.0. *The hypothesis of this section leads to a contradiction.*

LEMMA 3.1.

(a) *$Z_\alpha \cap Q_{\alpha'} = C_{Z_\alpha}(Z_{\alpha'})$; in particular b is even,*

(b) *$Z_{\alpha'} \cap Q_\alpha = C_{Z_{\alpha'}}(Z_\alpha)$,*

PROOF: See [**9**].

DEFINITION 3.2: $\varepsilon = 1$ if $Z_\beta \neq \Omega_1 Z(S)$ and $\varepsilon = 2$ if $Z_\beta = \Omega_1 Z(S)$.

LEMMA 3.3.

(a) *$L_\alpha/Q_\alpha \cong L_{\alpha'}/Q_{\alpha'} \cong \mathrm{SL}_2(9)$ and Z_α is an FF-module for L_α/Q_α.*

(b) *$Z_\alpha = [Z_\alpha, L_\alpha] \oplus \Omega_1 Z(L_\alpha)$ and $[Z_\alpha, L_\alpha]$ is the unique natural $\mathrm{SL}_2(9)$-module for L_α/Q_α.*

PROOF: (a) Since $[Z_\alpha, Z_{\alpha'}, Z_{\alpha'}] = 1$ and $[Z_\alpha, Z_{\alpha'}] \neq 1$ and as $Z_\alpha \not\leqslant Q_{\alpha'}$ we get that $L_{\alpha'}/Q_{\alpha'}$ cannot be 3-stable. Similarly L_α/Q_α is not 3-stable. Hence

$$L_\alpha/Q_\alpha \cong L_{\alpha'}/Q_{\alpha'} \cong \mathrm{SL}_2(9).$$

Without loss of generality we may assume that

$$|Z_\alpha Q_{\alpha'}/Q_{\alpha'}| \leqslant |Z_{\alpha'} Q_\alpha/Q_\alpha|.$$

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Let $V = Z_\alpha$ and $A = Z_{\alpha'}Q_\alpha/Q_\alpha$. Then

$$|V/C_V(A)| = |Z_\alpha/X_{Z_\alpha}(Z_{\alpha'})| = |Z_\alpha/Z_\alpha \cap Q_{\alpha'}| = |Z_\alpha Q_{\alpha'}/Q_{\alpha'}| \leqslant |Z_{\alpha'}Q_\alpha/Q_\alpha| = |A|.$$

Therefore Z_α is an FF-module for L_α/Q_α.

(b) Follows from 2.8. ∎

By 3.3, L_α fixes some symplectic form on Z_α with $\Omega_1 Z(L_\alpha)$ in its radical. In what follows "$\perp$" and "singular" is meant with respect to that form on Z_α (or also on $Z_{\alpha'}$).

LEMMA 3.4. *Let $X \leqslant G_{\alpha'}$. Then $C_{Z'_\alpha}(X)^\perp = [Z_{\alpha'}, X] + \Omega_1 Z(L_{\alpha'})$.*

PROOF: See [**2**, 22.1]. ∎

DEFINITION 3.5: Let $\Lambda(\alpha, \alpha') = \Lambda = \Delta(\alpha)\backslash\{\beta\}$. It is clear that $\Lambda \neq \emptyset$.

LEMMA 3.6. $\varepsilon = 2$. *In particular Z_α is a natural* $\mathrm{SL}_2(9)$*-module and $Z_\beta \leqslant Z_\alpha$.*

PROOF: Suppose $\varepsilon = 1$. Let $\alpha - 1 \in \Lambda$. ∎

If $Z_{\alpha-1} \not\leqslant Q_{\alpha'-1}$ then $(\alpha - 1, \alpha' - 1)$ has the same properties as (α, α'), which can't happen as the vertices alternate in terms of 3-stability. Hence

$$Z_{\alpha-1} \leqslant Q_{\alpha'-1} \leqslant G^{(1)}_{\alpha'-1} \leqslant G_{\alpha'}$$

and

$$[Z_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha, Z_{\alpha'} \cap Q_\alpha] \leqslant [G_{\alpha'}, Z_{\alpha'}, Z_{\alpha'}] = 1.$$

Now, 3-stability of $G_{\alpha-1}$ implies $[Z_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha] = 1$ which gives

$$C_{Z_{\alpha'}}(Z_\alpha) = Z_{\alpha'} \cap Q_\alpha \leqslant C_{Z_{\alpha'}}(Z_{\alpha-1}).$$

