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NON-ISOMORPHIC TENSOR PRODUCTS OF VON NEUMANN ALGEBRAS

J. J. WILLIAMS

1. Introduction. This paper investigates special conditions under which the tensor product of two von Neumann algebras will be non-isomorphic to the tensor product of two others. The main tools are the algebraic invariants property Λ_x $(x \geqq 0)$ (first defined by Powers [**18**]) and the r_∞ and ρ sets (defined by Araki and Woods [**3**]).

We show that if $\mathscr{A}_i$ is not purely infinite and $\mathscr{M}_i$ is a tensor product of finite type I factors with $r_\infty(\mathscr{M}_i) \supseteq \{0, 1\}$ $(i = 1, 2)$, then $\mathscr{A}_1 \otimes \mathscr{M}_1$ has property Λ_x if and only if $x \in r_\infty(\mathscr{M}_1)$; also $r_\infty(\mathscr{A}_1 \otimes \mathscr{M}_1) = r_\infty(\mathscr{M}_1) = r_\infty(\mathscr{M}_{11})$ for some countable sub-tensor product $\mathscr{M}_{11}$ of $\mathscr{M}_1$, and if $r_\infty(\mathscr{M}_1) \neq r_\infty(\mathscr{M}_2)$ *or* if $\rho(\mathscr{M}_1) \neq \rho(\mathscr{M}_2)$ and $\mathscr{M}_1$ and $\mathscr{M}_2$ are *countable* tensor products, then $\mathscr{A}_1 \otimes \mathscr{M}_1 \ncong \mathscr{A}_2 \otimes \mathscr{M}_2$ (Theorems 4.1 and 5.5). We show also that an algebra with property Λ_x $(0 < x < 1)$ is purely infinite (Theorem 4.5 (c)), and that there exists a continuum of non-isomorphic, non-hyperfinite, type III factors on a separable Hilbert space, each one having its r_∞ set equal to $\{0, 1\}$ (Theorem 5.6). This last result (with the exception of the r_∞ part) has also been obtained, using other methods, by Ching [**6**], Connes [**7**], and Sakai [**20**].

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2. Definitions and notations. If $\mathfrak{H}$ is a Hilbert space, then we denote the inner product on $\mathfrak{H}$ by $(. , .)$ which will be linear in the first argument and conjugate-linear in the second. We write $\mathscr{B}(\mathfrak{H})$, $1(\mathfrak{H})$ and $\mathbf{1}(\mathfrak{H})$ to denote the algebra of all bounded linear operators on $\mathfrak{H}$, the identity operator on $\mathfrak{H}$ and the algebra of all complex scalar multiples of the identity, respectively. If $K \subseteq \mathfrak{H}$ then we write Proj K to denote the projection operator from $\mathfrak{H}$ onto the closed, linear subspace of $\mathfrak{H}$ generated by K. If $z \in \mathfrak{H}$ then we define ω_z to be the linear functional on $\mathscr{B}(\mathfrak{H})$ defined by $\omega_z(T) = (T z, z)$. If $\mathscr{A}$ is a von Neumann algebra on $\mathfrak{H}$ then we say that z is a *trace vector* for $\mathscr{A}$ if ω_z defines a normalized, *faithful* trace on $\mathscr{A}$.

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If n is a positive integer, $\mathscr{A}$ is a type I_n factor on $\mathfrak{H}$ and $0 \neq z \in \mathfrak{H}$, then there exist Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ such that $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2, \mathscr{A} = \mathscr{B}(\mathfrak{H}_1) \otimes \mathbf{1}(\mathfrak{H}_2)$, and $z = \sum_{i=1}^{m} \lambda_i^{\frac{1}{2}} \varphi_i \otimes \psi_i$ for some positive integer $m \leqq n$, where $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_m > 0$ and $\{\varphi_i : i = 1, 2, \ldots, m\}$ and $\{\psi_i : i = 1, 2, \ldots, m\}$ are orthonormal sets in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively [**2**, pp. 164, 165]. Define $\mathrm{Sp}(z, \mathscr{A})$, *the spectrum of* z *in* $\mathscr{A}$, to be the "set" $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ together with $n - m$ zeroes. Although we use set notation, the elements of $\mathrm{Sp}(z, \mathscr{A})$ are understood to be taken with their multiplicity, so that, for example, two subsets of $\mathrm{Sp}(z, \mathscr{A})$ will be considered to be disjoint even if they contain the same value λ, providing that the total multiplicity of λ in these two subsets does not exceed the multiplicity of λ in $\mathrm{Sp}(z, \mathscr{A})$.

If we write $\mathfrak{H} = \otimes (\mathfrak{H}_\alpha, z_\alpha : \alpha \in I)$ and $\mathscr{A} = \otimes (\mathfrak{H}_\alpha, \mathscr{A}_\alpha, z_\alpha : \alpha \in I)$, then we will assume that we have been given an arbitrary, non-empty index set I such that for each $\alpha \in I$, $\mathfrak{H}_\alpha$ is a Hilbert space, $z_\alpha \in \mathfrak{H}_\alpha$ with $||z_\alpha|| = 1$, and $\mathscr{A}_\alpha$ is a von Neumann algebra on $\mathfrak{H}_\alpha$; $\mathfrak{H}$ is the tensor product of the Hilbert spaces $\{\mathfrak{H}_\alpha : \alpha \in I\}$ relative to the reference family $\{z_\alpha : \alpha \in I\}$ and $\mathscr{A}$ is the von Neumann algebra on $\mathfrak{H}$ generated by $\{\pi_\alpha \mathscr{A}_\alpha : \alpha \in I\}$ where π_α is the canonical imbedding of $\mathscr{B}(\mathfrak{H}_\alpha)$ into $\mathscr{B}(\mathfrak{H})$. If J is an arbitrary subset of I, then we define $\mathfrak{H}(J) = \otimes (\mathfrak{H}_\alpha, z_\alpha : \alpha \in J)$, $z(J) = \otimes (z_\alpha : \alpha \in J) \in \mathfrak{H}(J)$, and $\mathscr{A}(J) = \otimes(\mathfrak{H}_\alpha, \mathscr{A}_\alpha, z_\alpha : \alpha \in J)$. If J is a finite subset of I, and $w_\alpha \in \mathfrak{H}_\alpha$ for each $\alpha \in J$, then we define $w(J) = \otimes(w_\alpha : \alpha \in J) \in \mathfrak{H}(J)$. If J is finite and for each $\alpha \in J$, $\mathscr{A}_\alpha$ is a finite type $I_{n(\alpha)}$ factor on $\mathfrak{H}_\alpha$ and $\mathrm{Sp}(z_\alpha, \mathscr{A}_\alpha) = \{\lambda_{\alpha i} : i = 1, 2, \ldots, n(\alpha)\}$ then

$$\mathrm{Sp}(z(J), \mathscr{A}(J)) = \{\prod \lambda_{\alpha\, i(\alpha)} (\alpha \in J) : i(\alpha) \in \{1, 2, \ldots, n(\alpha)\}, \alpha \in J\}.$$

Suppose that $0 \leqq x \leqq 1$, I is a countably infinite index set, and that for each $\alpha \in I$, $\mathfrak{H}_\alpha$ is a four-dimensional Hilbert space, $\mathscr{R}_\alpha$ is a type I_2 factor on $\mathfrak{H}_\alpha$, $v_\alpha \in \mathfrak{H}_\alpha$ with $||v_\alpha|| = 1$ and $\mathrm{Sp}(v_\alpha, \mathscr{R}_\alpha) = \{(1 + x)^{-1}, x(1 + x)^{-1}\}$. Then the algebra $\otimes (\mathfrak{H}_\alpha, \mathscr{R}_\alpha, v_\alpha : \alpha \in I)$ depends up to spatial (product) isomorphism only on the value of x, and is denoted by $\mathscr{R}_x$. If $x > 1$ then we define $\mathscr{R}_x = \mathscr{R}_{1/x}$.

We write $\cong$ to denote an algebraic *-isomorphism and $\mathscr{N}$ to denote the set of positive integers.

General discussions are given in Dixmier [**8**] for von Neumann algebras and in von Neumann [**15**] for tensor products.

3. Property Λ_x and the r_∞ set.

Definition 3.1. (a) Suppose that $x \geqslant 0$, $\epsilon > 0$, $\mathscr{M}$ is a von Neumann algebra, ω is a normal positive linear functional (PLF) on $\mathscr{M}$, and $U \in \mathscr{M}$. Then *the pair* (ω, U) *is said to have property* (ϵ, Λ_x) *for* $\mathscr{M}$ *if*

(i) $U^2 = 0$ *and* $U^* U + UU^* = 1$, *and*

(ii) $|\omega(UT) - x\omega(TU)| \leqq \epsilon ||T||$, *for all* $T \in \mathscr{M}$.

(b) $\mathscr{M}$ *is said to have property* Λ_x *if for every* $\epsilon > 0$, *and for every normal* PLF ω *on* $\mathscr{M}$, *there exists a* $U \in \mathscr{M}$ *such that the pair* (ω, U) *has property* (ϵ, Λ_x) *for* $\mathscr{M}$.

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(c) *$\mathscr{M}$ is said to have property Λ_x' if for every $\epsilon > 0$, and for every finite set $\omega_1, \omega_2, \ldots, \omega_n$ of normal PLF's on $\mathscr{M}$, there exists a $U \in \mathscr{M}$ such that for each $i = 1, 2, \ldots, n$, the pair (ω_i, U) has property (ϵ, Λ_x) for $\mathscr{M}$.*

Remark. If $0 \leqq x \leqq 1$, then $x = \lambda/(1 - \lambda)$ for some $0 \leqq \lambda \leqq \frac{1}{2}$ and property Λ_x is equivalent to the property L_λ that was defined by Powers [**18**, Definition 3.1] where he used it to distinguish between the $\mathscr{R}_x$.

PROPOSITION 3.2. *Suppose that $x > 0$, $\epsilon > 0$, $\mathscr{M}$ is a von Neumann algebra, ω is a normal PLF on $\mathscr{M}$, $U \in \mathscr{M}$ and the pair (ω, U) has property (ϵ, Λ_x) for $\mathscr{M}$. Then the pair (ω, U^*) has property $(\epsilon x^{-1}, \Lambda_{1/x})$ for $\mathscr{M}$.*

Proof. For all $T \in \mathscr{M}$, the complex conjugate of $\omega(T)$ is $\omega(T^*)$. By hypothesis, $|\omega(UT) - x\omega(TU)| \leqq \epsilon||T||$, for all $T \in \mathscr{M}$. Take complex conjugates, divide by x, let $S = T^*$, and we obtain $|\omega(U^*S) - x^{-1}\omega(SU^*)| \leqq \epsilon x^{-1}||S||$, for all $S \in \mathscr{M}$.

COROLLARY 3.3. *If $x > 0$, then property Λ_x is equivalent to property $\Lambda_{1/x}$, and property Λ_x' is equivalent to property $\Lambda_{1/x}'$.*

The asymptotic ratio set (r_∞) was defined by Araki and Woods [**3**, Definition 6.1] where they used it to give a classification of tensor products of type I factors.

Definition 3.4. Suppose that $\mathscr{M}$ is a von Neumann algebra. Then we define

$$r_\infty(\mathscr{M}) = \{x \geqq 0 : \mathscr{M} \cong \mathscr{M} \otimes \mathscr{R}_x\},$$
$$\Lambda(\mathscr{M}) = \{x \geqq 0 : \mathscr{M} \text{ has property } \Lambda_x\},$$

and

$$\Lambda'(\mathscr{M}) = \{x \geqq 0 : \mathscr{M} \text{ has property } \Lambda_x'\}.$$

It is clear that property Λ_x' implies property Λ_x, that $r_\infty(\mathscr{A}) \subseteq r_\infty(\mathscr{A} \otimes \mathscr{B})$ for any von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$, and that if $\mathscr{A} \cong \mathscr{B}$ then $r_\infty(\mathscr{A}) = r_\infty(\mathscr{B})$, $\Lambda(\mathscr{A}) = \Lambda(\mathscr{B})$, and $\Lambda'(\mathscr{A}) = \Lambda'(\mathscr{B})$.

