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LIFTING THE COMMUTANT OF A SUBNORMAL OPERATOR

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Introduction. Let S be a subnormal operator on a Hilbert space $\mathscr{H}$ and let N be its minimal normal extension on the Hilbert space $\mathscr{K}$. (We refer the reader to [**5**, 15] for the basic material on subnormal operators.) Denote the commutant and double commutant of an operator T by $\{T\}'$ and $\{T\}''$, respectively.

One of the questions J. Bram considered in [**5**] is the following: does the commutant of S lift to the commutant of N? That is, if an operator A belongs to $\{S\}'$, does there exist an operator A_0 in $\{N\}'$ such that A_0 leaves $\mathscr{H}$ invariant and the restriction of A_0 to $\mathscr{H}$, denoted $A_0|_{\mathscr{H}}$, equals A? His answer appeared as follows:

THEOREM (Bram). *Let S be a subnormal operator on $\mathscr{H}$. Then necessary and sufficient conditions that an operator A on $\mathscr{H}$ have an extension A_0 on $\mathscr{K}$ such that A_0 commutes with N are that*

a) *A commutes with S, and*

b) *There exists a positive constant c such that for every finite set $x_0, x_1, \ldots x_r$ in $\mathscr{H}$ we have*

$$\sum_{m,n=0}^{r} (S^m A x_n, S^n A x_m) \leqq c \sum_{m,n=0}^{r} (S^m x_n, S^n x_m).$$

In the remark immediately after this theorem Bram states (without giving a proof) that condition (a) does not imply condition (b). [Later, Yoshino [**23**] did show that (b) does imply (a).] Based on the remark of Bram one deduces that the commutant of a subnormal operator does not always lift to that of its minimal normal extension. (The first example of this behaviour appears in [**13**]. Others have considered related lifting questions and we refer the reader to the literature [**1**, **2**, **3**, **13**, **16**, **17**, **21**, **22**].)

In the first part of the paper we shall show that this phenomenon occurs frequently. More specifically, let N be a normal operator and denote the collection of subnormal operators that have N as their minimal normal extension by $\mathscr{S}(N)$. We shall characterize (up to unitary equivalence) those normal operators which have the property that every S in $\mathscr{S}(N)$ has a commutant that lifts to the commutant of N. This class of normals is small.

To do this we study the problem of when the linear manifold of polynomials and their conjugates are dense in $L^\infty(\mu)$, for a finite measure μ. This problem

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has been studied extensively in [**11**, **12**]. The solution and technique of proof obtained here are different than those in [**11**, **12**].

In the second part of the paper we exhibit an irreducible subnormal operator S that has a commutative commutant (therefore $\{S\}' = \{S\}''$) that does not lift. This answers two questions by Abrahamse [**2**].

Lifting the commutant of every S in $\mathscr{S}(N)$. If N is a normal operator with scalar spectral measure μ then we say N is *antisymmetric* if the only projections in the weakly closed algebra generated by N and the identity are zero and one. Equivalently, if $P^\infty(\mu)$ denotes the weak-star closure of the polynomials in $L^\infty(\mu)$, then the constants are the only real-valued functions in $P^\infty(\mu)$.

In [**8**] there is a canonical decomposition of any normal operator N as a direct sum of a reductive normal operator and antisymmetric ones. Furthermore the weakly closed algebra generated by N splits with this decomposition. Every S in $\mathscr{S}(N)$ has a decomposition related to that of N and the ultraweakly closed algebra generated by S splits with this decomposition.

If T is an operator such that $T = T_0 \oplus T_1$ and $\mathscr{W}(T) = \mathscr{W}(T_0) \oplus \mathscr{W}(T_1)$ then $\{T\}' = \{T_0\}' \oplus \{T_1\}'$. (Here $\mathscr{W}(T)$ denotes the weakly closed algebra generated by T and the identity. See [**9**] for related results.) Hence, by the above discussion, if we want to classify those normal operators N for which every $S \in \mathscr{S}(N)$ has a commutant that lifts we can assume N is an antisymmetric normal operator.

Before we proceed we need to develop some function theory results about $P^\infty(\mu)$. The notation and terminology is consistent with the work of [**8**].