Hence

$$C_{Z_{\alpha'}}(Z_{\alpha-1})^\perp \leqslant C_{Z_{\alpha'}}(Z_\alpha)^\perp$$

and by 3.4,

$$[Z_{\alpha'}, Z_{\alpha-1} \leqslant [Z_{\alpha'}, Z_\alpha].$$

Since $Z_{\alpha-1}Z_\alpha$ is normalised by $Z_{\alpha'}$ and by $G_{\alpha-1} \cap G_\alpha$ we get by choice of $\alpha - 1$ that $Z_{\alpha-1}Z_\alpha \trianglelefteq G_\alpha$ and therefore

$$C_{G_\alpha}(Z_{\alpha-1}Z_\alpha) \trianglelefteq G_\alpha.$$

By [**9**] now, $O^3(C_{G_\alpha}(Z_{\alpha-1}Z_\alpha)) = Q_\alpha \cap Q_{\alpha-1}$ and so we conclude that

$$Q_{\alpha-1} \cap Q_\alpha \trianglelefteq G_\alpha \text{ and } Q_\beta \cap Q_\alpha \trianglelefteq G_\alpha.$$

Let $L = \langle Q_\beta^{G_\alpha} \rangle$. As $[Q_\beta, Q_\alpha] \leqslant Q_\beta \cap Q_\alpha \trianglelefteq G_\alpha$, $[L, Q_\alpha] \leqslant Q_\alpha \cap Q_\beta \leqslant Q_\beta$. Recall the definition of t_α (see 2.21) now. Since $Q_\beta \nleqslant Q_\alpha$ 2.19 (ix) implies that $t_\alpha \in O^3(L_\alpha) \leqslant L$. Hence

$$[t_\alpha, Q_\alpha] \leqslant Q_\alpha \cap Q_\beta \leqslant \langle t_\alpha \rangle (Q_\alpha \cap Q_\beta) \trianglelefteq \langle t_\alpha \rangle Q_\alpha$$

and

$$O^2\big(\langle t_\alpha \rangle (Q_\alpha \cap Q_\beta)\big) \leqslant \langle t_\alpha \rangle (Q_\alpha \cap Q_\beta).$$

Thus

$$[t_\alpha^S] \leqslant \langle t_\alpha S \rangle \cap Q_\alpha \leqslant O^2(\langle t_\alpha \rangle Q_\alpha) \cap Q_\alpha \leqslant \big(\langle t_\alpha \rangle Q_\alpha \cap Q_\beta\big) \cap Q_\alpha \leqslant Q_\alpha \cap Q_\beta \leqslant Q_\beta.$$

Hence t_α centralises S/Q_β, a contradiction by 2.22. Thus $\varepsilon = 2$. So $Z_\beta = \Omega_1 Z(L_\beta) = \Omega_1 Z(S)$ and by 2.17(e), $\Omega_1 Z(L_\alpha) = 1$. The last statement of the lemma follows from 3.3(b). ∎

Lemma 3.7. *$Z_\beta = C_{Z_\alpha}(Z_{\alpha'}) = [Z_\alpha, Z_{\alpha'}] + \Omega_1 Z(L_\alpha) = [Z_\alpha, Q_\beta] + \Omega_1 Z(L_\alpha) = C_{Z_\alpha}(Q_\beta) = C_{Z_\alpha}(S)$.*

Proof: As $\varepsilon = 2$, $[Z_\alpha, L_\alpha]$ is 2-dimensional over GF(9). Hence $[Z_\alpha, L_\alpha]$, $C_{Z_\alpha}(Z_{\alpha'})$, $[Z_\alpha, Z_{\alpha'}]$, $[Z_\alpha, Q_\beta]$ and $C_{[Z_\alpha, L_\alpha]}(Q_\beta)$ are all 1-dimensional over GF(9). Moreover $[Z_\beta, Q_\beta] = 1 = [Z_\beta, Z_{\alpha'}]$ and the lemma follows. ∎

Lemma 3.8. *Let $\alpha - 1 \in \Lambda$. Then $\langle G_{\alpha-1,\alpha}, Z_{\alpha'} \rangle G_\alpha$.*

Proof: Lemma 3.7 implies $[Z_{\alpha-1}, Z_{\alpha'}] \neq 1$ and so $Z_{\alpha'} \nleqslant G_{\alpha-1,\alpha}$. By 2.10, $G_{\alpha-1,\alpha}$ is maximal in G_α and so $\langle G_{\alpha-1,\alpha}, Z_{\alpha'} \rangle = G_\alpha$. ∎

Remark 3.9. The following are equivalent:

(i) $Z_{\alpha-1}, \nleq [Z_\alpha, Z_{\alpha'}]$;
(ii) $C_{Z_{\alpha-1}}(Z_{\alpha'}) = 1$;

Define now $Y_\beta^\bullet$ and Y_β by

$$Y_\beta^\bullet / Z_\beta = \langle C_{Z_\delta / Z_\beta}(Q_\beta) \mid \delta \in \Delta(\beta) \rangle$$

and

$$Y_\beta = C_{Z_\alpha}\big(O^3(L_\beta)\big).$$

Note that $[Y_\beta^\bullet, Q_\beta] \leqslant Z_\beta$.