THEOREM 3.5. *Suppose that $x \geqq 0$ and that $\mathscr{A}$ is any von Neumann algebra. Then $\mathscr{A} \otimes \mathscr{R}_x$ has property Λ_x'.*

Proof. This is an easy generalization of [**18**, Lemma 3.2], or follows from [**1**, Lemma 3.1].

COROLLARY 3.6. *Suppose that $\mathscr{A}$ is any von Neumann algebra. Then $r_\infty(\mathscr{A}) \subseteq \Lambda'(\mathscr{A}) \subseteq \Lambda(\mathscr{A})$.*

Araki [**1**, Theorem 1.3] showed that if $\mathscr{A}$ is a von Neumann algebra on a separable Hilbert space, then $r_\infty(\mathscr{A}) = \Lambda'(\mathscr{A})$. However, $r_\infty(\mathscr{A}) \neq \Lambda(\mathscr{A})$, in general. Let Φ_2 be the free group on two generators, and let $\mathscr{A}(\Phi_2)$ be the von Neumann algebra generated by the left regular representation of Φ_2. Note that $\mathscr{A}(\Phi_2)$ is a II_1 factor on a separable Hilbert space. Schwartz [**21**, Lemma 10, Corollary 12] showed that $1 \notin r_\infty(\mathscr{A}(\Phi_2))$, but [**1**, Lemma 6.1] $1 \in \Lambda(\mathscr{A}(\Phi_2))$. Part of our results are to give conditions under which r_∞ and Λ are the same (Theorem 4.1(*b*)).

Definition 3.7. Suppose that $\mathscr{A} = \otimes(\mathfrak{H}_\alpha, \mathscr{A}_\alpha, z_\alpha : \alpha \in I)$ with each $\mathscr{A}_\alpha$ a finite type *I* factor on $\mathfrak{H}_\alpha$, and that $x \geqq 0$. *We call a sequence* $(I_n, K_{n1}, K_{n2}, \varphi_n : n \in \mathscr{N})$ *an x-sequence for* $\mathscr{A}$ if $\{I_n : n \in \mathscr{N}\}$ *are pairwise disjoint, finite subsets of I, and for each* $n \in \mathscr{N}$, K_{n1} *and* K_{n2} *are disjoint subsets of* $\mathrm{Sp}(z(I_n), \mathscr{A}(I_n))$ *and* φ_n *is a bijection from* K_{n1} *to* K_{n2} *such that* $0 \notin K_{n1}$,

$$\sum_{n=1}^{\infty} \left[\sum \lambda(\lambda \in K_{n1})\right] = \infty$$

and

$$\lim_{n\to\infty} \max\{|x - \varphi_n(\lambda)/\lambda| : \lambda \in K_{n1}\} = 0.$$

THEOREM 3.8. *Suppose that* $x \geqq 0$, *that* $\mathscr{A}$ *is a countable tensor product of finite type I factors, and that there exists an x-sequence for* $\mathscr{A}$. *Then* $x \in r_\infty(\mathscr{A})$.

Proof. See [**3**, Definition 3.2, Corollary 5.5].

Remark. The converse of this theorem is also true [**3**, Lemma 5.8].

Definition 3.9. If $0 < x < 1$, define $S_x = \{0, x^n : n = 0, \pm 1, \pm 2, \ldots\}$. Define $S_0 = \{0\}$, $S_1 = \{1\}$, $S_{01} = \{0, 1\}$, and $S_\infty = [0, \infty)$.

It follows from Theorem 3.8 and its converse that for $0 \leqq x \leqq 1$, $r_\infty(\mathscr{R}_x) = S_x$ and that $r_\infty(\mathscr{R}_0 \otimes \mathscr{R}_1) = S_{01}$. There exists a tensor product of finite type *I* factors, $\mathscr{R}_\infty$, such that $r_\infty(\mathscr{R}_\infty) = S_\infty$ [**3**, Lemma 3.13].

For the remainder of this section, we will assume that we are given a von Neumann algebra $\mathscr{S}$ described as follows.

Let I_0 be an arbitrary index set and let $\mathscr{N}_1$ be a countably infinite index set such that I_0, $\mathscr{N}_1$ and $\mathscr{N}$ are pairwise disjoint, and let $I = I_0 \cup \mathscr{N}_1$. For each $\alpha \in I$, let $n(\alpha) \in \mathscr{N}$, let $\mathfrak{H}_{\alpha 1}$ and $\mathfrak{H}_{\alpha 2}$ be Hilbert spaces with orthonormal bases $\{\varphi_{\alpha i} : i = 1, 2, \ldots, n(\alpha)\}$ and $\{\Psi_{\alpha i} : i = 1, 2, \ldots, n(\alpha)\}$, respectively, let

$$w_\alpha = \sum_{i=1}^{n(\alpha)} (n(\alpha))^{-\frac{1}{2}} \varphi_{\alpha i} \otimes \Psi_{\alpha i},$$

and

$$v_\alpha = \sum_{i=1}^{n(\alpha)} (\lambda_{\alpha i})^{\frac{1}{2}} \varphi_{\alpha i} \otimes \Psi_{\alpha i},$$

with $\lambda_{\alpha 1} \geqq \lambda_{\alpha 2} \geqq \ldots \geqq \lambda_{\alpha n(\alpha)} \geqq 0$, $\|v_\alpha\|^2 = \sum_{i=1}^{n(\alpha)} \lambda_{\alpha i} = 1$, and for each $k \in \mathscr{N}_1$, let $n(k) = 2$.

Let $\mathscr{S}_\alpha = \mathscr{B}(\mathfrak{H}_{\alpha 1}) \otimes \mathbf{1}(\mathfrak{H}_{\alpha 2})$, let $\mathfrak{H} = \otimes(\mathfrak{H}_{\alpha 1} \otimes \mathfrak{H}_{\alpha 2}, v_\alpha : \alpha \in I)$ and let $\mathscr{S} = \otimes(\mathfrak{H}_{\alpha 1} \otimes \mathfrak{H}_{\alpha 2}, \mathscr{S}_\alpha, v_\alpha : \alpha \in I)$. Note that w_α is a trace vector for $\mathscr{S}_\alpha$.

LEMMA 3.10. *Suppose that* $0 \leqq x \leqq 1$, $\epsilon > 0$, *that* $\mathscr{B}$ *is a von Neumann algebra on a Hilbert space* $\mathfrak{K}$, *and that* $z \in \mathfrak{K} \otimes \mathfrak{H}$ *with* $\|z\| \leqq 1$. *Suppose that* *there exists a* $U \in \mathscr{B} \otimes \mathscr{S}$ *such that the pair* (ω_z, U) *has property* (ϵ, Λ_x) *for* $\mathscr{B} \otimes \mathscr{S}$. *Then there exist a finite subset* J *of* I *and a* $U_1 \in \mathscr{B} \otimes \mathscr{S}(J) \otimes \mathbf{1}(\mathfrak{H}(I - J))$ *such that the pair* (ω_z, U_1) *has property* $(2\epsilon, \Lambda_x)$ *for* $\mathscr{B} \otimes \mathscr{S}$ (cf. [**18**, Lemma 3.5]).

Proof. Choose some $k \in \mathscr{N}_1$ and define $W_1 \in \mathscr{B}(\mathfrak{H}_{k1})$ as follows: if $p \in \mathfrak{H}_{k1}$, then $W_1 p = (p, \varphi_{k1})\varphi_{k2}$. Then $W_1^* p = (p, \varphi_{k2})\varphi_{k1}$, $W_1^2 = 0$ and $W_1^* W_1 + W_1 W_1^* = 1(\mathfrak{H}_{k1})$. Let

$$Q = 1(\mathfrak{K}) \otimes (W_1 \otimes 1(\mathfrak{H}_{k2})) \otimes 1(\mathfrak{H}(I - \{k\})).$$

Then $Q \in \mathscr{B} \otimes \mathscr{S}(\{k\}) \otimes \mathbf{1}(\mathfrak{H}(I - \{k\}))$, $Q^2 = 0$ and $Q^*Q + QQ^* = 1(\mathfrak{K} \otimes \mathfrak{H})$. By hypothesis, $U^2 = 0$ and $U^*U + UU^* = 1(\mathfrak{K} \otimes \mathfrak{H})$. Therefore, $\{Q^*Q, QQ^*\}$ and $\{U^*U, UU^*\}$ are each a pair of orthogonal, equivalent, complementary projections in $\mathscr{B} \otimes \mathscr{S}$, and hence, it follows from [**12**, p. 25, Corollary] that Q^*Q and U^*U are equivalent. Hence, there exists a $W \in \mathscr{B} \otimes \mathscr{S}$ with $W^*W = U^*U$ and $WW^* = Q^*Q$.

Let $V = W + QWU^*$. Since $(WU)^*(WU) = 0$ and $(W^*Q)^*(W^*Q) = 0$, it follows that $WU = W^*Q = 0$ and $U^*W^* = Q^*W = 0$. From this, a straightforward calculation shows that V is a unitary in $\mathscr{B} \otimes \mathscr{S}$ and that $V^*QV = U$.

Using the spectral theory, $V = \exp(i\pi S)$ for some

$$S \in \mathscr{C} = \{T \in \mathscr{B} \otimes \mathscr{S} : T = T^*, \|T\| \leqq 1\}.$$

Let

$$\mathscr{D} = \bigcup \{\mathscr{B} \otimes \mathscr{S}(J) \otimes \mathbf{1}(\mathfrak{H}(I - J)) : J \text{ is a finite subset of } I\}.$$

Then it is easy to see that $\mathscr{D}$ is a *-algebra which is strongly dense in $\mathscr{B} \otimes \mathscr{S}$. Hence, it follows from the Kaplansky density theorem that S lies in the strong closure of $\mathscr{E} = \{T \in \mathscr{D} : T = T^*, \|T\| \leqq 1\}$. The mapping of $\mathscr{C}$ into $\mathscr{B} \otimes \mathscr{S}$ defined by $T \mapsto \exp(i\pi T)$ is strongly continuous [**11**, Lemma 2]. There exists a net $\{S_\beta : \beta \in \Gamma\} \subseteq \mathscr{E}$ such that $S =$ strong limit S_β, and hence, $V = \exp(i\pi S)$ = strong limit $\exp(i\pi S_\beta)$. Therefore, there exists a $\gamma \in \Gamma$ such that if we let $X = \exp(i\pi S_\gamma)$ then $\|(V - X)t\| < \epsilon/4$ for $t \in \{z, V^*QVz, V^*Q^*Vz\}$. $S_\gamma \in \mathscr{E}$, so S_γ and hence X lie in $\mathscr{B} \otimes \mathscr{S}(J_0) \otimes \mathbf{1}(\mathfrak{H}(I - J_0))$ for some finite subset J_0 of I, and X is unitary. Let $J = J_0 \cup \{k\}$.

Since X and V are unitary, we have, for any $T \in \mathscr{B}(\mathfrak{K} \otimes \mathfrak{H})$,

$$\|(V^*TV - X^*TX)z\| \leqq \|(X - V)(V^*TV)z\| + \|T\| \, \|(V - X)z\|.$$

By substituting first Q, then Q^* for T, we obtain

$$\|(V^*QV - X^*QX)z\| < \epsilon/2, \quad \|(V^*Q^*V - X^*Q^*X)z\| < \epsilon/2.$$

Let $U_1 = X^*QX$. Then $U_1 \in \mathscr{B} \otimes \mathscr{S}(J) \otimes \mathbf{1}(\mathfrak{H}(I - J))$, $U_1^2 = 0$ and $U_1^*U_1 + U_1U_1^* = 1(\mathfrak{K} \otimes \mathfrak{H})$.