LEMMA 1. *Let G be a bounded, simply connected open set in the plane. There exists a measure μ in the plane such that G equals a component of* int $\tilde{K}$ *if and only if the conformal map of the unit disk onto G is a weak-star generator of H^∞.*

Here $\tilde{K}$ is the set used in [**20**] to describe $P^\infty(\mu)$. H^∞ denotes the usual Hardy space of bounded analytic functions on the unit disk (or viewed on the unit circle in the usual way). To say that φ is a weak-star generator of H^∞ means that the polynomials in φ are weak-star dense in H^∞.

Proof of Lemma 1. Suppose G is a component of int $\tilde{K}$, for some measure μ. Without loss of generality we may assume $\tilde{K} = \bar{G}$ and $G =$ int $\tilde{K}$, i.e., $P^\infty(\mu)$ is antisymmetric. (See [**8**], Theorems 4.11 and 4.12.) Let $\{K_\alpha\}$ denote the collection of sets defined in section seven of [**20**] and α_0 the least ordinal for which the transfinite induction stops. That is, $K_{\alpha_0} = \tilde{K}$. Now let B be a domain which contains G so that

$$(*) \qquad \sup_{z\in B} |f(z)| = \sup_{z\in G} |f(z)|$$

for every f bounded and analytic on B. Note that these suprema are equal to the $L^\infty(\mu)$ norm of f because $\tilde{K} = \bar{G}$.

Suppose $B\backslash G$ is nonempty. (This will lead to a contradiction. Corollary 2 of [**19**] then shows that the conformal map of the disk to G is a weak-star generator.) Since $B\backslash G$ is nonempty there exists a first countable ordinal β such that $B\backslash \text{ int } K_\beta$ is nonempty. It follows from the definition of the sets K_α that β is not a limit ordinal. Let z belong to $B\backslash\text{int } K_\beta$. Then there exists a function f bounded and analytic on int K_β such that

$$(**) \quad |f(z)| > ||f||_\mu.$$

Since β was chosen to be the first ordinal for which $B\backslash\text{int } K_\beta$ is nonempty, f is analytic on B. Inequality (**) then contradicts equality (*).

Now suppose G is a bounded, simply connected open set in the plane that has the property that the conformal map, φ, of the disk to G is a weak-star generator of H^∞. Let μ be planar measure restricted to G. Since φ is a weak-star generator of H^∞, Corollary 2 of [**19**] implies that there is no domain B containing G properly such that

$$\sup_{z\in B} |f(z)| = \sup_{z\in G} |f(z)|$$

for every function f bounded and analytic in B. Therefore, by definition of the sets K_α with respect to μ, $\tilde{K} = \bar{G}$. Corollary 3 of [**19**] implies $G = \text{int } \tilde{K}$.

The following lemma is due to de Branges ([**11**], Theorem 18). The proof here is different.

LEMMA 2. *Suppose $P^\infty(\mu) = H^\infty(G, \mu)$ is antisymmetric. Then the linear manifold $\mathscr{A} \equiv \{p(z) + \overline{q(z)} : p, q \text{ polynomials}\}$ is weak-star dense in $L^\infty(\mu)$ if and only if $\mu(G) = 0$. That is, $\mathscr{A}$ is weak-star dense in $L^\infty(\mu)$ if and only if μ and harmonic measure on ∂G are mutually absolutely continuous. Here ∂G denotes the boundary of G.*

Proof. We first note that for every function f in $L^\infty(\partial G)$ the function $\tilde{f}$ defined on $\bar{G}$ via

$$\tilde{f}(z) = \begin{cases} f(z) & z \in \partial G \\ \int f d\lambda_z & z \in G \end{cases}$$

(where λ_z is harmonic measure at z) has the property that $\tilde{f}$ is harmonic in G, [**10**]. Let $\tilde{H} = \{\tilde{f} : f \in L^\infty(\partial G)\}$. Using the same reasoning as in Lemma 6.2 of [**20**] it follows that $\tilde{H}$ is weak-star closed in $L^\infty(\mu)$. Observe that $\tilde{H}$ contains $\mathscr{A}$.