Lemma 3.10. *$b = 2$.*

Proof: Suppose $b > 2$. Then, by 3.7 $C_{Z_\alpha / Z_\beta}(Q_\beta / Z_\beta) = Z_\alpha$. As by **[9]** $Y_\beta^\bullet \leqslant Z_\alpha$ for $b > 2$ we get

$$Z_\alpha \leqslant Y_\beta^\bullet \leqslant Z_\alpha$$

whence

$$Z_\alpha = Y_\beta^\bullet \trianglelefteq \langle G_\alpha,\, G_\beta\rangle,$$

a contradiction. ∎

PROOF OF THE PROPOSITION: Since $[t_\alpha,\, K] \leqslant Q_\alpha \cap K = 1$ we have $[t_\alpha,\, K_\beta] = 1$ and the order of t_α is 2. By [9], t_α induces an inner automorphism on L_β/Q_β.

By 2.22 t_α does not centralise L_β/Q_β. Also, as t_α is an inner automorphism we can pick $t \in K_\beta$ which acts on the same way on L_β/Q_β, that is, pick $t \in K_\beta$ so that $x_\beta = t_\alpha t$ and x_β centralises L_β/Q_β.

I now claim that the order of t is 2 as well. By choice of t,

$$|t| = |tQ_\alpha/Q_\alpha|$$

and the image of t in $L_\beta/\langle t_\beta\rangle Q_\alpha \cong L_\beta\langle x_\beta\rangle/\langle t_\beta,\, x_\beta\rangle Q_\alpha$ is t_α which has order two. Hence the claim holds if $t_\beta = 1$ and so we are done for the cases $\mathrm{PSL}_2(9)$, M_{11} or M_{12}. The only problem could appear in $2\cdot M_{12}$ since when we lift M_{12} to $2\cdot M_{12}$ the order of t could become 4. But this does not happen by 2.1(b). Moreover in any case x_β centralises L_β/Q_β and the order of x_β is also one or two.

Now t_α acts non-trivially on Z_α which is irreducible for L_α so t_α inverts Z_α. K_α acts on Y_β faithfully and K_β centralises Y_β so $[K_\alpha, K_\beta] = 1$.

As $Z_\beta \leqslant Y_\beta$ and $\left|Z_\beta^2\right| = |Z_\alpha|$ for $L_\alpha/Q_\alpha \cong \mathrm{SL}_2(9)$, we get that $|Z_\alpha| \leqslant |Y_\beta|^2$.

K_α acts on Y_β faithfully and K_β centralises Y_β so $[K_\alpha,\, K_\beta] = 1$. Since K_α centralises t and K_α centralises t_α we get that K_α centralises x_β. Thus $[x_\beta,\, K_\alpha] = 1$.

Now define $Y_\beta = C_{Z_\alpha}\big(O^2(L_\beta)\big)$. Let $A = Z_\beta$.

Since t centralises Y_β and t_α inverts Y_β, x_β inverts Y_β and so x_β inverts A. This means that if $x_\beta^\bullet$ is the image of x_β in $\mathrm{Aut}(A)$ then $x_\beta^\bullet \in Z\big(\mathrm{Aut}(A)\big)$ and so $[N_{G_\alpha}(A),\, x_\beta^\bullet]$ centralises A.

Let $L = N_{L_\alpha}(A)$ and $Q = C_{L_\alpha}(A)$. Since Z_α is a natural $\mathrm{SL}_2(9)$-module, $L/C_L(A) \cong GL_F(A)$ where $F = GF(9)$ and L acts irreducibly on A. Since $A = A^\perp$, $[Z_\alpha, Q] \leqslant A^\perp = A$. Hence $[Z_\alpha, Q, Q] = 1$ and Q is a 3-group. So $Q = O_3(L)$. Now $[L, x_\beta] \leqslant Q$ and so by a Frattini argument $L = C_L(x_\beta)Q$. Hence $C_L(x_\beta)$ acts irreducibly on A and on Z_α/A (which is isomorphic to the dual of A). In particular x_β inverts or centralises Z_α/A. Since

$$V_\beta = \langle Z_\alpha^{G_\beta}\rangle = \left\langle Z_\alpha^{C_{G_\beta}(x_\beta)}\right\rangle$$

we conclude that x_β inverts or centralises V_β/A.

Note that x_β inverts A so if x_β inverts V_β/A, x_β inverts V_β and V_β is Abelian, a contradiction to $1 \neq [Z_\alpha,\, Z_{\alpha'}] \leqslant V_\beta$.