Since $V^*QV = U$, we have $\|(U - U_1)z\| < \epsilon/2$ and $\|(U^* - U_1^*)z\| <$

$\epsilon/2$, and since the pair (ω_z, U) has property (ϵ, Λ_x) for $\mathscr{B} \otimes \mathscr{S}$, we have for any $T \in \mathscr{B} \otimes \mathscr{S}$,

$$\begin{aligned} &|\omega_z(U_1T) - x\omega_z(TU_1)| \\ &\leqq |\omega_z(U_1T) - \omega_z(UT)| + |\omega_z(UT) - x\omega_z(TU)| \\ &\quad + |x\omega_z(TU) - x\omega_z(TU_1)| \\ &\leqq |(Tz, (U^* - U_1^*)z)| + \epsilon||T|| + x|((U - U_1)z, T^*z)| \\ &\leqq 2\epsilon||T||. \end{aligned}$$

Definition 3.11. Suppose that $\mathscr{B}$ is a von Neumann algebra with a normalized finite trace (tr). For any $T \in \mathscr{B}$ we let Δ_T be the linear functional defined on $\mathscr{B}$ as follows: if $S \in \mathscr{B}$ then $\Delta_T(S) = \operatorname{tr}(TS)$. If $T = T^* \in \mathscr{B}$, then, by the spectral theory, T can be written as

$$T = \int_{-\infty}^{\infty} \lambda dE(\lambda)$$

with $E(\lambda) \in \mathscr{B}$ for all λ, and the $E(\lambda)$ are right strongly continuous. If $0 < \theta \leqq 1$, then we define

$$\epsilon_T(\theta) = \inf\{\lambda : \operatorname{tr}(E(\lambda)) \geqq \theta\}.$$

Remark 3.12. If $T = \sum_{i=1}^{n} \oplus \lambda_i P_i$, where $\lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n$ (real), $P_1, P_2, \ldots, P_n$ are orthogonal projections in $\mathscr{B}$ with $\sum_{i=1}^{n} \oplus P_i = 1$ and if we let $P_1 + \ldots + P_{k-1} = 0$ if $k = 1$, then for $k = 1, \ldots, n$, $\epsilon_T(\theta) = \lambda_k$ if $\operatorname{tr}(P_1 + \ldots + P_{k-1}) < \theta \leqq \operatorname{tr}(P_1 + \ldots + P_k)$.

LEMMA 3.13. *Suppose that $\mathscr{B}$ is a von Neumann algebra with a normal, normalized, finite trace* (tr), *and that S and T are self-adjoint operators in $\mathscr{B}$, and let Δ and ϵ be defined relative to this trace, as in Definition* 3.11. *Then*

$$\int_0^1 |\epsilon_S(\theta) - \epsilon_T(\theta)| d\theta \leqslant ||\Delta_S - \Delta_T||$$

(cf. [**17**, Lemma 5.5, Theorem 5.6]).

Proof. Let $\mathscr{B}$ act on the Hilbert space $\mathfrak{K}$. Let A be any self-adjoint operator in $\mathscr{B}$ and let

$$A = \int_{-\infty}^{\infty} \lambda dE(\lambda)$$

with the $E(\lambda)$ right strongly continuous. For any real λ_0, let $\lambda \to \lambda_0^+$. Then $E(\lambda) \to E(\lambda_0)$ strongly, and hence ultra-strongly, and hence ultra-weakly. We note that the strong and ultra-strong operator topologies coincide on bounded subsets of $\mathscr{B}(\mathfrak{K})$ [**8**, p. 34]. Since the trace is normal, it is also ultra-weakly continuous [**8**, p. 51, Théorème 1] and hence $\operatorname{tr} E(\lambda) \to \operatorname{tr} E(\lambda_0)$. This fact is needed in order to make the proofs of [**13**, Lemmas 15.2.1, 15.2.2] valid for $\mathscr{B}$ and its trace. Therefore, for $0 < \theta \leqq 1$,

$$\epsilon_A(\theta) = \inf \{\sup \{(Af,f):f \in P\mathfrak{K}, ||f|| = 1\}:P \text{ is a projection in } \mathscr{B} \text{ with tr } P \geqq \theta\}, \tag{3.1}$$

and

$$\int_0^1 \epsilon_A(\theta)d\theta = \text{tr }(A). \tag{3.2}$$

Let $A \in \mathscr{B}$ and let $A = WB$ be the polar decomposition of A [**13**, p. 142, § 4.4] where W is a partial isometry, $B \geqq 0$, W, $B \in \mathscr{B}$, and $W^*A = B = (A^*A)^{\frac{1}{2}}$. Then

$$\begin{aligned} ||\Delta_A|| &= \sup \{|\Delta_A(D)|:D \in \mathscr{B}, ||D|| \leqq 1\} \\ &\geqq |\Delta_A(W^*)| = |\text{tr }(W^*A)| = \text{tr }[(A^*A)^{\frac{1}{2}}] \end{aligned} \tag{3.3}$$

From the spectral theory, we can write $S - T = C_1 - C_2$ with $C_1, C_2 \in \mathscr{B}$, C_1 and $C_2 \geqq 0$, and $C_1C_2 = C_2C_1 = 0$. Let $C = S + C_2$. Then $C \in \mathscr{B}$, and it is easy to see that $C \geqq S$, $C \geqq T$, $2C - S - T = C_1 + C_2$, and that $(C_1 + C_2)^2 = (C_1 - C_2)^2 = (S - T)^2 = (S - T)^*(S - T)$.

If $A, B \in \mathscr{B}$ with $A = A^*$, $B = B^*$ and $A \leqq B$ then it follows from (3.1) that for each $0 < \theta \leqq 1$, $\epsilon_A(\theta) \leqq \epsilon_B(\theta)$. This, together with (3.2), (3.3) and the above shows that

$$\begin{aligned} \int_0^1 |\epsilon_S(\theta) - \epsilon_T(\theta)|d\theta &\leqslant \int_0^1 |\epsilon_S(\theta) - \epsilon_C(\theta)| + |\epsilon_C(\theta) - \epsilon_T(\theta)|d\theta \\ &= \int_0^1 (2\epsilon_C(\theta) - \epsilon_S(\theta) - \epsilon_T(\theta)d\theta = \text{tr }(2C - S - T) \\ &= \text{tr }(C_1 + C_2) \\ &= \text{tr }[\{(S - T)^*(S - T)\}^{\frac{1}{2}}] \\ &\leqslant ||\Delta_{S-T}|| = ||\Delta_S - \Delta_T||. \end{aligned}$$

LEMMA 3.14. *Suppose that* $0 \leqq x \leqq 1$, $\epsilon > 0$, *that* J *is a finite subset of* I, *and that* $\mathscr{B}$ *is a von Neumann algebra on a Hilbert space* $\mathfrak{K}$ *with a trace vector* $t \in \mathfrak{K}$. *Let* $\omega = \omega_z$ *where* $z = t \otimes v(I)$ *and suppose that there exists a*

$$U \in \mathscr{B} \otimes \mathscr{S}(J) \otimes \mathbf{1}(\mathfrak{H}(I - J))$$

such that the pair (ω, U) *has property* (ϵ, Λ_x) *for* $\mathscr{B} \otimes \mathscr{S}$. *Then there exist a finite-dimensional Hilbert space* $\mathfrak{G}$, *a finite type I factor* $\mathscr{G}$ *on* $\mathfrak{G}$, *a* $q \in \mathfrak{G}$ *such that* q *is a trace vector for* $\mathscr{G}$, *disjoint subsets* K_1 *and* K_2 *of*

$$\text{Sp}(v(J) \otimes q, \mathscr{S}(J) \otimes \mathscr{G}),$$

and a bijection $\varphi:K_1 \to K_2$ *such that* $0 \notin K_1$, $\sum\lambda(\lambda \in K_1) \geqq \frac{1}{4}$ *and*

$$\max \{|x - \varphi(\lambda)/\lambda|:\lambda \in K_1\} < 24\epsilon$$

(cf. [**18**, Lemmas 3.3, 3.4]).

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Proof. Let $\mathscr{R} = \mathscr{B} \otimes \mathscr{S}(J) \otimes \mathbf{1}(\mathfrak{H}(I - J))$. Let $E = F_{11} = U^*U$, $F_{21} = U$, $F_{12} = U^*$, and $F = F_{22} = UU^*$. Then for $i, j = 1, 2$, F_{ij} is a partial isometry in $\mathscr{R}$ from $F_{jj}(\mathfrak{K} \otimes \mathfrak{H})$ to $F_{ii}(\mathfrak{K} \otimes \mathfrak{H})$. Therefore, it follows from [**8**, p. 25, Proposition 5(ii)] that $\mathscr{R} \cong \mathscr{R}_E \otimes \mathscr{L}$ where $\mathscr{L}$ is the von Neumann algebra spanned by $\{E, U, U^*, F\}$. We will identify operators that correspond under this isomorphism, hence, if $T \in \mathscr{R}$, then

$$(3.4) \qquad T = (ETE \otimes E) + (U^*TE \otimes U) + (ETU \otimes U^*) + (U^*TU \otimes F)$$

so that, in particular, $U = E \otimes U$ and

$$(3.5) \qquad ETE + UTU^* = ETE \otimes 1.$$

For any $S \in \mathscr{R}_E$ it is easy to see that $U(S \otimes E) = S \otimes U$, $(S \otimes E)U = 0$, $U(S \otimes U^*) = S \otimes F$, and $(S \otimes U^*)U = S \otimes E$. From the hypothesis, $|\omega(UT) - x\omega(TU)| \leqq \epsilon||T||$, for all $T \in \mathscr{R}$. Hence, by substituting first $S \otimes E$, then $S \otimes U^*$ for T, we obtain that, for any $S \in \mathscr{R}_E$,

$$(3.6) \qquad |\omega(S \otimes U)| \leqq \epsilon||S||, \quad |\omega(S \otimes F) - x\omega(S \otimes E)| \leqq \epsilon||S||.$$

Let β be the linear functional on $\mathscr{R}$ that is defined as follows: if $T \in \mathscr{R}$, then $\beta(T) = (1 + x)^{-1}\{\omega(ETE \otimes 1) + x\omega(U^*TU \otimes 1)\}$. Since $E + F = 1$, and the complex conjugate of $\omega(T)$ is $\omega(T^*)$, we see from (3.4) and (3.6) that for any $T \in \mathscr{R}$,

$$\begin{aligned}(3.7) \qquad |\omega(T) - \beta(T)| &\leqq (1 + x)^{-1}|x\omega(ETE \otimes E) - \omega(ETE \otimes F)| \\ &\quad + |\omega(U^*TE \otimes U)| + |\omega(U^*T^*E \otimes U)| \\ &\quad + (1 + x)^{-1}|\omega(U^*TU \otimes F) - x\omega(U^*TU \otimes E)| \\ &\leqq 4\epsilon||T||.\end{aligned}$$

We shall now express our functionals ω and β in terms of a trace (tr) on $\mathscr{R}$.