Now if $\mu(G) \neq 0$ we can find a compact set K inside G such that $\mu(K) \neq 0$. Clearly the characteristic function of K does not belong to $\tilde{H}$, so $\mathscr{A}$ is not dense in $L^\infty(\mu)$.

To see the other direction suppose $\mu(G) = 0$. That is, μ and harmonic mea-

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sure are mutually absolutely continuous. Then by Lemma 4.4 of [**20**] it suffices to show

$$\mathscr{B} = \{p(\varphi) + \overline{q(\varphi)} : p, q \text{ polynomials and } \varphi \text{ a conformal map from the disk to } G\}$$

is weak-star dense in $L^\infty(m)$, where m is normalized Lebesque measure on the unit circle. This now follows because φ is a weak-star generator of $H^\infty(m)$ and $H^\infty + \overline{H^\infty}$ is dense in $L^\infty(m)$.

These last two lemmas will provide enough information concerning $P^\infty(\mu)$ for the theorem. However we need two more elementary facts.

Definition. Let μ be a measure in the plane with compact support. Let $H^2(\mu)$ denote the closure of the polynomials in $L^2(\mu)$. We define the operators T_μ and M_μ on $H^2(\mu)$ and $L^2(\mu)$, respectively, as follows:

$$T_\mu(f) = zf \text{ for } f \in H^2(\mu), \quad \text{and} \quad M_\mu(f) = zf \text{ for } f \in L^2(\mu).$$

If μ and ν are mutually singular measures then there does not exist a nonzero bounded operator which intertwines the operators M_μ and M_ν ([**13**], Theorem 3). With this fact we leave it to the reader to show the following lemma.

LEMMA 3. *Let μ and ν be mutually singular measures. The following statements are equivalent:*

(a) *The commutant of $T_\mu \oplus T_\nu$ on $H^2(\mu) \oplus H^2(\nu)$ lifts to the commutant of $M_{\mu+\nu}$.*

(b) $\{T_\mu \oplus T_\nu\}' = \{T_\mu\}' \oplus \{T_\nu\}'$

(c) *The only bounded operators A and B that satisfy the equations*

$$AT_\mu = T_\nu A$$
$$BT_\nu = T_\mu B$$

are the zero operators.

Remark. If μ and ν are mutually singular there may be a nonzero operator that intertwines T_μ and T_ν. In fact the proof of our theorem shows how to construct many of them.

A point λ in the plane is called a *bounded point evaluation for $H^2(\mu)$* if there exists a constant C such that for all polynomials p it follows that

$$|p(\lambda)| \leqq C||p||_2$$

where $||p||_2$ denotes the $L^2(\mu)$ norm of p. If λ is a point evaluation then the smallest such C is called the norm of the point evaluation. The construction mentioned above will depend on the following lemma. This fact is well-known but the authors do not know if it appears in the literature.

LEMMA 4. *Let U be a bounded component of the complement of the support of a measure μ. Suppose every point in U is a bounded point evaluation for $H^2(\mu)$.*

Then the norms of the point evaluations are uniformly bounded on compact subsets of U.

Proof. Fix $\lambda_0 \in U$. Since the spectrum of T_μ contains U it follows that $1/(z - \lambda_0) \in L^2(\mu)\backslash H^2(\mu)$, [**5**]. Choose $g \in L^2(\mu)$ such that $\int fgd\mu = 0$ for all $f \in H^2(\mu)$ and $\int (g/(z - \lambda_0))d\mu = 1$. Let p be a polynomial. Then

$$\int (p - p(\lambda_0))g/(z - \lambda_0)d\mu = 0$$

implies

$$p(\lambda_0) = \int pg/(z - \lambda_0)d\mu.$$

Therefore

$$|p(\lambda_0)| \leqq ||p||_2||g/(z - \lambda_0)||_2.$$

For λ sufficiently close to λ_0 clearly

$$\left\| \frac{1}{z - \lambda} g - \frac{1}{z - \lambda_0} g \right\|_2 < \frac{1}{2}.$$

But for such λ we also have

$$\int \frac{p - p(\lambda)}{z - \lambda} gd\mu = 0,$$

which implies

$$p(\lambda) = \left[\frac{1}{\int \frac{g}{z - \lambda} d\mu} \right] \cdot \int p \frac{g}{z - \lambda} d\mu.$$

Therefore,

$$|p(\lambda)| \leqq 2||p||_2 \left\| \frac{g}{z - \lambda} \right\|_2 \leqq 2||p||_2 \left(\frac{1}{2} + \left\| \frac{g}{z - \lambda_0} \right\|_2 \right).$$

A compactness argument finishes the proof.