If x_β centralises V_β/A then $V_\beta = C_{V_\beta}(Z_\beta)A = C_{V_\beta}(Z_\beta)\times A$. Hence $V_\beta^\bullet \leqslant (C_{V_\beta}(Z_\beta))^\bullet$ (as $A \leqslant \bigcap_{g\in G_\beta} Z_\alpha^g \leqslant Z(V_\beta)$) and so

$$V_{\beta'}^\bullet \cap Z_\beta \leqslant (C_{V_\beta}(Z_\beta))^\bullet \cap A = 1.$$

Hence $C_{V_{\beta'}}(S) = 1$ and $V_{\beta'} = 1$, again a contradiction.

4. The case $[Z_\alpha, Z_{\alpha'}] = 1$

In this section we will deal with the case $Z_{\alpha'} \leqslant Q_\alpha$.

It follows from the hpothesis that there is no symmetry between α and α' any more. Also $[Z_\alpha, Z_{\alpha'}] \leqslant [Z_\alpha, Q_\alpha] = 1$ gives

$$C_{Z_\alpha}(Z_{\alpha'}) = Z_\alpha.$$

Now notice that $Z_\alpha \cap Q_{\alpha'} \neq Z_\alpha$ (otherwise we get $Z_\alpha \leqslant Q_{\alpha'}$, a contradiction). Hence, $C_{Z_\alpha}(Z_{\alpha'}) \neq Z_\alpha \cap Q_{\alpha'}$ and by 2.19 (viii), $Z_{\alpha'} \leqslant Z(L_{\alpha'})$, α and α' are not conjugate and b is odd. Therefore we have

$$Z_\beta = \Omega_1 Z(L_\beta) \text{ and } Z_{\alpha'} = \Omega_1 Z(L_{\alpha'}).$$

Lemma 4.1. $L_{\alpha'}/Q_{\alpha'} \cong L_\beta/Q_\beta \cong \mathrm{SL}_2(9)$.

Proof: If $b > 1$ then $[V_{\alpha'}, Z_\alpha, Z_\alpha] \leqslant [V_{\alpha'}, V_\beta, V_\beta] \leqslant [V_\beta, V_\beta] = 1$ by 2.19 (iv), so, since $Z_\alpha \not\leqslant Q_{\alpha'}$, we conclude that L_β/Q_β is not 3-stable and the claim follows by 2.5.

If $b = 1$, $Z_\alpha \not\leqslant Q_\beta$ and $[Q_\beta, Z_\alpha, Z_\alpha] = 1$ (by [**9**]) again imply that L_β/Q_β is not 3-stable and the claim follows by 2.5. ∎

Notation 4.2. For $\gamma \in \Gamma$ let $D_\gamma = C_{Q_\gamma}(O^3(L_\gamma))$.

Lemma 4.3. $Z(L_\alpha) = D_\alpha = 1$.

Proof: See [**9**] ∎

Proposition 4.4. $b = 1$.

Proof: Assume that $b > 1$. Since b is odd, $b \geqslant 3$.

4.4.1. V_β has a unique non-central L_β-composition factor; moreover, this composition factor is the natural module for L_β/Q_β.

Proof: By [**9**], $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = 1$ and hence $V_\beta \cap Q_{\alpha'} \leqslant C_{V_\beta}(V_{\alpha'})$. By a similar argument we also have that $V_{\alpha'} \cap Q_\beta \leqslant C_{V_{\alpha'}}(V_\beta)$. Without loss of generality, assume

$$|V_\beta Q_{\alpha'}/Q_{\alpha'}| \leqslant |V_{\alpha'} Q_\beta/Q_\beta|.$$

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Now let $X = Y/Z$ be a non-central chief factor in V_β. As

$$C_Y(V_{\alpha'})Z/Z \leqslant C_{Y/Z}(V_{\alpha'})$$

we get that

$$\begin{aligned}|X/C_X(V_{\alpha'})| &= \left|Y/Z/C_{Y/Z}(V_{\alpha'})\right| \leqslant |Y/Z/C_Y(V_{\alpha'})Z/Z| \\ &= |Y/C_Y(V_{\alpha'})Z| \leqslant |Y/C_Y(V_{\alpha'})| = \left|Y/Y \cap C_{V_\beta}(V_{\alpha'})\right| \\ &= \left|Y \cdot C_{V_\beta}(V_{\alpha'})/C_{V_\beta}(V_{\alpha'})\right| \leqslant \left|V_\beta/C_{V_\beta}(V_{\alpha'})\right| \\ &\leqslant |V_\beta Q_{\alpha'}/Q_{\alpha'}| \leqslant |V_{\alpha'} Q_\beta/Q_\beta|\end{aligned}$$

so X is an FF-module; similarly, the direct sum of the L_β chief factors on V_β is still an FF-module for L_β/Q_β and the lemma follows by 2.6. □

4.4.2. $[V_\beta, Q_\beta] \leqslant D_\beta$.