For each $\alpha \in I$ and each $i = 1, 2, \ldots, n(\alpha)$, define $P_{\alpha i} = \operatorname{Proj}\{\varphi_{\alpha i}\} \otimes 1(\mathfrak{H}_{\alpha 2})$. Then $\{P_{\alpha i}: i = 1, 2, \ldots, n(\alpha)\}$ are orthogonal, equivalent projections in $\mathscr{S}_\alpha$, each having trace equal to $1/n(\alpha)$. For each $\alpha \in I$, let

$$R_\alpha = \sum_{i=1}^{n(\alpha)} \oplus\ (n(\alpha)\lambda_{\alpha i})P_{\alpha i}.$$

Then $R_\alpha \geqq 0$ and $v_\alpha = R_\alpha^{\frac{1}{2}} w_\alpha$. Let

$$w = t \otimes w(J) \otimes v(I - J),$$

and

$$R = 1(\mathfrak{K}) \otimes \{\otimes(R_\alpha: \alpha \in J)\} \otimes 1(\mathfrak{H}(I - J)).$$

Note that $z = t \otimes v(J) \otimes v(I - J)$. Then $w \in \mathfrak{K} \otimes \mathfrak{H}$, $R \in \mathscr{R}$, $R \geqq 0$, and $z = R^{\frac{1}{2}}w$. It is straightforward to see that w is a trace vector for $\mathscr{R}$, and that for any $T \in \mathscr{R}$, $\omega(T) = \operatorname{tr}\ (TR)$ and $\beta(T) = \operatorname{tr}\ (TD_0)$, i.e., $\omega = \Delta_R$ and

$\beta = \Delta_{D_0}$, where, by using (3.4) and (3.5),

$$\begin{aligned} D_0 &= (1+x)^{-1}\{(ERE + U^*RU) + x(URU^* + FRF)\} \\ &= (1+x)^{-1}\{(ERE \otimes E) + (U^*RU \otimes E) + x(ERE \otimes F) \\ &\quad + x(U^*RU \otimes F)\} \\ &= (ERE + U^*RU) \otimes \{(1+x)^{-1}(E + xF)\}. \end{aligned}$$

Let $D = ERE + U^*RU$ and let $S = (1+x)^{-1}(E \oplus xF)$. Then $D \in \mathscr{R}_E$, $S \in \mathscr{L}$, and $\beta(T) = \text{tr}\,(T(D \otimes S))$, for all $T \in R$.

We shall now approximate D by a finite sum of projections. Since $R \geqq 0$ we have $D \geqq 0$, and by the spectral theory,

$$D = \int_0^\infty \lambda dE(\lambda),$$

with $E(\lambda) \in \mathscr{R}_E$, for all λ. Choose a positive integer $p \geqq 1/\epsilon$ and let $D_1 = \sum_{n=0}^\infty (n/p)\{E((n+1)/p) - E(n/p)\}$. Since $E(\lambda) = E$ for all $\lambda \geqq ||D||$, this is a finite sum and hence, we may write

$$(3.8)\qquad D_1 = \sum_{i=1}^m \oplus \nu_i Q_i$$

with $m \in \mathscr{N}$, $\nu_1, \nu_2, \ldots, \nu_m \geqq 0$, $\{Q_1, Q_2, \ldots, Q_m\}$ orthogonal, non-zero projections in $\mathscr{R}_E$ with $\sum_{i=1}^m \oplus Q_i = E$ and $||D - D_1|| \leqq 1/p \leqq \epsilon$.

Let $\beta_1 = \Delta_{D_1 \otimes S}$ on $\mathscr{R}$. Then, for any $T \in \mathscr{R}$,

$$(3.9)\qquad \begin{aligned} |\beta(T) - \beta_1(T)| &= |\text{tr}\,[T\{(D - D_1) \otimes S\}]| \\ &= |(T\{(D - D_1) \otimes S\}w, w)| \\ &\leqq ||T||\,||D - D_1||\,||S||\,||w||^2 \\ &\leqq \epsilon ||T||. \end{aligned}$$

Let $N = \prod n(\alpha)$ $(\alpha \in J)$. From the definition, R can be written as $R = N \sum_{i=1}^N \oplus \rho_i E_i$, where $\{\rho_1, \ldots, \rho_N\} = \text{Sp}\,(v(J), \mathscr{S}(J))$, ordered so that $0 \leqq \rho_1 \leqq \rho_2 \leqq \ldots \leqq \rho_N$, $\{E_1, E_2, \ldots, E_N\}$ are orthogonal projections in $\mathscr{R}$, and for each $i = 1, 2, \ldots, N$, $\text{tr}\,(E_i) = \prod 1/n(\alpha)$ $(\alpha \in J) = 1/N$.

If $\mathscr{I}$ is a subset of real numbers, we write $\mathscr{X}(\mathscr{I})$ to denote its characteristic function, and if $\mathscr{I}$ is an interval, then we write $||\mathscr{I}||$ to denote its length.

For each $i = 1, 2, \ldots, N$, let $\mathscr{E}_i = ((i-1)/N, i/N]$. Let $f = \epsilon_R$. Then, using Remark 3.12, $f = \sum_{i=1}^N N\rho_i \mathscr{X}(\mathscr{E}_i)$. From (3.8),

$$D_1 \otimes S = \sum_{i=1}^m \oplus (1+x)^{-1}\nu_i\{(Q_i \otimes E) \oplus x(Q_i \otimes F)\},$$

where $\{Q_i \otimes E, Q_j \otimes F : i, j = 1, 2, \ldots, m\}$ are pairwise orthogonal projections in $\mathscr{R}$, and for each $i = 1, 2, \ldots, m$, $\text{tr}\,(Q_i \otimes E) = \text{tr}\,(EQ_iE) = \text{tr}\,(UEQ_iEU^*) = \text{tr}\,(Q_i \otimes F) \neq 0$, since $Q_i \neq 0$. Using Remark 3.12, we see that we have the following situation.

There exists a partition of $(0, 1]$, $\{\mathscr{C}_i, \mathscr{D}_i : i = 1, 2, \ldots, m\}$, such that for each $i = 1, 2, \ldots, m$, $\mathscr{C}_i$ and $\mathscr{D}_i$ are each of the form $(a, b]$ for some $0 \leqq a < b \leqq 1$, and $\|\mathscr{C}_i\| = \|\mathscr{D}_i\| \neq 0$, and if we let $g = \epsilon_{D1\otimes S}$, then

$$g = \sum_{i=1}^{m} (1 + x)^{-1}\nu_i\{\mathscr{X}(\mathscr{C}_i) + x\mathscr{X}(\mathscr{D}_i)\}.$$

We wish to compare f and g, and, as a first step, we will begin to subdivide the $\mathscr{C}_i$, $\mathscr{D}_i$ and $\mathscr{E}_i$ in order to obtain common end points.

For $i = 1, 2, \ldots, m$, let $l_i = \|\mathscr{C}_i\| = \|\mathscr{D}_i\|$. Then $l_i > 0$ and $\sum_{i=1}^{m} l_i = \frac{1}{2}$. Let $\{u_i : i = 0, 1, \ldots, 2m\}$ be the end points of the intervals

$$\{\mathscr{C}_i, \mathscr{D}_i : i = 1, 2, \ldots, m\}$$

so that $0 = u_0 < u_1 < \ldots < u_{2m-1} < u_{2m} = 1$. Let

$$\delta = \min \{2l_m, \epsilon/(1 + 2 \textstyle\sum_{i=1}^{m}\nu_i)\}.$$

Then $\delta > 0$. For each $i = 1, 2, \ldots, m - 1$, let r_i be a rational number such that $r_i > 0$ and $|r_i - l_i| < \delta/(2m^2)$. Then

$$|(\textstyle\sum_{i=1}^{m-1} r_i) - (\sum_{i=1}^{m-1} l_i)| < \delta/(2m) \leqslant l_m.$$

Hence,

$$\textstyle\sum_{i=1}^{m-1} r_i < (\sum_{i=1}^{m-1} l_i) + l_m = \frac{1}{2}.$$

Let $r_m = \frac{1}{2} - (\sum_{i=1}^{m-1} r_i)$. Then $r_m > 0$ and

$$|r_m - l_m| = |\tfrac{1}{2} - \sum_{i=1}^{m-1} r_i - (\tfrac{1}{2} - \sum_{i=1}^{m-1} l_i)| < \delta/(2m).$$

We wish to define a partition of $(0, 1]$, $\{\mathscr{C}_{i1}, \mathscr{D}_{i1} : i = 1, 2, \ldots, m\}$, such that for each $i = 1, 2, \ldots, m$, $\mathscr{C}_{i1}$ and $\mathscr{D}_{i1}$ are each of the form $(a, b]$ for some $0 \leqq a < b \leqq 1$, a, b rational numbers, and $\|\mathscr{C}_{i1}\| = \|\mathscr{D}_{i1}\| = r_i > 0$, and the relative ordering of the $\{\mathscr{C}_{i1}, \mathscr{D}_{i1} : i = 1, 2, \ldots, m\}$ is the same as that of the $\{\mathscr{C}_i, \mathscr{D}_i : i = 1, 2, \ldots, m\}$. The end points of the intervals $\{\mathscr{C}_{i1}, \mathscr{D}_{i1} : i = 1, 2, \ldots, m\}$ will be $\{d_i : i = 0, 1, \ldots, 2m\}$ so that $0 = d_0 < d_1 < \ldots < d_{2m-1} < d_{2m} = 1$.

Let $d_0 = 0$. Suppose that $k \in \{0, 1, \ldots, 2m - 1\}$ and that $d_0, d_1, \ldots, d_k$ have been chosen. Then $(u_k, u_{k+1}] = \mathscr{C}_i$ (or $\mathscr{D}_i$) for some $i \in \{1, 2, \ldots, m\}$. Define $d_{k+1} = d_k + r_i$ and define $\mathscr{C}_{i1}$ (respectively, $\mathscr{D}_{i1}$) $= (d_k, d_{k+1}]$. It is clear that $d_{2m} = 2 \sum_{i=1}^{m} r_i = 1$, and that our intervals and end points exist as required.

For each $i = 1, 2, \ldots, 2m$, $d_i = \sum r_j$, the sum taken over some set of j's in which each r_j may occur twice, and $u_i = \sum l_j$, the sum taken over the same set of j's. Hence $|d_i - u_i| \leqq 2 \sum_{j=1}^{m} |r_j - l_j| < \delta$.

If $a < b$ and $c < d$ then it is easy to see that

$$\int_0^1 |\mathscr{X}((a, b])(\theta) - \mathscr{X}((c, d])(\theta)| d\theta \leqslant |a - c| + |b - d|.$$

Since for each $i = 1, 2, \ldots, m$, there exists a $j \in \{1, 2, \ldots, 2m\}$ such that

$\mathscr{C}_i = (u_{j-1}, u_j]$ and $\mathscr{C}_{i1} = (d_{j-1}, d_j]$, we have that

$$\int_0^1 |\mathscr{X}(\mathscr{C}_i)(\theta) - \mathscr{X}(\mathscr{C}_{i1})(\theta)|d\theta \leqslant |u_{j-1} - d_{j-1}| + |u_j - d_j| < 2\delta,$$

and similarly for $\mathscr{D}_i$ and $\mathscr{D}_{i1}$.

Let $h = \sum_{i=1}^m (1+x)^{-1}\nu_i\{\mathscr{X}(\mathscr{C}_{i1}) + x\mathscr{X}(\mathscr{D}_{i1})\}$. Then,

$$\begin{aligned}(3.10)\quad \int_0^1 |g(\theta) - h(\theta)|d\theta &\leqslant \sum_{i=1}^m (1+x)^{-1}\nu_i \int_0^1 \{|\mathscr{X}(\mathscr{C}_i)(\theta) - \mathscr{X}(\mathscr{C}_{i1})(\theta)| \\ &\qquad + x|\mathscr{X}(\mathscr{D}_i)(\theta) - \mathscr{X}(\mathscr{D}_{i1})(\theta)|\}d\theta \\ &\leqslant \sum_{i=1}^m (1+x)^{-1}\nu_i(2\delta + 2\delta x) \\ &< \epsilon.\end{aligned}$$

For each $i = 1, 2, \ldots, 2m$, d_i is rational and so $d_i = a_i/b_i$ for $a_i, b_i \in \mathscr{N}$. Let b = least common multiple of $\{b_1, b_2, \ldots, b_{2m}\}$. Then, $b \in \mathscr{N}$ and there exist $c_1, c_2, \ldots, c_{2m} \in \mathscr{N}$ so that $d_i = c_i/(2bN)$.