We are now ready to solve the problem mentioned in the introduction.

THEOREM. *Let N be an antisymmetric normal operator and $\mathscr{S}(N)$ denote the collection of subnormal operators that have N as their minimal normal extension. Let μ be a scalar spectral measure for N. The following statements are equivalent.*

1. *Every $S \in \mathscr{S}(N)$ has a commutant that lifts to the commutant of N.*

2. *The linear manifold $\mathscr{A} \equiv \{p + \bar{q} : p$ and q are polynomials$\}$ is weak-star dense in $L^\infty(\mu)$.*

3. *N is unitarily equivalent to $\varphi(U)$ where φ is a weak-star generator of the Hardy space H^∞ and U is a unitary operator.*

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(This unitary operator U must contain a direct sum of some bilateral shift because of the assumption of antisymmetry.)

Proof. An argument based on Lemma 2, the spectral mapping theorem for normal operators, and Lemmas 8.6 and 8.7 in [**8**], shows the equivalence of (2) and (3). We leave the details to the reader.

To show that (1) implies (2) we consider the case where the normal operator N is equal to M_μ, i.e., N has multiplicity one. The arbitrary case is a slight modification of this one and we leave the details to the reader. We show (1) implies (2) by proving the contrapositive. If $P^\infty(\mu) = H^\infty(G, \mu)$ then the negation of (2) implies $\mu(G) \neq 0$ via Lemma 2. Choose open disks B_1 and B_2 such that $\mu(B_1) \neq 0$, $\bar{B}_1 \subset B_2$, and $\bar{B}_2 \subset G$. Let ν be the measure obtained by restricting μ to $\bar{G}\backslash B_2$. By the maximum-modulus principal for analytic functions, $\tilde{K}_\nu = \tilde{K}_\mu = \bar{G}$ so that $H^\infty(\text{int } \tilde{K}_\nu) = H^\infty(\text{int } \tilde{K}_\mu) = H^\infty(G)$.

Fix $w \in B_2$. Using Proposition 3.8 of [**4**] there exists a representing measure, β, for evaluation of the polynomials at w, such that β and ν are mutually absolutely continuous. It is easy to verify that $H^2(\beta)$ has a bounded point evaluation at w; hence, it must have bounded point evaluations at every point in B_2, since the spectrum of T_β contains B_2 (see [**6**]). By Lemma 4, there exists a constant C such that for all polynomials p

$$|p(z)| \leqq C||p||_2^\beta$$

for all $z \in \bar{B}_1$. (The superscript means the norm is taken in $L^2(\beta)$.)

Let $\gamma = \mu|_{B_1}$ and let $\beta' = \beta + \mu|_{B_2 \backslash B_1}$. We now have, for any polynomial p,

$$\int_{B_1} |p(z)|^2 d\gamma \leqq C^2 \mu(B_1)(||p||_2^\beta)^2.$$

Therefore the map A which sends a polynomial p in $H^2(\beta')$ to the polynomial p in $H^2(\gamma)$ is bounded, so we extend it to all of $H^2(\beta')$. Clearly A intertwines the operators $T_{\beta'}$ and T_γ. Since β' and γ are mutually singular, Lemma 3 implies the commutant of $T_{\beta'} \oplus T_\gamma$ does not lift to the commutant of $M_{\beta'+\gamma}$. The proof that (1) implies (2) (for the case N has multiplicity 1) is done by observing $\beta' + \gamma$ and μ are mutually absolutely continuous and therefore the operators $M_{\beta'+\gamma}$ and M_μ are unitarily equivalent.