PROOF: Assume that $[V_\beta, Q_\beta] \not\leqslant D_\beta$. Then by 4.4.1, $Z_\alpha[V_\beta, Q_\beta]$ is normalised by $G_{\alpha\beta}O^3(L_\beta) = G_\beta$ and we get that $Z_\alpha[V_\beta, Q_\beta] = V_\beta$. Hence $V_\beta/Z_\alpha = [V_\beta/Z_\alpha, Q_\beta]$. Since Q_β is a 3-group acting on the 3-group V_β/Z_α in the above manner, we conclude that $V_\beta/Z_\alpha = 1$. Therefore $V_\beta = Z_\alpha$, a contradiction. Hence $[V_\beta, Q_\beta] \leqslant D_\beta$. □

4.4.3. Let $Q^*_\beta = [Q_\beta, O^3(L_\beta)]$.

By 4.4.2, $[V_\beta, Q^*_\beta] \leqslant [V_\beta, Q_\beta] \leqslant D_\beta$. Note that $Q^*_\beta \leqslant O^3(L_\beta)$ and therefore

$$[V_\beta, Q^*_\beta, Q^*_\beta] \leqslant [D_\beta, Q^*_\beta] \leqslant [D_\beta, O^3(L_\beta)] = 1.$$

Hence $[Z_\alpha, Q^*_\beta, Q^*_\beta] = 1$ and 3-stability of L_α gives that $[Z_\alpha, Q^*_\beta] = 1$ whence $Q^*_\beta \leqslant Q_\alpha$. □

4.4.4. The hypothesis that $b > 1$ gives a contradiction.

PROOF: By 4.4.3, Q^*_β centralises Z_α and so it centralises $\langle Z_\alpha^{G_\beta}\rangle = V_\beta$ as well. Since $[t_\beta, Q_\beta] \leqslant Q^*_\beta$, t_β is the unique involution in $t_\beta Q_\beta/Q^*_\beta$ and so $t_\beta Q^*_\beta \in Z\left(L_\beta/Q^*_\beta\right)$. In particular, L_β normalises $[V_\beta, t_\beta]$. By 4.4.1, $[V_\beta, t_\beta] \neq 1$ and so $C_{[V_\beta, t_\beta]}(S) \neq 1$. Hence

$$Z_\beta \cap [V_\beta, t_\beta] \neq 1.$$

On the other hand, since

$$V_\beta - C_{V_\beta}(t_\beta) \times [V_\beta, t_\beta]$$

and $[Z_\beta, t_\beta] \leqslant [Z_\beta, L_\beta] = 1$, we have a contradiction. □

NOTATION 4.5. For $\gamma \in \Gamma$ let F_γ be a normal 3-subgroup of L_γ minimal with respect to the property $F_\gamma \not\leqslant D_\gamma$.

REMARK 4.6. As F_γ is a 3-group we get $F_\gamma \leqslant Q_\gamma$ and $F_{\gamma'} \neq F_\gamma$. Also, the definition implies $F_\gamma \neq 1$. Since Q_γ is a 3-group acting on the 3-group $F_\gamma, F_\gamma \neq [F_\gamma, Q_\gamma]$ and by minimality of F_γ, $[F_\gamma, Q_\gamma] \leqslant D_\gamma$. Also it is clear from the definitions that $F_\beta = [F_\beta, O^3(L_\beta)] \leqslant O^3(L_\beta)$ and therefore:

$$[D_\beta, F_\beta] \leqslant [D_\beta, O^3(L_\beta)] = 1.$$

LEMMA 4.7. *$F_\beta \not\leqslant Q_\alpha$ and $D_\beta \leqslant Q_\alpha$.*

PROOF: See [9]. ∎

LEMMA 4.8. *Q_α is elementary Abelian, $[Q_\alpha, O^3(L_\alpha)]$ is an irreducible L_α-module and $F_\alpha = Z_\alpha = [Q_\alpha, O^3(L_\alpha)]$. In particular, $\phi(D_\beta) = 1$.*

PROOF: See [9]. ∎

COROLLARY 4.9. *$C_{G_\alpha}(Q_\alpha) = Q_\alpha$. In particular, if $X \leqslant G_\alpha$ then $Z \cap Q_\alpha = C_X(Q_\alpha)$.*

PROOF: See [9]. ∎

PROPOSITION 4.10. *$\Theta \not\cong (2) \cdot M_{12}$; in particular, $\Theta \cong \mathrm{PSL}_2(9)$ or M_{11}.*

PROOF: By 4.1, $L_\beta/Q_\beta \cong \mathrm{SL}_2(9)$. Also from $Q_\alpha Q_\beta = S$ we get

$$[F_\beta Q_\alpha/Q_\alpha, S] = [F_\beta Q_\alpha/Q_\alpha, Q_\beta]$$

and as $[F_\beta Q_\beta] \leqslant D_\beta \leqslant Q_\alpha$ (see 4.6 and 4.7) 23 conclude that

$$[F_\beta Q_\alpha/Q_\alpha, Q_\beta] = 1.$$

Hence $F_\beta Q_\alpha/Q_\alpha \leqslant Z(S/Q_\alpha)$.