We now subdivide the $\mathscr{C}_{i1}$ and $\mathscr{D}_{i1}$ into subintervals of length $1/(2bN)$. Hence, there exists a partition of $(0, 1]$, $\{\mathscr{C}_{i2}, \mathscr{D}_{i2}: i = 1, 2, \ldots, bN\}$, such that each $\mathscr{C}_{i2}$ and $\mathscr{D}_{i2}$ is of the form $((k-1)/(2bN), k/(2bN)]$ for some $k \in \mathscr{N}$, and there exists a partition of $\{1, 2, \ldots, bN\}$, $\{L(i): i = 1, 2, \ldots, m\}$, such that for each $i = 1, 2, \ldots, m$,

$$\mathscr{C}_{i1} = \bigcup \mathscr{C}_{j2} \quad (j \in L(i)),$$
$$\mathscr{D}_{i1} = \bigcup \mathscr{D}_{j2} \quad (j \in L(i)).$$

For each $j = 1, 2, \ldots, bN$, there exists exactly one $i \in \{1, 2, \ldots, m\}$ such that $j \in L(i)$, and we define $\sigma_j = \nu_i$. Hence,

$$(3.11)\quad h = \sum_{j=1}^{bN} (1+x)^{-1}\sigma_j\{\mathscr{X}(\mathscr{C}_{j2}) + x\mathscr{X}(\mathscr{D}_{j2})\}.$$

For each $i = 1, 2, \ldots, N$, we define

$$L(i, 1) = \{j \in \{1, 2, \ldots, bN\}: \mathscr{C}_{j2} \subseteq \mathscr{E}_i\}$$

and

$$L(i, 2) = \{j \in \{1, 2, \ldots, bN\}: \mathscr{D}_{j2} \subseteq \mathscr{E}_i\}.$$

Then, $\{L(i, 1): i = 1, 2, \ldots, N\}$ and $\{L(i, 2): i = 1, 2, \ldots, N\}$ are each a partition of $\{1, 2, \ldots, bN\}$, and if card stands for cardinality, then,

$$(3.12)\quad \text{card } L(i, 1) + \text{card } L(i, 2) = 2b.$$

For each $j = 1, 2, \ldots, bN$, there exists exactly one i and one k such that $j \in L(i, 1)$ and $j \in L(k, 2)$, and we define $\lambda_{j1} = \rho_i$ and $\lambda_{j2} = \rho_k$. Hence,

$$(3.13)\quad f = \sum_{j=1}^{bN} \{N\lambda_{j1}\mathscr{X}(\mathscr{C}_{j2}) + N\lambda_{j2}\mathscr{X}(\mathscr{D}_{j2})\}.$$

For any σ, $\lambda_1, \lambda_2 \geqq 0$, let $a = \min\{\lambda_1, \lambda_2\}$, let $A = \max\{\lambda_1, \lambda_2\}$, and let $\Omega = |N\lambda_1 - (1+x)^{-1}\sigma| + |N\lambda_2 - x(1+x)^{-1}\sigma|$. Then, since $0 \leqq x \leqq 1$,

$$\begin{aligned}\Omega &\geqq |Nx\lambda_1 - x(1+x)^{-1}\sigma| + |N\lambda_2 - x(1+x)^{-1}\sigma| \\ &\geqq N|x\lambda_1 - \lambda_2| \\ &\geqq N|xA - a| \\ &= NA|x - a/A|,\end{aligned}$$

if we define $0/0 = 0$.

For $j = 1, 2, \ldots, bN$, let

$$\mu_{j1} = \max\{\lambda_{j1}, \lambda_{j2}\}/(2b), \qquad \mu_{j2} = \min\{\lambda_{j1}, \lambda_{j2}\}/(2b).$$

Then, using (3.11), (3.13), and the above, we obtain

$$(3.14)\quad \begin{aligned}&\int_0^1 |f(\theta) - h(\theta)|d\theta \\ &= \sum_{j=1}^{bN} \{|N\lambda_{j1} - (1+x)^{-1}\sigma_j| + |N\lambda_{j2} - x(1+x)^{-1}\sigma_j|\}/(2bN) \\ &\geqslant \sum_{j=1}^{bN} \mu_{j1}|x - \mu_{j2}/\mu_{j1}|.\end{aligned}$$

Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be Hilbert spaces with orthonormal bases $\{e_i : i = 1, 2, \ldots, 2b\}$ and $\{f_i : i = 1, 2, \ldots, 2b\}$, respectively. Let $q = \sum_{i=1}^{2b}(2b)^{-\frac{1}{2}}e_i \otimes f_i$, and let $\mathscr{G} = \mathscr{B}(\mathfrak{G}_1) \otimes \mathbf{1}(\mathfrak{G}_2)$. Then q is a trace vector for $\mathscr{G}$. Let $\mathfrak{G} = \mathfrak{G}_1 \otimes \mathfrak{G}_2$.

The elements of $\mathrm{Sp}(v(J) \otimes q, \mathscr{S}(J) \otimes \mathscr{G})$ are identical to those obtained by taking the elements of $\mathrm{Sp}(v(J), \mathscr{S}(J)) = \{\rho_1, \ldots, \rho_N\}$, multiplying each one by $1/(2b)$ and repeating it $2b$ times. Using (3.12) and the definitions of λ_{j1}, λ_{j2}, μ_{j1} and μ_{j2}, this set is the same as

$$\{\lambda_{j1}/(2b), \lambda_{j2}/(2b) : j = 1, 2, \ldots, bN\} = \{\mu_{j1}, \mu_{j2} : j = 1, 2, \ldots, bN\}.$$

Using Lemma 3.13 together with (3.7), (3.9), (3.10), and (3.14), we obtain

$$(3.15)\quad \begin{aligned}&\sum_{j=1}^{bN} \mu_{j1}|x - \mu_{j2}/\mu_{j1}| \\ &\leqslant \int_0^1 |f(\theta) - h(\theta)|d\theta \\ &\leqslant \int_0^1 |f(\theta) - g(\theta)|d\theta + \int_0^1 |g(\theta) - h(\theta)|d\theta \\ &< \int_0^1 |\epsilon_R(\theta) - \epsilon_{D_1 \otimes S}(\theta)|d\theta + \epsilon \\ &\leqslant ||\Delta_R - \Delta_{D_1 \otimes S}|| + \epsilon \\ &= ||\omega - \beta_1|| + \epsilon \\ &\leqslant ||\omega - \beta|| + ||\beta - \beta_1|| + \epsilon \\ &\leqslant 6\epsilon.\end{aligned}$$

For each $j = 1, 2, \ldots, bN$, let $a_j = |x - \mu_{j2}/\mu_{j1}|$. Let

$$L = \{j \in \{1, 2, \ldots, bN\} : \mu_{j1} \neq 0\},$$

let

$$L_1 = \{j \in L : a_j \geqq 24\epsilon\},$$

and let

$$L_2 = \{j \in L : a_j < 24\epsilon\}.$$

Then, from (3.15),

$$\begin{aligned} \sum \mu_{j1}(j \in L_1) &\leqslant (24\epsilon)^{-1} \sum \mu_{j1} a_j \quad (j \in L_1) \\ &\leqslant (24\epsilon)^{-1} \sum_{j=1}^{bN} \mu_{j1} a_j \\ &\leqslant \tfrac{1}{4}. \end{aligned}$$

Since $\mu_{j1} \geqq \mu_{j2}$ for each $j = 1, 2, \ldots, bN$, we have

$$\begin{aligned} \sum \mu_{j1}(j \in L_2) &= \left\{ \sum_{j=1}^{bN} \mu_{j1} \right\} - \left\{ \sum \mu_{j1} \quad (j \in L_1) \right\} \\ &\geqslant \tfrac{1}{2} \left\{ \sum_{j=1}^{bN} (\mu_{j1} + \mu_{j2}) \right\} - \tfrac{1}{4}. \\ &= \tfrac{1}{2} ||v(J) \otimes q||^2 - \tfrac{1}{4} \\ &= \tfrac{1}{4}. \end{aligned}$$

Let $K_1 = \{\mu_{j1} : j \in L_2\}$ and let $K_2 = \{\mu_{j2} : j \in L_2\}$. Define a mapping $\varphi : K_1 \to K_2$ by $\varphi(\mu_{j1}) = \mu_{j2}$ for $j \in L_2$. Then φ is a bijection. Hence, K_1, K_2 and φ satisfy the requirements of the statement of the lemma.

We now come to the key theorem of the paper.

THEOREM 3.15. *Suppose that $x \geqq 0$, that $\mathscr{A}$ is a von Neumann algebra with a trace vector, and that $\mathscr{A} \otimes \mathscr{S}$ has property Λ_x. Then there exists a countable subset I_∞ of I such that $x \in r_\infty(\mathscr{S}(I_\infty) \otimes \mathscr{R}_1)$.*

Proof. Suppose first that $0 \leqq x \leqq 1$. Let $\mathscr{A}$ act on the Hilbert space $\mathfrak{K}_0$ and let $t_0 \in \mathfrak{K}_0$ be a trace vector for $\mathscr{A}$. We will prove, by induction, the following: there exists a sequence $\{J_n : n \in \mathscr{N}\}$ of pairwise disjoint, finite subsets of I, and for each $n \in \mathscr{N}$, there exist a finite-dimensional Hilbert space $\mathfrak{G}_n$, a finite type I factor $\mathscr{G}_n$ on $\mathfrak{G}_n$, a $q_n \in \mathfrak{G}_n$ such that q_n is a trace vector for $\mathscr{G}_n$, disjoint subsets K_{n1} and K_{n2} of $\operatorname{Sp}(v(J_n) \otimes q_n, \mathscr{S}(J_n) \otimes \mathscr{G}_n)$ and a bijection $\varphi_n : K_{n1} \to K_{n2}$ such that $0 \notin K_{n1}$, $\sum \lambda(\lambda \in K_{n1}) \geqq \frac{1}{4}$, and $\max\{|x - \varphi_n(\lambda)/\lambda| : \lambda \in K_{n1}\} < 1/n$.

Suppose that $n \in \mathscr{N}$ and that the $J_1, \ldots, J_{n-1}$ have been chosen as required. Let $K = \bigcup_{k=1}^{n-1} J_k$ (K is empty when $n = 1$). Let $\omega = \omega_z$ with $z = t_0 \otimes w(K) \otimes$

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$v(I - K)$. Then ω is a normal PLF on $\mathscr{A} \otimes \mathscr{S}$, and by hypothesis, there exists a $U_{n0} \in \mathscr{A} \otimes \mathscr{S}$ such that the pair (ω, U_{n0}) has property $((48n)^{-1}, \Lambda_x)$ for $\mathscr{A} \otimes \mathscr{S}$. We apply Lemma 3.10 with $\epsilon = (48n)^{-1}$, $\mathscr{B} = \mathscr{A} \otimes \mathscr{S}(K)$, $\mathscr{S}(I - K)$ in place of $\mathscr{S}$ and $U = U_{n0}$. Thus, there exist a finite subset J_n of $I - K$ and a $U_{n1} \in \mathscr{A} \otimes \mathscr{S}(K) \otimes \mathscr{S}(J_n) \otimes \mathbf{1}(\mathfrak{H}(I - K - J_n))$ such that the pair (ω, U_{n1}) has property $((24n)^{-1}, \Lambda_x)$ for $\mathscr{A} \otimes \mathscr{S}$. We now apply Lemma 3.14 with $\epsilon = (24n)^{-1}$, $\mathscr{B}$ and $\mathscr{S}$ as above, $J = J_n$, $t = t_0 \otimes w(K)$, and $U = U_{n1}$. Therefore, there exist $\mathfrak{G}_n$, $\mathscr{G}_n$, q_n, K_{n1}, K_{n2}, and φ_n as required.

