OPERATORS WITH A SINGLE SPECTRUM

by T. T. WEST

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1. Introduction

Let X be an infinite dimensional normed linear space over the complex field Z. X will not be complete, in general, and its completion will be denoted by $\bar{X}$. If $\mathscr{B}(X)$ is the algebra of all bounded linear operators in X then $T \in \mathscr{B}(X)$ has a unique extension $\bar{T} \in \mathscr{B}(\bar{X})$ and $\| T \| = \| \bar{T} \|$. The resolvent set of $T \in \mathscr{B}(X)$ is defined to be

$$\rho(T) = \{z \in Z: (zI-T)^{-1} \in \mathscr{B}(X)\};$$

and the spectrum of T is the complement of $\rho(T)$ in Z.

Lemma 1.1. *If $T \in \mathscr{B}(X)$ then $\sigma(T) \supset \sigma(\bar{T})$.*

Proof. Let $z \in \rho(T)$; then $(zI-T)^{-1} \in \mathscr{B}(X)$ and

$$(zI-T)^{-1}(zI-T) = (zI-T)(zI-T)^{-1} = I. \tag{1}$$

Hence, considering the unique bounded extension of (1),

$$\overline{(zI-T)^{-1}}(z\bar{I}-\bar{T}) = (z\bar{I}-\bar{T})\overline{(zI-T)^{-1}} = \bar{I}.$$

Thus $$(z\bar{I}-\bar{T})^{-1} = \overline{(zI-T)^{-1}} \in \mathscr{B}(\bar{X}),$$

and $z \in \rho(\bar{T})$, showing that $\rho(T) \subset \rho(\bar{T})$, which completes the proof.

Example 1 (§ 5) shows that we cannot replace inclusion by equality in Lemma 1.1. Incidentally, this invalidates Problem 5 on p. 311 of (**1**) and suggests the following definition. If $T \in \mathscr{B}(X)$, T has a *single spectrum* if $\sigma(T) = \sigma(\bar{T})$. In § 2 an operational calculus is developed for such operators. Compact operators have a single spectrum and our results are applied to them in § 3. Riesz operators, which form a natural generalisation of compact operators are considered in § 4, while § 5 consists of a list of the examples of operators in normed spaces which are referred to in the text.

Before proceeding, we sound a warning note. The definition of the spectrum of a bounded operator as given in (**2**) is not equivalent to that given here. In fact Taylor's definition is so designed that $\sigma(T) = \sigma(\bar{T})$ for all $T \in \mathscr{B}(X)$. All definitions coincide if X is complete.

2. Single spectrum operators

Here we develop a restricted functional calculus for operators of this type.

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Theorem 2.1. *If $T \in \mathscr{B}(X)$ has a single spectrum then $R(z; T) = (zI-T)^{-1}$ is an analytic function of z in $\rho(T)$.*

Proof. By hypothesis $\rho(T) = \rho(\bar{T})$; hence $\rho(T)$ is an open set, neither empty nor the whole plane. $R(z; T) \in \mathscr{B}(X)$ if and only if $R(z; \bar{T}) \in \mathscr{B}(\bar{X})$ and the uniqueness of the extension shows that

$$\overline{R(z; T)} = R(z; \bar{T}) \qquad (z \in \rho(T)).$$

Let $z_0 \in \rho(\bar{T})$. If $| z-z_0 | < \| R(z_0; \bar{T})\|^{-1}$, then by (**1**), p. 310

$$R(z; \bar{T}) = \sum_{n=0}^{\infty} (z-z_0)^n (R(z_0; \bar{T}))^{n+1}.$$

But $z_0 \in \rho(T)$; hence $R(z; T) \in \mathscr{B}(X)$ and

$$\left\| R(z; T) - \sum_{n=0}^{N} (z-z_0)^n (R(z_0; T))^{n+1} \right\|$$

$$= \left\| R(z; \bar{T}) - \sum_{n=0}^{N} (z-z_0)^n (R(z_0; \bar{T}))^{n+1} \right\|$$

which $\to 0$ as $N \to \infty$.