Since $|S/Q_\alpha| = 3^3$ and S/Q_α is not Abelian we get that

$$|Z(S/Q_\alpha)| = 3$$

and therefore $F_\beta Q_\alpha/Q_\alpha = Z(S/Q_\alpha)$. But $F_\beta \leqslant G_\alpha$ and therefore 4.9 gives $|F_\beta/C_{F_\beta}(Q_\alpha)| = 3$. In particular F_β/D_β is an FF-module for L_β/Q_β. As $L_\beta/Q_\beta \cong \mathrm{SL}_2(9)$, by 2.8, F_β/D_β is a natural $\mathrm{SL}_2(9)$-module, a contradiction to $|F_\beta/C_{F_\beta}(Q_\alpha)| = 3$. ∎

REMARK 4.11. Since a Sylow 3-subgroup of Θ is elementary Abelian we have

$$\Phi(Q_\beta) \leqslant Q_\alpha.$$

Similarly $\Phi(Q_\alpha) \leqslant Q_\beta$.

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Lemma 4.12. *If $N \leqslant S$, $N \trianglelefteq B$, $\delta \in \{\alpha, \beta\}$ then $N \leqslant Q_\delta$ or $NQ_\delta = S$. In particular, $S = Z_\alpha Q_\beta$.*

Proof: See [9]. ∎

Lemma 4.13. *Let $X_\beta = \bigcap_{\delta \in \Delta(\beta)} Q_\delta$. Then:*

(a) *Q_β / X_β is an irreducible G_β-module,*
(b) *$[Q_\beta/D_\beta, t_\beta] = Q_\beta/D_\beta$ and $C_{Q_\beta/D_\beta} = 1$,*
(c) *$C_{Q_\beta}(t_\beta) \leqslant D_\beta$ and*
(d) *$X_\beta = D_\beta$.*

Proof: Let $X_\beta < A \leqslant Q_\beta$ with $A \trianglelefteq G_\beta$. Then $A \not\leqslant Q_\alpha$ (since if $A \leqslant Q_\alpha$ and $\gamma = \beta^g$ with $g \in G_\beta$ then since $A \trianglelefteq G_\beta$ we get

$$A + A^g \leqslant Q_\beta^g = Q_{\beta^g} = Q_\gamma$$

which gives $A \leqslant X_\beta$, a contradiction). Hence by 8.2, $AQ_\alpha = S$ and therefore $[Z_\alpha, Q_\beta] \leqslant [Z_\alpha, A] \leqslant A$. By 4.11 $Q_{\beta'} \leqslant X_\beta$ and so

$$[L_\beta, Q_\beta] = \left[\langle Z_\alpha^{G_\beta}\rangle Q_\beta, Q_\beta\right] \leqslant A.$$

Let $\widetilde{Q}_\beta + Q_\beta/X_\beta$. Then $\widetilde{Q}_\beta$ is Abelian. Now $\widetilde{Q} = C_{\widetilde{Q}_\beta}(t_\beta) \times [\widetilde{Q}_\beta, t_\beta]$ and both parts are normalised by L_β.

If $C_{\widetilde{Q}_\beta}(t_\beta) \neq 1$, we may assume $A = C_{Q_\beta}(t_\beta)X_\beta$ (since then $A \leqslant Q_\beta$, $A \trianglelefteq G_\beta$ and as $C_{\widetilde{Q}_\beta}(t_\beta) \neq 1$ we also have $X_\beta \neq A$). Hence

$$\widetilde{A} = C_{\widetilde{Q}_\beta}(t_\beta)$$

and we get $[[\widetilde{Q}_\beta, t_\beta], t_\beta] \leqslant [[L_\beta, Q_\beta], t_\beta] \leqslant [A, t_\beta, t_\beta] = 1$. Hence (element of order 2 acting on a 3-group) $[\widetilde{Q}_\beta, t_\beta] = 1$, a contradiction to $[\widetilde{Q}_\beta, Q_\alpha, Q_\alpha] = 1$ and the 3-stability of $L_\beta/\langle t_\beta Q_\beta\rangle$. Therefore $C_{\widetilde{Q}_\beta}(t_\beta) = 1$ and $\widetilde{Q}_\beta = [\widetilde{Q}_\beta, t_\beta] = [\widetilde{Q}_\beta, L_\beta]$. Thus $\widetilde{Q}_\beta \leqslant [L_\beta, \widetilde{Q}_\beta] \leqslant \widetilde{A}$ which implies $\widetilde{A} = \widetilde{Q}_\beta$ and Q_β/X_β is an irreducible G_β-module.