For each $n \in \mathscr{N}$, let $I_n = J_n \cup \{n\}$. Let

$$\mathscr{G} = \otimes (\mathfrak{G}_n, \mathscr{G}_n, q_n : n \in \mathscr{N}).$$

It is straightforward to show that $\otimes q_n$ is a trace vector for $\mathscr{G}$, and thus that $\mathscr{G}$ is a hyperfinite, finite factor on a separable Hilbert space. Similarly, $\mathscr{R}_1$ is a hyperfinite II_1 factor on a separable Hilbert space, and so, $\mathscr{G} \otimes \mathscr{R}_1 \cong \mathscr{R}_1$ [**14**, p. 760, Theorem XI and p. 778, Theorem XII].

Let $I_\infty = \cup_{n=1}^{\infty} J_n$. It is clear that $(I_n, K_{n1}, K_{n2}, \varphi_n : n \in \mathscr{N})$ is an x-sequence for $\mathscr{S}(I_\infty) \otimes \mathscr{G}$, which is a countable tensor product of finite type I factors. Hence, by Theorem 3.8, $x \in r_\infty(\mathscr{S}(I_\infty) \otimes \mathscr{G}) \subseteq r_\infty(\mathscr{S}(I_\infty) \otimes \mathscr{G} \otimes \mathscr{R}_1) = r_\infty(\mathscr{S}(I_\infty) \otimes \mathscr{R}_1)$.

If $x > 1$, then by Corollary 3.3 $\mathscr{A} \otimes \mathscr{S}$ has property $\Lambda_{1/x}$. By the above, x^{-1} and hence x lie in $r_\infty(\mathscr{S}(I_\infty) \otimes \mathscr{R}_1)$, for some countable subset I_∞ of I.

4. The main result.

THEOREM 4.1. *Suppose that $\mathscr{A}$ is a von Neumann algebra that is not purely infinite, and that $\mathscr{M} = \otimes (\mathfrak{H}_\alpha, \mathscr{M}_\alpha, z_\alpha : \alpha \in J)$ with $\mathscr{M}_\alpha$ a finite type I factor on $\mathfrak{H}_\alpha$ for each $\alpha \in J$.*

(a) *Suppose that $x \geqq 0$ and that $\mathscr{A} \otimes \mathscr{M}$ has property Λ_x. There then exists a countable subset $J(x)$ of J such that $x \in r_\infty(\mathscr{M}(J(x)) \otimes \mathscr{R}_0 \otimes \mathscr{R}_1)$.*

(b) *Suppose that $0, 1 \in r_\infty(\mathscr{M})$. Then there exists a countable subset J_0 of J such that J_0 is independent of $\mathscr{A}$, and*

$$\begin{aligned} r_\infty(\mathscr{A} \otimes \mathscr{M}) &= \Lambda'(\mathscr{A} \otimes \mathscr{M}) = \Lambda(\mathscr{A} \otimes \mathscr{M}) = r_\infty(\mathscr{M}) \\ &= \Lambda'(\mathscr{M}) = \Lambda(\mathscr{M}) = r_\infty(\mathscr{M}(J_0)) = \Lambda'(\mathscr{M}(J_0)) = \Lambda(\mathscr{M}(J_0)). \end{aligned}$$

LEMMA 4.2. *Suppose that $x \geqq 0$, that I is an index set, and that for each $i \in I$, $\mathscr{A}_i$ is a von Neumann algebra. Let $\mathscr{A} = \sum \oplus \mathscr{A}_i$ $(i \in I)$. Then $\mathscr{A}$ has property Λ_x if and only if for each $i \in I$, $\mathscr{A}_i$ has property Λ_x.*

Proof. Suppose that $\mathscr{A}$ has property Λ_x. Choose any $j \in I$, any $\epsilon > 0$, and any normal PLF ω on $\mathscr{A}_j$. Let ρ be the normal PLF on $\mathscr{A}$ defined as follows: if $T \in \mathscr{A}$, then $T = \sum \oplus T_i$ with $T_i \in \mathscr{A}_i$ for each $i \in I$, and let $\rho(T) = \omega(T_j)$. Then, there exists a $U \in \mathscr{A}$ such that the pair (ρ, U) has property (ϵ, Λ_x) for $\mathscr{A}$. $U = \sum \oplus U_i$ with $U_i \in \mathscr{A}_i$ for each $i \in I$. It is clear that the pair (ω, U_j) has property (ϵ, Λ_x) for $\mathscr{A}_j$. Hence, $\mathscr{A}_j$ has property Λ_x.

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Conversely, suppose that each $\mathscr{A}_i$ has property Λ_x. Choose any $\epsilon > 0$ and any normal PLF ω on $\mathscr{A}$. For each $i \in I$, we may consider $\mathscr{A}_i$ to be a subset of $\mathscr{A}$, and we define ω_i to be the restriction of ω to $\mathscr{A}_i$. Then ω_i is a normal PLF on $\mathscr{A}_i$. If $T \in \mathscr{A}$, then $T = \sum \oplus T_i$ with $T_i \in \mathscr{A}_i$ for each $i \in I$, and $\omega(T) = \sum \omega_i(T_i)$. In particular, $\omega(1) = \sum \omega_i(1)$, and hence, since $\|\omega_i\| = \omega_i(1)$, at most a countable number of the ω_i are non-zero, which, we may assume, occurs only for $i \in \mathscr{N}_1 \subseteq \mathscr{N} \cap I$. For each $k \in \mathscr{N}_1$, there exists a $U_k \in \mathscr{A}_k$ such that the pair (ω_k, U_k) has property $(\epsilon 2^{-k}, \Lambda_x)$ for $\mathscr{A}_k$. Let $U = \sum \oplus U_k \quad (k \in \mathscr{N}_1)$. Then it is easy to show that the pair (ω, U) has property (ϵ, Λ_x) for $\mathscr{A}$. Hence, $\mathscr{A}$ has property Λ_x.

Remark. The above proof can be modified easily to show that $\mathscr{A}$ has property Λ_x' if and only if for each $i \in I, \mathscr{A}_i$ has property Λ_x'.

LEMMA 4.3. *Suppose that $\mathscr{A}$ is a countably decomposable, finite von Neumann algebra. Then there exists a von Neumann algebra $\mathscr{A}_1$ with a trace vector, such that $\mathscr{A} \cong \mathscr{A}_1$.*

Proof. Suppose that $\mathscr{A}$ acts on the Hilbert space $\mathfrak{R}$. Then there exists a faithful, normal, normalized, finite trace (tr) on $\mathscr{A}$ [**8**, p. 99, Proposition 9(ii)], and there exists a sequence $x_1, x_2, \ldots \in \mathfrak{R}$ such that $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$ and for each $T \in \mathscr{A}$, tr $(T) = \sum_{n=1}^{\infty} (Tx_n, x_n)$ [**8**, p. 51, Théorème 1]. Let $\mathfrak{R}_2$ be a Hilbert space with orthonormal basis $\{e_n : n \in \mathscr{N}\}$. Let $\mathfrak{R}_1 = \mathfrak{R} \otimes \mathfrak{R}_2$, let $\mathscr{A}_1 = \mathscr{A} \otimes \mathbf{1}(\mathfrak{R}_2)$, and let $t = \sum_{n=1}^{\infty} (x_n \otimes e_n)$. Then $t \in \mathfrak{R}_1$, t is a trace vector for $\mathscr{A}_1$ and $\mathscr{A} \cong \mathscr{A}_1$.

LEMMA 4.4. *Suppose that $\mathscr{A}$ is a von Neumann algebra that is not purely infinite. Then $\mathscr{A} \cong (\mathscr{A}_1 \otimes \mathscr{B}(\mathfrak{R})) \oplus \mathscr{D}$ where $\mathscr{A}_1$ is a von Neumann algebra with a trace vector, $\mathfrak{R}$ is a Hilbert space, and $\mathscr{D}$ is a (possibly zero) von Neumann algebra.*

Proof. There exist e_1, e_2, e_3 orthogonal, central projections in $\mathscr{A}$ such that $1 = e_1 \oplus e_2 \oplus e_3$, e_1 is finite, e_2 is properly infinite and semi-finite, and e_3 is purely infinite. By hypothesis, $e_3 \neq 1$, i.e., $e_1 \oplus e_2 \neq 0$.

We claim that there exists a non-zero, central projection e in $\mathscr{A}$, a finite von Neumann algebra $\mathscr{G}$, and a Hilbert space $\mathfrak{R}$ such that $\mathscr{A}_e \cong \mathscr{G} \otimes \mathscr{B}(\mathfrak{R})$. If $e_1 \neq 0$, then this follows if we let $e = e_1$, let $\mathscr{G} = \mathscr{A}_e$, and let $\mathfrak{R}$ be a one-dimensional Hilbert space. If $e_2 \neq 0$, then this follows from [**8**, p. 242, Exercice 5(a), (d)]. It follows from [**8**, p. 99, Proposition 9(iii)] that there exists a non-zero, central projection p in $\mathscr{G}$ such that $\mathscr{G}_p$ is finite and countably decomposable. By Lemma 4.3, there exists a von Neumann algebra $\mathscr{A}_1$ with a trace vector such that $\mathscr{G}_p \cong \mathscr{A}_1$. The result now follows if we let

$$\mathscr{D} = (\mathscr{G}_{1-p} \otimes \mathscr{B}(\mathfrak{R})) \oplus \mathscr{A}_{1-e}.$$

THEOREM 4.5. *Suppose that $\mathscr{A}$ is a von Neumann algebra.*

(a) *Suppose that* $x \geqq 0$, *that* $\mathscr{A}$ *is finite, and that* $\mathscr{A}$ *has property* Λ_x. *Then* $x = 1$.

(b) *The following are equivalent:* (i) $\mathscr{A}$ *is properly infinite,* (ii) $0 \in r_\infty(\mathscr{A})$, (iii) $0 \in \Lambda'(\mathscr{A})$, *and* (iv) $0 \in \Lambda\ (\mathscr{A})$.

(c) *Suppose that* $x > 0$, $x \neq 1$ *and that* $\mathscr{A}$ *has property* Λ_x. *Then* $\mathscr{A}$ *is purely infinite.*

Proof. (a). Let ω be a finite, normalized, normal trace on $\mathscr{A}$. Choose any $\epsilon > 0$. Then there exists a $U \in \mathscr{A}$ such that $U^*U + UU^* = 1$ and $|\omega(UT) - x\omega(TU)| \leqq (\epsilon/2)||T||$, for any $T \in \mathscr{A}$. Hence, $\omega(U^*U) = \omega(UU^*) = \frac{1}{2}$, and letting $T = U^*$, we have that $|1 - x| \leqq \epsilon$. Thus, $x = 1$.

(b). (i) $\Rightarrow$ (ii): It follows from [**8**, p. 25, Proposition 5(ii) and p. 298, Corollaire 2] that there exists a projection e in $\mathscr{A}$, equivalent to 1, such that $\mathscr{A} \cong \mathscr{A}_e \otimes \mathscr{B}(l_2(\mathscr{N})) \cong \mathscr{A} \otimes \mathscr{R}_0$. Hence, $0 \in r_\infty(\mathscr{A})$. (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (iv) by Corollary 3.6. (iv) $\Rightarrow$ (i): There exist central projections e and f in $\mathscr{A}$ such that $1 = e \oplus f$, e is finite, and f is properly infinite. Then $\mathscr{A} = \mathscr{A}_e \oplus \mathscr{A}_f$. If $e \neq 0$, then, by Lemma 4.2, $\mathscr{A}_e$ has property Λ_0. However, this contradicts part (a), and hence, $e = 0$, $f = 1$, and $\mathscr{A}$ is properly infinite.