Hence $R(z; T)$ is an analytic function of z in $\rho(T)$, and

$$R(z; T) = \sum_{n=0}^{\infty} (z-z_0)^n (R(z_0; T))^{n+1}.$$

Theorem 2.2. *If $T \in \mathscr{B}(X)$ has a single spectrum then*

$$\sup \{| \lambda |: \lambda \in \sigma(T)\} = \lim_{n\to\infty} \| T^n \|^{1/n}.$$

Proof. This follows directly from the corresponding formula for $\bar{T}$ ((**2**), p. 263) and the fact that $\| T \| = \| \bar{T} \|$.

Let $T \in \mathscr{B}(X)$ have a single spectrum. If f is a function which is analytic in a neighbourhood of $\sigma(T)$ then for a suitable contour Γ surrounding $\sigma(T)$, $f(\bar{T})$ is defined by

$$f(\bar{T}) = \frac{1}{2\pi i}\int_{\Gamma} f(z) R(z; \bar{T}) dz, \tag{2}$$

((**2**), p. 289). f is called a *restricted function* of T if f is analytic in some neighbourhood of $\sigma(T)$ and if $f(\bar{T})$ has a restriction to X which we denote by $f(T)$. The restricted functions of T form an algebra, written $\mathscr{F}_r(T)$.

Theorem 2.3. *If $T \in \mathscr{B}(X)$ has a single spectrum and $f \in \mathscr{F}_r(T)$ then $f(T)$ is given by the formula*

$$f(T) = \frac{1}{2\pi i}\int_{\Gamma} f(z) R(z; T) dz,$$

for a suitable contour Γ.

Proof. $f(\bar{T})$ in $\mathscr{B}(\bar{X})$ is given by (2); hence

$$\left\| f(\bar{T}) - \frac{1}{2\pi i}\sum_{n=0}^{N} f(z_n) R(z_n; \bar{T})(z_n - z_{n+1}) \right\| \to 0 \quad (N \to \infty),$$

for suitable partitions $\{z_0, \ldots, z_N\}$ of Γ. Thus

$$\left\| f(T) - \frac{1}{2\pi i} \sum_{n=0}^{N} f(z_n) R(z_n; T)(z_n - z_{n+1}) \right\| \to 0 \quad (N \to \infty),$$

and the result follows.

To obtain a spectral mapping theorem we need a further limitation upon the class of functions we consider. If T has a single spectrum an algebra $\mathscr{A}(T)$ of restricted functions of T is *full* if it contains the identity, and if, whenever $f \in \mathscr{A}(T)$ does not vanish on $\sigma(T)$ then $1/f \in \mathscr{A}(T)$.

Theorem 2.4. *If $T \in \mathscr{B}(X)$ has a single spectrum and $\mathscr{A}(T)$ is a full algebra of restricted functions of T then $f \in \mathscr{A}(T)$ implies that $f(T)$ has a single spectrum.*

Proof. Since f is a restricted function $f(T) \in \mathscr{B}(X)$. Let $\mu \in \sigma(f(T))$ and suppose that $f(z) - \mu$ has no zeros on $\sigma(T)$. Defining $h(z) = (\mu - f(z))^{-1}$ on a suitable neighbourhood of $\sigma(T)$ we have $h(z) \in \mathscr{A}(T)$. Thus $h(T) \in \mathscr{B}(X)$,

$$h(\bar{T})(\mu \bar{I} - f(\bar{T})) = (\mu \bar{I} - f(\bar{T}))h(\bar{T}) = \bar{I}$$

by the operational calculus for $\bar{T}$; hence

$$h(T)(\mu I - f(T)) = (\mu I - f(T))h(T) = I.$$

Thus $\mu \in \rho(f(T))$, which is a contradiction, giving

$$\sigma(f(T)) \subset f(\sigma(T)).$$

But, by the spectral mapping theorem for $\bar{T}$,

$$f(\sigma(T)) = f(\sigma(\bar{T})) = \sigma(f(\bar{T})) = \sigma(\overline{f(T)}).$$

Thus we have

$$\sigma(f(T)) \subset \sigma(\overline{f(T)}),$$

and the result follows from Lemma 1.1.

In the course of the above proof we have obtained our spectral mapping theorem.