Now by 4.7, $D_\beta \leqslant Q_\alpha$ and as $D_\beta \trianglelefteq G_\beta$ we get $D_\beta \leqslant X_\beta$. But $[X_\beta, Z_\alpha] \leqslant [Q_\alpha, Z_\alpha] = 1$ and $Z_\alpha \not\leqslant Q_\beta$ give $X_\beta \leqslant D_\beta$. Hence $X_\beta = D_\beta$.

Lemma 4.14. *There is $g \in G_\beta$ such that $t_\beta \in \langle Z_\alpha, Z_\alpha^g\rangle Q_\beta$.*

Proof: If $\Psi \cong 2 \cdot A_5$ it is clear since in this case

$$L_\beta = \langle Z_\alpha, Z_\alpha^g\rangle Q_\beta$$

for some $g \in G_\beta$ and $t_\beta \in L_\beta$ by defintion. Since inside $SL_2(9)$ we can generate a $2 \cdot A_5$ this case is also clear. ∎

NOTATION 4.15. $\overline{Q_\gamma} = Q_\gamma / D_\gamma$.

LEMMA 4.16. $|\overline{Q_\beta}| = 3^4$.

PROOF: By 4.14, pick $g \in G_\beta$ such that

$$t_\beta \in \langle Z_\alpha, Z_\alpha^g \rangle Q_\beta.$$

Since $|Q_\beta / C_{Q_\beta}(Z_\alpha)| = |Q_\beta Q_\alpha / Q_\alpha| + |S/Q_\alpha| = 3^2$, we get

$$\left|\overline{Q_\beta} / C_{\overline{Q_\beta}}(t_\beta)\right| \leqslant 3^4.$$

By 4.13(b), $C_{\overline{Q_\beta}}(t_\beta) = 1$ and therefore $|\overline{Q_\beta}| \leqslant 3^4$. Suppose $|\overline{Q_\beta}| < 3^4$. Since 5 does not divide $|\mathrm{GL}\,3(3)|$ we conclude that $L_\beta / Q_\beta \not\cong \mathrm{SL}_2(9)$, and contradiction. Hence $|\overline{Q_\beta}| = 3^4$. ∎

LEMMA 4.17. $\left|[Z_\alpha, \overline{Q_\beta}]\right| = |\overline{Q_{\alpha\beta}}| = |\overline{Q_\beta \cap Z_\alpha}| = 9$.

PROOF: If $|[Z_\alpha, \overline{Q_\beta}]| = 3$, then, with same argument as before, we get

$$|\overline{Q_\beta}| = |[\overline{Q_\beta}, t_\beta]| \leqslant 3^2,$$

a contradiction. Hence

$$9 \leqslant \left|[Z_\alpha, \overline{Q_\beta}]\right| \leqslant |\overline{CQ_\beta \cap A_\alpha}| \leqslant |\overline{Q_{\alpha\beta}}| \leqslant 9$$

and the lemma is proved. ∎

LEMMA 4.18. $D_\beta = Z_\beta$.

PROOF: First, show $D_\beta \leqslant Z_\beta$. Let $L = \langle Z_\alpha^{G_\beta} \rangle$. Then by 2.19 (ix), $O^3(L_\beta) \leqslant L$ and $L_\beta = LQ_\beta$. Since $\overline{Q_\beta}$ is irreducible for G_β we get $[\overline{Q_\beta}, L] = 1$ or $\overline{Q_\beta}$. If $[\overline{Q_\beta}, L] = 1$ then $[Q_\beta, L] \leqslant D_\beta$ so $[Q_\beta, O^3(L_\beta)] = 1$ a contradiction. Therefore $[\overline{Q_\beta}, L] = \overline{Q_\beta}$ which gives $[Q_\beta, L] D_\beta = Q_\beta$.

Also, as $L \trianglelefteq G_\beta$, we have $Q_\beta \leqslant N_{G_\beta}(L)$. Hence $[Q_\beta, L] \subseteq L$, $Q_\beta \leqslant D_\beta L$ and $L_\beta = LD_\beta$. But from 4.9 now, $[D_\beta, D_\beta] \leqslant \Phi(D_\beta) = 1$. As $D_\beta \leqslant Q_\alpha$, $[L, D_\beta] = 1$ so D_β and L both centralise D_β. But then, we also get $[D_\beta, L_\beta] = [D_\beta, LD_\beta] = 1$. Thus $D_\beta \leqslant Z(L_\beta) \leqslant Z_\beta$. Therefore $D_\beta \leqslant Z_\beta$.