(c). Assume that $\mathscr{A}$ is not purely infinite. Apply Theorem 4.1(a) with $J = \{1\}$ and $\mathscr{M}_1$ a type I_1 factor. Then $x \in r_\infty(\mathscr{R}_0 \otimes \mathscr{R}_1) = S_{01}$. This is a contradiction and hence, $\mathscr{A}$ is purely infinite.

Remark. Part (c) and Theorem 3.5 show that for $x > 0$ and $x \neq 1$, $\mathscr{R}_x$ is a type *III* factor. This was first shown by von Neumann [**16**] and Pukánszky [**19**]. Part (c) will not be used in the following.

Proof of Theorem 4.1. We may assume that J is disjoint from $\mathscr{N}$. For each $\alpha \in J$, $\mathscr{M}_\alpha$ is a type $I_{n(\alpha)}$ factor on $\mathfrak{H}_\alpha$ for some $n(\alpha) \in \mathscr{N}$. Hence, there exist Hilbert spaces $\mathfrak{H}_{\alpha 1}$ and $\mathfrak{K}_\alpha$ with orthonormal bases $\{\varphi_{\alpha i}: i = 1, 2, \ldots, n(\alpha)\}$ and $\{\chi_{\alpha i}: i \in N_\alpha\}$, respectively, for some index set N_α, a $p(\alpha) \in \mathscr{N}$ with $p(\alpha) \leqq \min\{n(\alpha), \text{card } N_\alpha\}$, and real numbers $\lambda_{\alpha 1} \geqq \lambda_{\alpha 2} \geqq \ldots \geqq \lambda_{\alpha p(\alpha)} > 0$ such that $\mathfrak{H}_\alpha = \mathfrak{H}_{\alpha 1} \otimes \mathfrak{K}_\alpha$, $\mathscr{M}_\alpha = \mathscr{B}(\mathfrak{H}_{\alpha 1}) \otimes \mathbf{1}(\mathfrak{K}_\alpha)$, $\{1, 2, \ldots, p(\alpha)\} \subseteq N_\alpha$, and $z_\alpha = \sum_{i=1}^{p(\alpha)} (\lambda_{\alpha i})^{\frac{1}{2}} \varphi_{\alpha i} \otimes \chi_{\alpha i}$. Let $\mathfrak{H}_{\alpha 2}$ be a Hilbert space of dimension $n(\alpha)$ with orthonormal basis $\{\Psi_{\alpha i}: i = 1, 2, \ldots, n(\alpha)\}$, let $\lambda_{\alpha i} = 0$ if $p(\alpha) < i \leqq n(\alpha)$, and let

$$v_\alpha = \sum_{i=1}^{n(\alpha)} (\lambda_{\alpha i})^{\frac{1}{2}} \varphi_{\alpha i} \otimes \Psi_{\alpha i}.$$

For any $T \in \mathscr{B}(\mathfrak{H}_{\alpha 1})$,

$$(4.1) \qquad ((T \otimes 1(\mathfrak{K}_\alpha))z_\alpha, z_\alpha) = ((T \otimes 1(\mathfrak{H}_{\alpha 2}))v_\alpha, v_\alpha).$$

By Lemma 4.4, $\mathscr{A} \cong (\mathscr{A}_1 \otimes \mathscr{B}(\mathfrak{K})) \oplus \mathscr{D}$ where $\mathscr{A}_1$ is a von Neumann algebra with a trace vector and $\mathfrak{K}$ is a Hilbert space. Let I_1 be an index set disjoint from J and $\mathscr{N}$ such that card $I_1 = \dim \mathfrak{K}$. Let $\mathscr{N}_1$ be a countably infinite index set disjoint from J, I_1, and $\mathscr{N}$, and let $I = J \cup I_1 \cup \mathscr{N}_1$. For each $k \in I_1 \cup \mathscr{N}_1$ let $\mathfrak{H}_{k1}$ and $\mathfrak{H}_{k2}$ be two-dimensional Hilbert spaces and choose $v_k \in \mathfrak{H}_{k1} \otimes \mathfrak{H}_{k2}$ such that Sp $(v_k, \mathscr{B}(\mathfrak{H}_{k1}) \otimes \mathbf{1}(\mathfrak{H}_{k2})) = \{1, 0\}$. Let $\mathscr{S} = \otimes (\mathfrak{H}_{\alpha 1} \otimes \mathfrak{H}_{\alpha 2}, \mathscr{B}(\mathfrak{H}_{\alpha 1}) \otimes \mathbf{1}(\mathfrak{H}_{\alpha 2}), v_\alpha : \alpha \in I)$. Then, using (4.1) and [**5**, Corollary 3.5], we have that if K is any subset of J then $\mathscr{S}(K) \cong \mathscr{M}(K)$. $\mathscr{S}(I_1) \cong \mathscr{B}(\mathfrak{K})$ and $\mathscr{S}(\mathscr{N}_1) \cong \mathscr{R}_0$ [**4**, Proposition 5.3]. Hence, $\mathscr{S} \cong \mathscr{M} \otimes \mathscr{B}(\mathfrak{K}) \otimes \mathscr{R}_0$. Also, $\mathscr{S}$ satisfies the conditions imposed on the $\mathscr{S}$ of Theorem 3.15. We are now prepared to prove (a) and (b).

(a) We are assuming that $\mathscr{A} \otimes \mathscr{M}$ has property Λ_x.

$$\mathscr{A} \otimes \mathscr{M} \cong (\mathscr{A}_1 \otimes \mathscr{B}(\mathfrak{K}) \otimes \mathscr{M}) \oplus (\mathscr{D} \otimes \mathscr{M}),$$

and hence, by Lemma 4.2, $\mathscr{A}_1 \otimes \mathscr{B}(\mathfrak{K}) \otimes \mathscr{M}$ has property Λ_x. If $\mathscr{B}(\mathfrak{K}) \otimes \mathscr{M}$ is finite, then so is $\mathscr{A}_1 \otimes \mathscr{B}(\mathfrak{K}) \otimes \mathscr{M}$, and by Theorem 4.5 (a), $x = 1$, and hence, $x = 1 \in r_\infty(\mathscr{R}_1) \subseteq r_\infty(\mathscr{M}(J(x)) \otimes \mathscr{R}_0 \otimes \mathscr{R}_1)$ for any subset $J(x)$ of J. If $\mathscr{B}(\mathfrak{K}) \otimes \mathscr{M}$ is infinite, then by Theorem 4.5 (b), $0 \in r_\infty(\mathscr{B}(\mathfrak{K}) \otimes \mathscr{M})$ and hence,

$$\mathscr{A}_1 \otimes \mathscr{B}(\mathfrak{K}) \otimes \mathscr{M} \cong \mathscr{A}_1 \otimes \mathscr{B}(\mathfrak{K}) \otimes \mathscr{M} \otimes \mathscr{R}_0 \cong \mathscr{A}_1 \otimes \mathscr{S}$$

and so $\mathscr{A}_1 \otimes \mathscr{S}$ has property Λ_x. Therefore, by Theorem 3.15, there exists a countable subset $I(x)$ of I such that $x \in r_\infty(\mathscr{S}(I(x)) \otimes \mathscr{R}_1)$. Let $J(x) = I(x) \cap J$, and let $I_2 = (I(x) \cap I_1) \cup \mathscr{N}_1$. Then $I(x) \subseteq J(x) \cup I_2$, $J(x)$ is countable, $\mathscr{S}(J(x)) \cong \mathscr{M}(J(x))$, and $\mathscr{S}(I_2) \cong \mathscr{R}_0$. Therefore,

$$x \in r_\infty(\mathscr{M}(J(x)) \otimes \mathscr{R}_0 \otimes \mathscr{R}_1).$$

(b) We are assuming that $0, 1 \in r_\infty(\mathscr{M})$. We will first show that there exists a countable subset K_0 of J such that $0, 1 \in r_\infty(\mathscr{M}(K_0))$.

Let $K_1 = \{\alpha \in J : n(\alpha) \geqq 2\}$. Since $n(\alpha) = 1$ if and only if $\mathscr{M}_\alpha = \mathbf{1}(\mathfrak{H}_\alpha)$, it follows that $\mathscr{M} \cong \mathscr{M}(K_1) \cong \mathscr{S}(K_1)$. K_1 is infinite, for otherwise, $\mathscr{M}$ would be a finite type I factor and $r_\infty(\mathscr{M})$ would be empty.

Assume that for every countably infinite subset K of K_1 that $0 \notin r_\infty(\mathscr{S}(K))$. By [**3**, Lemma 3.8], $r_\infty(\mathscr{S}(K))$ is non-empty. Since $0, 1 \in r_\infty(\mathscr{R}_x)$ for any x with $x > 0$, $x \neq 1$, it follows that $r_\infty(\mathscr{S}(K)) = \{1\}$. Therefore, $\mathscr{S}(K) \cong \mathscr{R}_1$ [**3**, Theorem 9.1], a II_1 factor, and hence, by [**23**, Theorem],

$$\sum_\alpha \left\{ \sum_{i=1}^{n(\alpha)} (n(\alpha)^{-\frac{1}{2}} - \lambda_{\alpha i}^{\frac{1}{2}})^2 \right\} < \infty \quad (\alpha \in K).$$

Since this is true for every countably infinite subset K of K_1, it follows that the above sum taken over $\alpha \in K_1$ is finite. Therefore, by [**4**, Proposition 5.4], $\mathscr{S}(K_1)$ is a II_1 factor, and hence, $0 \notin r_\infty(\mathscr{S}(K_1)) = r_\infty(\mathscr{M})$. This is a contradiction, and so there exists a countable subset K_2 of J such that $0 \in r_\infty(\mathscr{S}(K_2))$.

Assume that for every countably infinite subset K of K_1 that $1 \notin r_\infty(\mathscr{S}(K))$. Then by [**3**, Lemma 3.8], $\sum |1 - \lambda_{\alpha 1}|$ $(\alpha \in K) < \infty$. Hence,

$$\sum |1 - \lambda_{\alpha 1}| \ (\alpha \in K_1) < \infty,$$

and by [**4**, Proposition 5.3], $\mathscr{S}(K_1)$ is a type I factor. Therefore, $1 \notin$

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$r_\infty(\mathscr{S}(K_1)) = r_\infty(\mathscr{M})$. This is a contradiction, and so there exists a countable subset K_3 of J such that $1 \in r_\infty(\mathscr{S}(K_3))$. Therefore, $0, 1 \in r_\infty(\mathscr{M}(K_0))$ if $K_0 = K_2 \cup K_3$.

By Corollary 3.6, $r_\infty(\mathscr{A} \otimes \mathscr{M}) \subseteq \Lambda'(\mathscr{A} \otimes \mathscr{M}) \subseteq \Lambda(\mathscr{A} \otimes \mathscr{M})$. Suppose that $x \in \Lambda(\mathscr{A} \otimes \mathscr{M})$. By part (a) above, there exists a countable subset $J(x)$ of J such that $x \in r_\infty(\mathscr{M}(J(x)) \otimes \mathscr{R}_0 \otimes \mathscr{R}_1)$. Let $K(x) = J(x) \cup K_0$. Then $x \in r_\infty(\mathscr{M}(K(x)) \otimes \mathscr{R}_0 \otimes \mathscr{R}_1) = r_\infty(\mathscr{M}(K(x))) \subseteq r_\infty(\mathscr{M}) \subseteq r_\infty(\mathscr{A} \otimes \mathscr{M})$.