Corollary 2.5. *Under the hypotheses of Theorem 2.4 if $f \in \mathscr{A}(T)$ then $\sigma(f(T)) = f(\sigma(T))$.*

3. Compact operators

The linear operator T in X is compact if it maps each bounded set into a relatively compact set. Equivalently ((**1**), p. 311) T is compact if for any bounded sequence (x_n) in X the sequence (Tx_n) has a cluster point in X. If $\mathscr{C}(X)$ denotes the set of compact operators in X, $\mathscr{C}(X)$ is an ideal of $\mathscr{B}(X)$ and $T \in \mathscr{C}(X)$ implies that $\bar{T} \in \mathscr{C}(\bar{X})$ ((**1**) Chapter XI). If X is complete $\mathscr{C}(X)$ is closed in the uniform topology of $\mathscr{B}(X)$. Example 2 shows that this is not true in normed spaces.

Theorem 3.1. *If $T \in \mathscr{C}(X)$ then T has a single spectrum.*

Proof. Given in (**1**), p. 321.

The Riesz theory of compact operators is valid in normed spaces (see

(1), (2)). This fact combined with Theorem 3.1 allows us to deduce results for the Laurent expansion of $R(z;\ T)$ in a punctured neighbourhood of a non-zero eigenvalue of T corresponding to those for $R(z;\ \bar{T})$.

Theorem 3.2. *If $T \in \mathscr{C}(X)$ and λ is a non-zero eigenvalue of T then the associated spectral projection is given by the formula*

$$P(\lambda;\ T) = \frac{1}{2\pi i}\int_{\gamma} R(z;\ T)dz,$$

where γ is a sufficiently small circle of centre λ.

Proof. $P(\lambda;\ T) \in \mathscr{B}(X)$ by (1), p. 319 and as the corresponding formula for $P(\lambda;\ \bar{T})$ is well known the result follows as in Theorem 2.3.

It is known that in the situation of Theorem 3.2 λ has a finite index ν relative to T and that ν is also the index of λ relative to $\bar{T}$ ((1), p. 321). Also λ is a pole of order ν of $R(z;\ \bar{T})$ ((1), p. 319). The next theorem shows that this is also the case for $R(z;\ T)$.

Theorem 3.3. *If $T \in \mathscr{C}(X)$ and if λ is a non-zero eigenvalue of T with index ν then λ is a pole of $R(z;\ T)$ of order ν.*

Proof. In a punctured neighbourhood of λ

$$R(z;\ \bar{T}) = \sum_{n=0}^{\infty} A_n(z-\lambda)^n + \sum_{n=1}^{\nu} A_{-n}(z-\lambda)^{-n},$$

where $A_n,\ A_{-n} \in \mathscr{B}(\bar{X})$. It will be sufficient to show that we can choose B_n, $B_{-n} \in \mathscr{B}(X)$ satisfying

$$\overline{B_n} = A_n \quad (n = 0, 1, \ldots), \tag{3}$$

and

$$\overline{B_{-n}} = A_{-n} \quad (n = 1, \ldots, \nu). \tag{4}$$

For, then, by the usual method

$$R(z;\ T) = \sum_{n=0}^{\infty} B_n(z-\lambda)^n + \sum_{n=1}^{\nu} B_{-n}(z-\lambda)^{-n},$$

in a punctured neighbourhood of λ.

Firstly, using the results of (2), § 5.8, we shall express the A_n's in terms of the spectral projection $P(\lambda;\ \bar{T})$. Thus

$$A_{-1} = P(\lambda;\ \bar{T})$$

and

$$A_{-n-1} = (\bar{T}-\lambda\bar{I})^n A_{-1} = (\bar{T}-\lambda\bar{I})^n P(\lambda;\ \bar{T}) \quad (n = 0, \ldots, \nu-1).$$

Hence (4) is satisfied on putting

$$B_{-n-1} = (T-\lambda I)^n P(\lambda;\ T) \quad (n = 0, \ldots, \nu-1).$$

Equations 5.8-5 and 5.8-7 of (2) give

$$(\bar{T}-\lambda\bar{I})A_0 = \bar{I}-A_{-1} = \bar{I}-P(\lambda;\ \bar{T}), \tag{5}$$

and

$$(\bar{T}-\lambda\bar{I})A_{n+1} = A_n \quad (n = 0, 1, \ldots). \tag{6}$$

To solve these equations we write $N(\lambda;\ \bar{T})$ and $F(\lambda;\ \bar{T})$ for the range and null-space of $P(\lambda;\ \bar{T})$ respectively. Then