Since $Z_\beta = \Omega_1 Z(L_\beta) \leqslant C_{Q_\beta}(O^3(L_\beta)) D_\beta$ the lemma follows. ∎

LEMMA 4.19. $Q_\alpha \cap Q_\beta = Z_\alpha \cap Q_\beta$.

PROOF: It is enough to show that $Q_\alpha \cap Q_\beta \leqslant Z_\alpha \cap Q_\beta$. Let $X \in Q_\alpha \cap Q_\beta$. Then $xD_\beta \in Q_\alpha \cap Q_\beta / D_\beta = \overline{Q_{\alpha\beta}} = \overline{Q_\beta \cap Z_\alpha} = Z_\alpha \cap Q_\beta / D_\beta$. Therefore, $xD_\beta = yD_\beta$, where $y \in Z_\alpha \cap Q_\beta$. Then $x = yd$, $d \in D_\beta$. 2.19 (v) gives

$$Z_\beta \leqslant Z_\alpha.$$

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By 4.18, $D_\beta = Z_\beta \leqslant Z_\alpha$. Therefore $x \in Z_\alpha$ and hence $x \in Z_\alpha \cap Q_\beta$. ∎

COROLLARY 4.20. $Q_\alpha = Z_\alpha$.

PROOF: Since $Q_\alpha \subseteq S = Z_\alpha Q_\beta$ we get $Q_\alpha \subseteq Z_\alpha Q_\beta \cap Q_\alpha = Z_\alpha(Q_\alpha \cap Q_\beta)$ and hence $Q_\alpha = Z_\alpha(Q_\alpha \cap Q_\beta) = Z_\alpha$. ∎

LEMMA 4.21.

(1) *$Q_\alpha = Z_\alpha$ is irreducible as an L_α-module*

(2) *If $\Theta \cong \mathrm{PSL}_2(9)$ then $|Z_\alpha| = 3^6$, $|Z_\beta| = 3^2$ and $|Q_\beta| = 3^6$; moreover $(L_\alpha, L_\beta) \sim (3^6\,\mathrm{PSL}_2(9),\ 3^{1+1+4}\,\mathrm{SL}_2(9))$.*

(3) *If $\Theta \cong M_{11}$ then $|Z_\alpha| = 3^5$, $|Q_\beta| = 3^5$ and $|Z_\beta| = 3$; moreover $(L_\alpha, L_\beta) \sim (3^5 M_{11}, 3^{1+4}\,\mathrm{PSL}_2(9))$.*

PROOF: 2.19 (v) and 4.18 give $D_\beta = Z_\beta \leqslant Z_\alpha$. Hence

$$|Z_\alpha/Z_\beta| = |Z_\alpha Q_\beta/Q_\beta|\,|Z_\alpha \cap Q_\beta/Z_\alpha \cap D_\beta| = |Z_\alpha Q_\beta/Q_\beta|\,|Z_\alpha \cap Q_\beta/Z_\beta|.$$

Recall now 4.17 to get $|Z_\alpha \cap Q_\beta/Z_\beta| = 3^2$ and hence

$$|Z_\alpha/Z_\beta| = 3^2\,|Z_\alpha Q_\beta/Q_\beta| = 3^2\,|S/Q_\beta|.$$

Since $S/Q_\beta \in \mathrm{Syl}_3(\Psi)$ we get that

$$|S/Q_\beta| = 3^2.$$

Hence $|Z_\alpha/Z_\beta|3^4$; in particular, $|Z_\alpha/Z_\beta| \leqslant 3^4$. Since by [9] we can generate L_α by two Sylow 3-subgroups we get $|Z_\alpha| \leqslant 3^8$.

By 4.8, Z_α is irreducible as an L_α-module.

CASE $\Theta \cong \mathrm{PSL}_2(9)$. Then by 2.8 $|Z_\alpha| = 3^4$ or 3^6 and since $|Z_\alpha/Z_\beta| = 3^4$ we get that $|Z_\alpha| = 3^6$ and $|Z_\beta| = 3^2$.

CASE $\Theta \cong M_{11}$. 2.8 gives that $|Z_\alpha| = 3^5$ and $|Z_\beta| = 3$.

Notice now that in both cases, D_β is central as by 4.18 we have $D_\beta = Z_\beta$. Moreover if $\Theta \cong \mathrm{PSL}_2(9)$ then $|D_\beta| = 3$ and if $\Theta \cong M_{11}$ then $|D_\beta| = 3^2$. Finally, in both cases, $|Q_\beta/Z_\beta| = 3^4$ and hence Q_β/Z_β is an irreducible L_β-module. This completes the proof of the lemma. ∎

PROOF OF THEOREM: It follows from 4.10 and 4.21. ∎

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