Can. J. Math., Vol. XXVIII, No. 6, 1976, pp. 1132–1145

INFINITE SYSTEMS OF DIFFERENTIAL EQUATIONS

J. P. McCLURE AND R. WONG

1. Introduction. In an earlier paper [7], we have studied the existence, uniqueness and asymptotic behavior of solutions to certain infinite systems of linear differential equations with constant coefficients. In the present paper we are interested in systems of nonlinear equations whose coefficients are not necessarily constants; more specifically, we are concerned with infinite systems of the form

$$(1.1)\qquad \begin{cases} \dot{x}_i(t) = \sum_{j=1}^{\infty} a_{ij}(t)x_j(t) + f_i(t, \tilde{x}(t)), \quad t \in \mathbf{R}_s \\ x_i(s) = c_i, \end{cases}$$

$i = 1, 2, \ldots$. Here s is a nonnegative real number, $\mathbf{R}_s = \{t \in \mathbf{R} : t \geqq s\}$, and $\tilde{x}(t)$ denotes the sequence-valued function $\tilde{x}(t) = (x_1(t), x_2(t), \ldots)$. Such a sequence defines a *strongly continuous function with values in* l^1 if and only if each $x_i(t)$ is continuous and $||\tilde{x}(t)|| = \sum_{i=1}^{\infty}|x_i(t)|$ converges uniformly on compact subsets of $\mathbf{R}_s$. For simplicity we shall call such functions *strongly continuous*. We wish to find conditions on the coefficient matrix $A(t) = [a_{ij}(t)]$ and the nonlinear perturbation $\tilde{f}(t, \tilde{x}(t)) = (f_i(t, \tilde{x}(t))$ which guarantee the existence of a strongly continuous solution for the system (1.1). We are also interested in the asymptotic behavior of solutions.

In Section 2, we first consider the system of linear equations

$$(1.2)\qquad \begin{cases} \dot{x}_i(t) = \sum_{j=1}^{\infty} a_{ij}(t)x_j(t), \quad t \in \mathbf{R}_s \\ x_i(s) = c_i, \end{cases}$$

$i = 1, 2, \ldots$. Under certain conditions on the coefficient matrix $A(t)$, which nevertheless allow the diagonal entries $a_{ii}(t)$ to be unbounded, we show that for each initial value $\tilde{c} = (c_i) \in l^1$, the system (1.2) has a unique strongly continuous solution. This leads to a natural definition of the *fundamental matrix* (or *evolution operator*) for the system (1.2), which turns out to be very useful in the study of the nonlinear system (1.1). In Section 3, we show that the solution to (1.2) can be approximated by the solutions to finite systems obtained from (1.2) by truncation.

Time-dependent linear systems have been the subject of considerable research, and the results of Sections 2 and 3 give some improvement over some

Received November 1, 1974 and in revised form, August 12, 1976.

of this earlier work (cf. [**1**; **8**]). At the same time, the result of the present paper for linear systems does not entirely subsume that of [**7**]. There, in the case of constant coefficients, we were able to use results in the abstract theory of differential equations in Banach spaces to show that, for initial values in a dense subspace of l^1, the differential equations are satisfied in a much stronger sense (i.e. that of strong derivative). To apply the abstract results that we know to time-dependent problems, we would have to put much stronger conditions on $A(t)$ than those we actually use.

In Section 4, we use the fundamental matrix of Section 2 to show the existence of a strongly continuous solution to certain non-linear systems. Our theorem requires conditions on the perturbation $\tilde{f}(t, \tilde{x}(t))$ which are natural extensions of conditions known for finite systems. Finally, in Section 5, we investigate the asymptotic behavior of solutions. Using the approximation result of Section 3, we show that when $A(t)$ satisfies a diagonal dominance condition, the solution to (1.2) decays exponentially to zero. Furthermore, under certain assumptions which are again natural extensions of those used in the case of finite systems, we are able to extend this result to the nonlinear system.