$$\bar{X} = N(\lambda;\ \bar{T}) \oplus F(\lambda;\ \bar{T}),$$

and

$$P(\lambda;\ \bar{T}) = \bar{I} \oplus 0,$$

using an obvious direct sum notation for operators. In this notation (5) reads

$$(\bar{T} - \lambda\bar{I})A_0 = 0 \oplus \bar{I}.$$

By the Riesz theory $\bar{T} - \lambda\bar{I}$ restricted to $F(\lambda;\ \bar{T})$ is a homeomorphism, thus the solution of this equation is

$$A_0 = 0 \oplus (\bar{T} - \lambda\bar{I})^{-1}.$$

Similarly, from (6)

$$A_n = 0 \oplus (\bar{T} - \lambda\bar{I})^{-n-1} \quad (n = 0, 1, \ldots).$$

Write $N(\lambda;\ T)$ and $F(\lambda;\ T)$ for the range and null-space of $P(\lambda;\ T)$. Then

$$X = N(\lambda; T) \oplus F(\lambda; T)$$

and

$$P(\lambda; T) = I \oplus 0.$$

$T - \lambda I$ restricted to $F(\lambda; T)$ is a homeomorphism ((**1**), p. 319) so we may put

$$B_n = 0 \oplus (T - \lambda I)^{-n-1} \quad (n = 0, 1, \ldots).$$

The B_n's satisfy (3) and this ends the proof.

We end this section by giving an example of a full algebra of restricted functions of T for $T \in \mathscr{C}(X)$.

Theorem 3.4. *Let $T \in \mathscr{C}(X)$. $\mathscr{R}(\sigma(T))$ the algebra of rational functions on $\sigma(T)$ is a full algebra of restricted functions of T.*

Proof. Each $f \in \mathscr{R}(\sigma(T))$ is of the form $f = p/q$ where p, q are polynomials and $q(z)$ does not vanish on $\sigma(T)$. Since p is a restricted function of T and the product of two such functions is also a restricted function it suffices to show that $r = 1/q$ is restricted.

Since X is infinite dimensional, $0 \in \sigma(T)$; thus $q(z) = a_0 + q_1(z)$ where $a_0 \neq 0$ and $q_1(T) \in \mathscr{C}(X)$. Since $r(z)$ is analytic on $\sigma(T)$,

$$r(\bar{T}) = (a_0\bar{I} + q_1(\bar{T}))^{-1} \in \mathscr{B}(X).$$

Thus $-a_0 \in \rho(q_1(\bar{T})) = \rho(\overline{q_1(T)})$. But $q_1(T)$ has a single spectrum, being in $\mathscr{C}(X)$; hence $-a_0 \in \rho(q_1(T))$ and

$$(a_0 I + q_1(T))^{-1} \in \mathscr{B}(X).$$

This last operator is clearly a restriction of $r(\bar{T})$.

4. Riesz operators

A Riesz operator in $\mathscr{B}(X)$ is one which possesses a Riesz spectral theory commonly associated with the compact operators. For the case of Banach spaces the class of Riesz operators has been examined in (**3**). However the Riesz theory does not depend on the completeness of the underlying space, and here we sketch a theory of Riesz operators in normed spaces.

Let $K \in \mathscr{B}(X)$. A *Riesz point* of $\sigma(K)$ is a point $\lambda \in \sigma(K)$ such that

$$X = N(\lambda; K) \oplus F(\lambda; K)$$

where $N(\lambda; K)$ is a finite dimensional subspace of X and $F(\lambda; K)$ is closed, N is invariant under K and $\lambda I - K$ restricted to N is nilpotent, while F is also invariant under K and $\lambda I - K$ restricted to F is a homeomorphism. K is a *Riesz operator* in X if each non-zero point in $\sigma(K)$ is a Riesz point. The class of such operators will be written $\mathscr{K}(X)$.

Theorem 4.1. *If $K \in \mathscr{K}(X)$ the non-zero points of $\sigma(K)$ are the same as those of $\sigma(\overline{K})$.*

Proof. Let $\lambda(\neq 0) \in \sigma(K)$. The definition above implies that λ is an eigenvalue of K and hence of $\overline{K}$. Thus $\lambda \in \sigma(\overline{K})$. As $\sigma(K) \supset \sigma(\overline{K})$ (Lemma 1.1) the result follows.