Remark. If $P^\infty(\mu_{|B_1})$ is antisymmetric then T_γ can be constructed to be a pure subnormal operator so that the operator $T_{\beta'} \oplus T_\gamma$ on $H^2(\beta') \oplus H^2(\gamma)$ is also pure. (A *pure subnormal operator* is one that is nonnormal on every nonzero invariant subspace.)

The proof of the theorem is finished if we show (3) implies (1). Suppose $N = \varphi(U)$ where U is a unitary and φ is a weak-star generator of H^∞. Let $S \in \mathscr{S}(N)$ act on the space $\mathscr{H}$ and let T commute with S. Then T commutes with every operator in the ultraweakly closed algebra generated by S; and since φ is a weak * generator of H^∞, this implies T commutes with $U|_{\mathscr{H}}$. (See [**8**], Theorem 2.1 and Theorem 4.9.)

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But $U|_{\mathscr{H}}$ is an isometry and its minimal normal extension is U. ([**8**], Theorem 6.1.) Therefore T lifts to an operator T_0 commuting with U. ([**13**]) But T_0 commutes with every operator in the weakly closed algebra generated by U, so T_0 commutes with $\varphi(U) = N$.

An irreducible subnormal operator S with a commutative commutant that does not lift to the commutant of its minimal normal extension. J. Bram in [**5**] showed that every operator in the weakly closed algebra generated by a subnormal S lifts to an operator in the von Neuman algebra generated by its minimal normal extension. In general, the commutant of S does not lift. It is natural then to ask if every operator in the double commutant of S lifts to an operator in the von Neuman algebra generated by the minimal normal extension (see [**2**].) The answer is no as shown below. Furthermore, at least to the authors' knowledge, in each example where a lifting phenomenon fails for a subnormal operator, the example entails a reducible subnormal operator. (This also is the case in the proof of the theorem.) The operator given in this section is irreducible. This example, and other evidence, suggests many possible conjectures concerning these lifting questions.

Our example will proceed in a series of steps. First, some notation:

Definitions. Define the sets D, Π, Γ, and I by $D = \{|z| < 1\}$, $\Pi = \{|z| = 1\}$, $\Gamma = \{|z - 3/2| = 1\}$ and $I = D \cap \Gamma$.

Let the measures m_1, m_2 and ν be defined as follows: m_1 is normalized Lebesque measure on Π; m_2 is normalized Lebesque measure on Γ; and $\nu = m_1 + m_2|_I$. For $f \in H^2(m_1)$, the usual Hardy space, $\tilde{f}$ denotes the extension of f to the disk in the obvious way.

Step A. Observe $\nu_{|I}$ is a Carleson measure. Therefore, *there exists a constant $c > 0$ such that $||f||_2^{m_1} \geqq c||f||_2^{\nu}$ for each f in $H^2(\nu)$.* (Consult [**14**] pp. 156–158 for the appropriate definitions and facts.)

Step B. $H^2(\nu)$ and $H^2(m_2)$ are non-orthogonal invariant subspaces inside $L^2(m_1) \oplus L^2(m_2)$ such that $H^2(\nu) + H^2(m_2)$ is closed. (Invariant, of course, means with respect to the operator $M_{m_1} \oplus M_{m_2}$.)

Proof. (We identify $H^2(\nu)$ inside $L^2(m_1) \oplus L^2(m_2)$ by letting these functions be zero on $\Gamma\backslash I$.) Clearly $H^2(\nu)$ and $H^2(m_2)$ have a zero intersection and are not orthogonal. To show $H^2(\nu) + H^2(m_2)$ is closed suppose $f_n + g_n$ converges in $L^2(m_1) \oplus L^2(m_2)$ where $\{f_n\} \subset H^2(\nu)$ and $\{g_n\} \subseteq H^2(m_2)$. Clearly then $\{f_n\}$ is a Cauchy sequence in $H^2(m_1)$. Therefore, by Step A, the sequence $\{f_n\}$ converges in $H^2(\nu)$ and then it follows $\{g_n\}$ converges in $H^2(m_2)$.

Definition. Let $H = H^2(\nu) + H^2(m_2)$. H is a closed invariant subspace for $N \equiv M_{m_1} \oplus M_{m_2} = M_{m_1+m_2}$.