This shows that $r_\infty(\mathscr{A} \otimes \mathscr{M}) = \Lambda'(\mathscr{A} \otimes \mathscr{M}) = \Lambda(\mathscr{A} \otimes \mathscr{M}) = r_\infty(\mathscr{M})$. Letting $\mathscr{A}$ be a type I_1 factor, we have that $r_\infty(\mathscr{M}) = \Lambda'(\mathscr{M}) = \Lambda(\mathscr{M})$. If J_0 is any subset of J with $J_0 \supseteq K_0$, then $0, 1 \in r_\infty(\mathscr{M}(J_0))$ and $\mathscr{M}(J_0)$ satisfies the conditions of Theorem 4.1(b). Therefore, $r_\infty(\mathscr{M}(J_0)) = \Lambda'(\mathscr{M}(J_0)) = \Lambda(\mathscr{M}(J_0))$. Thus, it remains to show that there exists a countable subset J_0 of J such that $J_0 \supseteq K_0$ and $r_\infty(\mathscr{M}) = r_\infty(\mathscr{M}(J_0))$.

There exists a countable set of numbers $\{y_n : n \in \mathscr{N}\}$ contained in $r_\infty(\mathscr{M})$, and whose closure contains $r_\infty(\mathscr{M})$. For each $n \in \mathscr{N}$, $y_n \in r_\infty(\mathscr{M}) \subseteq \Lambda(\mathscr{M})$. Thus, by the above (with $\mathscr{A}$ a type I_1 factor), there exists a countable subset $K(y_n)$ of J such that $K(y_n) \supseteq K_0$ and $y_n \in r_\infty(\mathscr{M}(K(y_n)))$. Let

$$J_0 = \cup_{n=1}^{\infty} K(y_n).$$

Then, each $y_n \in r_\infty(\mathscr{M}(J_0))$ which is closed by [**3**, Lemma 3.7, Theorem 5.9]. Therefore,

$$r_\infty(\mathscr{M}) \subseteq \text{closure } \{y_n : n \in \mathscr{N}\} \subseteq r_\infty(\mathscr{M}(J_0)) \subseteq r_\infty(\mathscr{M}).$$

Remark. Both parts of Theorem 4.1 fail if $\mathscr{A}$ is purely infinite, and part (b) fails if $0 \notin r_\infty(\mathscr{M})$ of if $1 \notin r_\infty(\mathscr{M})$ as evidenced by the following: let $0 < x < 1$; then $\mathscr{R}_\infty \otimes \mathscr{R}_x$ has property Λ_y for every $y \geqq 0$, but

$$r_\infty(\mathscr{R}_x \otimes \mathscr{R}_0 \otimes \mathscr{R}_1) = S_x \neq S_\infty; r_\infty(\mathscr{R}_\infty \otimes \mathscr{R}_x) = S_\infty \neq r_\infty(\mathscr{R}_x) = S_x;$$
$$r_\infty(\mathscr{R}_0 \otimes \mathscr{R}_1) = S_{01} \neq r_\infty(\mathscr{R}_1) = S_1; r_\infty(\mathscr{R}_1 \otimes \mathscr{R}_0) = S_{01} \neq r_\infty(\mathscr{R}_0) = S_0.$$

COROLLARY 4.6. *$r_\infty(\mathscr{M})$ is closed for $\mathscr{M}$ an arbitrary tensor product of finite type I factors.*

5. Non-hyperfinite factors.

Definition 5.1. A von Neumann algebra $\mathscr{A}$ is said to be *hyperfinite* if there exists a sequence $\{\mathscr{A}_n : n \in \mathscr{N}\}$ of von Neumann sub-algebras of $\mathscr{A}$ such that for each $n \in \mathscr{N}$, $\mathscr{A}_n$ is finite-dimensional as a linear space and $\mathscr{A}_n \subseteq \mathscr{A}_{n+1}$, and the von Neumann algebra generated by $\{\mathscr{A}_n : n \in \mathscr{N}\}$ is $\mathscr{A}$.

Definition 5.2. A von Neumann algebra $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$ is said to have *property* AP if there exists a linear projection of norm one from $\mathscr{B}(\mathfrak{H})$ onto $\mathscr{A}'$ (the commutant of $\mathscr{A}$).

Let Φ_2 be the free group with two generators, and let $\mathscr{A}(\Phi_2)$ be the von Neumann algebra generated by the left regular representation of Φ_2. $\mathscr{A}(\Phi_2)$

acts on the separable Hilbert space $l_2(\Phi_2)$, and both $\mathscr{A}(\Phi_2)$ and its commutant are type II_1 factors [**14**, Lemmas 5.3.4, 5.3.5, 6.2.2].

LEMMA 5.3. *Suppose that $\mathscr{R}$ is any von Neumann algebra. Then $\mathscr{A}(\Phi_2) \otimes \mathscr{R}$ is non-hyperfinite.*

Proof. Suppose that $\mathscr{A}(\Phi_2) \otimes \mathscr{R}$ is hyperfinite. Then it has property AP [**22**, pp. 168-171]. Therefore, $\mathscr{A}(\Phi_2)$ has property AP [**9**, Theorem 3.2], i.e., there exists a linear projection of norm one, φ, from $\mathscr{B}(l_2(\Phi_2))$ onto $\mathscr{A}(\Phi_2)'$. Therefore, for any $T \in \mathscr{B}(l_2(\Phi_2))$ and any $A \in \mathscr{A}(\Phi_2)'$, $\varphi(AT) = A\varphi(T)$, $\varphi(TA) = \varphi(T)A$, $\varphi(T^*) = \varphi(T)^*$, and if $T \geqq 0$ then $\varphi(T) \geqq 0$ [**10**, p. 330, proof of Lemma 8, p. 331 (bottom); **24**, Theorem 1]. From this, it follows that Φ_2 admits a finite, non-zero, non-negative, finitely additive, right invariant measure [**22**, pp. 171, 172, proof of Lemma 3]; however, this is impossible [**22**, p. 172 (bottom)].

Definition 5.4. For any von Neumann algebra $\mathscr{A}$, define

$$\rho(\mathscr{A}) = \{0 \leqq x \leqq 1 : \mathscr{R}_x \cong \mathscr{R}_x \otimes \mathscr{A}\}.$$

This was defined by Araki and Woods [**3**, Definition 11.1] where they used it to distinguish factors in the S_{01} class.

THEOREM 5.5. *Suppose that for $i = 1, 2$, $\mathscr{A}_i$ is a von Neumann algebra that is not purely infinite, $\mathscr{M}_i$ is a tensor product of finite type I factors indexed by a set J_i, and $r_\infty(\mathscr{M}_i) \supseteq \{0, 1\}$. Suppose that either*

(a) *$r_\infty(\mathscr{M}_1) \neq r_\infty(\mathscr{M}_2)$, or*

(b) *$\rho(\mathscr{M}_1) \neq \rho(\mathscr{M}_2)$ and J_1 and J_2 are countable.*

Then $\mathscr{A}_1 \otimes \mathscr{M}_1 \ncong \mathscr{A}_2 \otimes \mathscr{M}_2$.

Proof. (a) By Theorem 4.1 (b),

$$r_\infty(\mathscr{A}_1 \otimes \mathscr{M}_1) = r_\infty(\mathscr{M}_1) \neq r_\infty(\mathscr{M}_2) = r_\infty(\mathscr{A}_2 \otimes \mathscr{M}_2).$$

(b) Suppose, if possible, that $\mathscr{A}_1 \otimes \mathscr{M}_1 \cong \mathscr{A}_2 \otimes \mathscr{M}_2$. Choose any $x \in \rho(\mathscr{M}_1)$. Then $\mathscr{M}_1 \otimes \mathscr{R}_x \cong \mathscr{R}_x$. Hence, using Theorem 4.1 (b),

$$\begin{aligned} r_\infty(\mathscr{M}_2 \otimes \mathscr{R}_x) &= r_\infty(\mathscr{A}_2 \otimes \mathscr{M}_2 \otimes \mathscr{R}_x) = r_\infty(\mathscr{A}_1 \otimes \mathscr{M}_1 \otimes \mathscr{R}_x) \\ &= r_\infty(\mathscr{M}_1 \otimes \mathscr{R}_x) = r_\infty(\mathscr{R}_x) = S_x. \end{aligned}$$

Therefore, $\mathscr{M}_2 \otimes \mathscr{R}_x \cong \mathscr{R}_x$ [**3**, Theorem 9.1], and $x \in \rho(\mathscr{M}_2)$. By symmetry, we have that $\rho(\mathscr{M}_1) = \rho(\mathscr{M}_2)$, a contradiction.

THEOREM 5.6. (a) *$\{\mathscr{A}(\Phi_2) \otimes \mathscr{R}_x : 0 < x < 1\}$ is a continuum of pairwise non-isomorphic, non-hyperfinite, type III factors on a separable Hilbert space. $r_\infty(\mathscr{A}(\Phi_2) \otimes \mathscr{R}_x) = S_x$.*

(b) *There exists a continuum of pairwise non-isomorphic, non-hyperfinite, type III factors on a separable Hilbert space, each one having its r_∞ set equal to S_{01}.*

Proof. (a) For any x with $0 < x < 1$, $\mathscr{A}(\Phi_2) \otimes \mathscr{R}_x$ is a non-hyperfinite type III

factor (Theorems 3.5, 4.5 (c) and Lemma 5.3), and $r_\infty(\mathscr{A}(\Phi_2) \otimes \mathscr{R}_x) = r_\infty(\mathscr{R}_x) = S_x$ (Theorem 4.1 (b)). The result now follows from Theorem 5.5 (a).

(b) Araki and Woods have constructed a family $\{\mathscr{S}_k : 0 \leqq k \leqq 1\}$ of type *III* factors on a separable Hilbert space such that for each $k \in [0, 1]$, $\mathscr{S}_k$ is a tensor product of type I_2 factors, and hence is hyperfinite,

$$r_\infty(\mathscr{S}_k) = r_\infty(\mathscr{S}_k \otimes \mathscr{S}_k) = S_{01},$$

and for any j, $k \in [0, 1]$, $e^{j-k} \in r_\infty(\mathscr{S}_j \otimes \mathscr{S}_k)$ [**3**, Lemma 10.1, proof of Theorem 10.10].

We claim that the family $\{\mathscr{A}(\Phi_2) \otimes \mathscr{S}_k : 0 \leqq k \leqq 1\}$ satisfies the conditions of this theorem. Using Theorem 4.1 (b), we see that for any $k \in [0, 1]$, $\mathscr{A}(\Phi_2) \otimes \mathscr{S}_k$ is non-hyperfinite (Lemma 5.3), $r_\infty(\mathscr{A}(\Phi_2) \otimes \mathscr{S}_k) = r_\infty(\mathscr{S}_k) = S_{01}$, and $r_\infty(\mathscr{A}(\Phi_2) \otimes \mathscr{S}_k \otimes \mathscr{A}(\Phi_2) \otimes \mathscr{S}_k) = r_\infty(\mathscr{S}_k \otimes \mathscr{S}_k) = S_{01}$. If j, $k \in [0, 1]$ with $j \neq k$, then $e^{j-k} \notin S_{01}$, but

$$e^{j-k} \in r_\infty(\mathscr{S}_j \otimes \mathscr{S}_k) \subseteq r_\infty(\mathscr{A}(\Phi_2) \otimes \mathscr{S}_j \otimes \mathscr{A}(\Phi_2) \otimes \mathscr{S}_k).$$

Therefore, $\mathscr{A}(\Phi_2) \otimes \mathscr{S}_j \ncong \mathscr{A}(\Phi_2) \otimes \mathscr{S}_k$.

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University of Manitoba,
Winnipeg, Manitoba.