If T is an operator in X we denote the range and null-space of T by $R\{T\}$ and $N\{T\}$ respectively. We shall need the following result.

Lemma 4.2. *If $T \in \mathscr{B}(X)$ then $R\{T\}$ is dense in $R\{\overline{T}\}$.*

Proof. Clearly $R\{T\} \subset R\{\overline{T}\}$. We shall show that $R\{\overline{T}\} \subset \mathrm{cl}\, R\{T\}$ (the closure of $R\{T\}$ in $\overline{X}$). If $\bar{x} \in R\{\overline{T}\}$ there is a $\bar{y} \in \overline{X}$ such that $\overline{T}\bar{y} = \bar{x}$, and there exists a Cauchy sequence (y_n) in X such that $y_n \to \bar{y}$. Putting $x_n = Ty_n$ we have $x_n \to \bar{x}$, and as $x_n \in R\{T\}$ for each n, this gives $\bar{x} \in \mathrm{cl}\, R\{T\}$ which completes the proof.

If λ is a Riesz point of $\sigma(K)$ for $K \in \mathscr{B}(X)$ the spectral projection $P(\lambda; K)$ of λ associated with K has range $N(\lambda; K)$ and null-space $F(\lambda; K)$.

Theorem 4.3. *If $K \in \mathscr{K}(X)$ and $\lambda(\neq 0) \in \sigma(K)$ then λ is a Riesz point of $\sigma(\overline{K})$, λ is isolated in $\sigma(K)$ and $P(\lambda; \overline{K}) = \overline{P(\lambda; K)}$.*

Proof. Basic spectral properties valid in normed spaces show that

$$K - KP(\lambda; K) \in \mathscr{K}(X),$$

and that

$$\sigma(\overline{K} - \overline{KP(\lambda; K)}) = \sigma(\overline{K}) \setminus \{\lambda\}.$$

Hence by Theorem 4.1

$$\sigma(\overline{K} - \overline{KP(\lambda; K)}) \setminus \{0\} = \sigma(\overline{K}) \setminus \{0, \lambda\}. \tag{7}$$

By Lemma 4.2 $N(\lambda; K)$ is dense in $R\{\overline{P(\lambda; K)}\}$; hence, as $N(\lambda; K)$ is finite dimensional, they are equal. Thus $\overline{P(\lambda; K)}$ decomposes $\overline{X}$ into

$$\overline{X} = N(\lambda; K) \oplus M$$

where $\lambda \bar{I} - \overline{K}$ restricted to $N(\lambda; K)$ is nilpotent, and restricted to the closed subspace M is, by (7), a homeomorphism. Thus λ is a Riesz point of $\sigma(\overline{K})$. Hence by (3) Theorem 2.1 λ is isolated in $\sigma(\overline{K})$ and is thus isolated in $\sigma(K)$. The uniqueness of the spectral projection of λ associated with $\overline{K}$ gives

$$P(\lambda; \overline{K}) = \overline{P(\lambda; K)}.$$

The main result follows at once.

Corollary 4.4. *If $K \in \mathscr{K}(X)$ then K has a single spectrum and $\bar{K} \in \mathscr{K}(\bar{X})$.*

Proof. Theorems 4.1 and 4.3 combine to give the first part of the result. To see that K has a single spectrum all we need is the fact that $0 \in \sigma(\bar{K})$. Since $\bar{K} \in \mathscr{K}(\bar{X})$ this follows from (**3**), Theorem 3.3.

Let λ be a non-zero point of $\sigma(K)$ where $K \in \mathscr{K}(X)$; then the arguments of (**1**) and (**2**), which are independent of completeness, show that there is a positive integer $\nu = \nu(\lambda; K)$ called the index of λ such that

$$N(\lambda; K) = N\{(\lambda I - K)^{\nu}\},$$

and

$$F(\lambda; K) = R\{(\lambda I - K)^{\nu}\}.$$

The eigenspace of λ associated with K is

$$E(\lambda; K) = N\{\lambda I - K\}.$$

Theorem 4.5. *If $K \in \mathscr{K}(X)$ and λ is a non-zero point of $\sigma(K)$ then*

(i) $E(\lambda; K) = E(\lambda; \bar{K})$;

(ii) $N(\lambda; K) = N(\lambda; \bar{K})$;

(iii) $F(\lambda; K)$ *is dense in* $F(\lambda; \bar{K})$;

(iv) $\nu(\lambda; K) = \nu(\lambda; \bar{K})$.