The equality follows because m_1 and m_2 are mutually singular. Let $S = N|_H$ and observe $S \in \mathscr{S}(N)$.

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Step C. Let $S_1 = T_{m_1} \oplus T_{m_2}$. *$S_1$ is similar to S.*

Proof. Let $T : H^2(m_1) \oplus H^2(m_2) \to H$ be defined by the following: If $f \in H^2(m_1)$ and $g \in H^2(m_2)$ then

$$T(f + g) = f \text{ on } \Pi, \quad \tilde{f} + g \text{ on } I, \text{ and } \quad g \text{ on } \Gamma$$

It is easy to show T induces the similarity.

Step D. $\{S_1\}' = \{\theta(T_{m_1}) \oplus \psi(T_{m_2}) : \theta \in H^\infty(m_1), \psi \in H^\infty(m_2)\}$.

Proof. By Lemma 3 it suffices to show there is no nonzero bounded operator A from $H^2(m_1)$ to $H^2(m_2)$ which intertwines the operators T_{m_1} and T_{m_2}. (That there is no nonzero operator intertwining T_{m_1} and T_{m_2} follows by a similar argument.)

Suppose such an A exists. Then since A intertwines T_{m_1} and T_{m_2} and A is bounded there exists a constant C such that for all polynomials p,

$$(*) \qquad \int_\Gamma |p|^2 |A1|^2 dm_2 \leqq C \int_\Pi |p|^2 dm_1.$$

Let $J = \Gamma \cap \{\operatorname{Re} z > 3/2\}$. By Runge's Theorem [**7**, p. 198], there exists a sequence of polynomials $\{p_n\}$ that converges uniformly to 0 on Π and 1 on J. Clearly such a sequence of polynomials contradicts the inequality (*) unless $A1 = 0$, in which case $A = 0$.

Step E. *$\{S\}'$ is commutative.*

Proof. That $\{S\}'$ is commutative follows by Steps D and C.

Step F. *$\{S\}'$ does not lift to $\{N\}'$.*

Proof. Observe the natural idempotent $Q : H \to H^2(\nu)$ (via $Q(f + g) = f$ for $f \in H^2(\nu)$ and $g \in H^2(m_2)$) commutes with S and is not a projection because of Step B. We shall show Q does not lift to an operator commuting with N.

If Q did lift, say to Q_0, then $Q_0 = \beta(M_{m_1+m_2})$ on $L^2(m_1 + m_2)$, a multiplication operator where $\beta \in L^\infty(m_1 + m_2)$. By the Fuglede Theorem [**18**, Cor. 1.18], Q_0 must also commute with N^* so that for all $x_j \in H$,

$$Q_0^2\left(\sum_{j=1}^n N^{*j} x_j\right) = \sum_{j=1}^n N^{*j} Q^2 x_j = Q_0\left(\sum_{j=1}^n N^{*j} x_j\right).$$

Therefore Q_0 is an idempotent because the vectors

$$\sum_{j=1}^n N^{*j} x_j$$

are dense in $L^2(m_1 + m_2)$. Hence $\beta^2 = \beta$ which implies Q_0 is a projection. Because $Q_0 H \subseteq H$ and Q_0 is self adjoint, this implies H reduces Q_0. Therefore $Q = Q_0|_H$ is a projection, which is a contradiction.

Step G: S is irreducible.

Proof. Suppose there exists a projection $P \neq 0, 1$ such that $PS = SP$. Then $T^{-1}PT$ commutes with S_1. (S_1 and T are as in Step C.) Clearly $R \equiv T^{-1}PT$ is an idempotent, so by Step D, R is a projection. The fact that $R \neq 0, 1$ implies $R = 0 \oplus 1$ or $R = 1 \oplus 0$. Therefore, via the operator T, P is the idempotent onto $H^2(\nu)$ or $H^2(m_2)$, neither of which is a projection.

The operator S, via steps E, F, and G, satisfies the conditions promised.

Addendum. A. Lubin has independently discovered an example of an irreducible subnormal whose double commutant does not lift to its minimal normal extension.

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