Proof. Part (ii) has been demonstrated in Theorem 4.3 and a similar argument disposes of (i). $F(\lambda; K)$ is the range of the projection $I - P(\lambda; K)$; hence (iii) follows from Lemma 4.2 and Theorem 4.3. The respective indices are the same since, by the arguments of parts (i) and (ii), the null-spaces of $(\lambda I - K)^k$ and $(\lambda \bar{I} - \bar{K})^k$ are equal for any integer $k \geqq 1$.

We remark that Theorem 3.3 is obviously true for Riesz operators.

If $T \in \mathscr{C}(X)$ it is known that $R\{\bar{T}\} \subset X$ ((**1**), p. 314). Example 3 shows that this is not true for Riesz operators.

5. Examples

Example 1. Take X to be the space of complex sequences with only a finite number of non-zero terms in the l^1 norm. Then $\bar{X} = l^1$. Let T be the shift operator in X defined by

$$x = (x_1, x_2, \ldots, x_k, 0, 0, \ldots),$$
$$Tx = (0, x_1, x_2, \ldots, x_k, 0, \ldots).$$

For any complex λ the point $(1, 0, 0, \ldots)$ is not in the range of $\lambda I - T$; hence $\sigma(T)$ is the whole complex plane. However $\bar{T}$ is the shift operator in l^1; hence $\sigma(\bar{T}) = \{\lambda : |\lambda| \leqq 1\}$ ((**2**) p. 266).

Example 2. Take X to be as in Example 1. If $x = (x_r) \in X$ define T to be the operator $Tx = \left(\frac{x_r}{r}\right)$, then $T \in \mathscr{B}(X)$. Also define the operators T_n in X by

$$T_n x = \left(\frac{x_1}{1}, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, 0, 0, \ldots\right).$$

Then T_n is of finite rank, and thus compact, for each n and $\| T_n - T \| \to 0$ $(n \to \infty)$.
Now consider the sequence of points y_n in X given by

$$y_n = \left(\frac{1}{1^2}, \frac{1}{2^2}, \ldots, \frac{1}{n^2}, 0, 0, \ldots\right).$$

This is a bounded sequence in X as $\| y_n \| \leq \frac{\pi^2}{6}$ $(n = 1, 2, \ldots)$. However the sequence (Ty_n) has only one cluster point

$$\left(\frac{1}{1^3}, \frac{1}{2^3}, \frac{1}{3^3}, \ldots\right)$$

and this is not in X. Thus T is not a compact operator.

Example 3. Take X to be the space of infinite sequences of complex numbers $x = (x_r)$ such that $\sum_1^\infty | x_r | < \infty$. Under the norm

$$\| x \| = \left(\sum_1^\infty | x_r |^2\right)^{\frac{1}{2}}$$

X is incomplete with completion l^2.

If $$x = (x_1, x_2, x_3, x_4, x_5, \ldots),$$
and $$Qx = (0, x_1, 0, x_3, 0, \ldots),$$
the operator Q defined for each x in X is in $\mathscr{B}(X)$ and its bounded extension $\bar{Q}$ is defined similarly for each x in $\bar{X}$. Now $Q^2 = 0$; hence for $\lambda \neq 0$

$$(\lambda I - Q)(\lambda I + Q) = \lambda^2 I = (\lambda I + Q)(\lambda I - Q);$$

thus $\lambda \in \rho(Q)$ and $\sigma(Q) = \{0\}$. Hence $Q \in \mathscr{K}(X)$. Choose $\bar{x} = \left(\frac{1}{r}\right) \in \bar{X}$, then

$$\bar{Q}\bar{x} = \left(0, \frac{1}{1}, 0, \frac{1}{3}, 0, \ldots\right)$$

and $\sum_1^\infty \frac{1}{2r-1}$ is divergent. Hence $\bar{Q}\bar{x} \notin X$; thus $R\{\bar{Q}\}$ is not contained in X.

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Glasgow University
and
Trinity College, Dublin