Throughout, the scalars may be either real or complex. As indicated by our notation, differentiation of l^1-valued functions will be considered only in the sense of differentiation of each coordinate function. In contrast with this, we find it convenient to use strong, as well as coordinatewise integrals. We adopt the convention that, whenever we write an integral of a vector-valued function (as opposed to writing the integrals of the coordinate functions), we mean the strong Riemann integral.

2. The linear system. The linear system (1.2) can be regarded as a special case of (1.1), and we consider this case first. The existence theorem for (1.1), proved in Section 4, depends on the solutions to (1.2). Throughout the remainder of the paper, we assume that $A(t)$ satisfies the following conditions:

(A_1) *each* $a_{ij}(t)$ *is continuous on* $\mathbf{R}_0$*;*

(A_2) *the function* $\omega(t) = \sup\{\operatorname{Re} a_{ii}(t) : i = 1, 2, \ldots\}$ *is locally bounded above on* $\mathbf{R}_0$*;*

(A_3) $\sum_{i\neq j}|a_{ij}(t)|$ *converges uniformly on compact subsets of* $\mathbf{R}_0$ *for each* j, *and* $\mu(t) = \sup\{\sum_{i\neq j}|a_{ij}(t)| : j = 1, 2, \ldots\}$ *is locally bounded on* $\mathbf{R}_0$.

As in [**7**], we decompose the matrix $A(t)$ into its diagonal and off-diagonal parts, and write $D(t)$ for the diagonal matrix diag $[a_{ii}(t)]$, and $B(t) = A(t) - D(t)$. By condition (A_1), the uniform convergence of $\sum_{i\neq j}|a_{ij}(t)|$ on compact subsets of $\mathbf{R}_0$ is equivalent to the continuity of $\sum_{i\neq j}|a_{ij}(t)|$ on $\mathbf{R}_0$ for each j. Furthermore, (A_1) and (A_3) together imply that $B(t)$ is a bounded operator on l^1 for each $t \in \mathbf{R}_0$, and that $t \mapsto B(t)$ is a strongly continuous operator-valued function (i.e., $t \mapsto B(t)\tilde{c}$ is a strongly continuous l^1-valued function for each $\tilde{c} \in l^1$). If we define

$$K(t, s) = \operatorname{diag}\left[\exp\left(\int_s^t a_{ii}(\tau)d\tau\right)\right]$$

for s and t in $\mathbf{R}_0$, then (A_1) and (A_2) imply that $K(t, s)$ is a bounded operator on l^1. It is easily seen that $K(t, s)$ is also strongly continuous (in the operator sense) as a function of two variables.

LEMMA 2.1. *Assume that conditions* (A_1), (A_2) *and* (A_3) *hold, and let* $\tilde{c} = (c_i) \in l^1$. *Then a strongly continuous function* $\tilde{x}(t)$ *is a solution of* (1.2) *if and only if it is a solution of the integral equation*

$$\tilde{x}(t) = K(t, s)\tilde{c} + \int_s^t K(t, \tau)B(\tau)\tilde{x}(\tau)d\tau. \tag{2.1}$$

Proof. Since $\tilde{x}(t)$ is strongly continuous and the operators $K(t, s)$ and $B(t)$ are continuous, the equation (2.1) is equivalent to the system of integral equations

$$x_i(t) = \exp\left(\int_s^t a_{ii}(\tau)d\tau\right)c_i + \int_s^t \exp\left(\int_\tau^t a_{ii}(\sigma)d\sigma\right) \sum_{j\neq i} a_{ij}(\tau)x_j(\tau)d\tau.$$

In turn, these equations are clearly equivalent to the system (1.2). The proof is complete.

THEOREM 2.1. *Assume that conditions* (A_1), (A_2) *and* (A_3) *hold. Then for each* $\tilde{c} \in l^1$, *the system* (1.2) *has a unique strongly continuous solution.*

Proof. We shall work with the integral equation (2.1). Define a sequence $\{\tilde{x}_n(t) : n = 0, 1, 2, \ldots\}$ as follows:

$$\begin{aligned} \tilde{x}_0(t) &= K(t, s)\tilde{c} \\ \tilde{x}_{n+1}(t) &= K(t, s)\tilde{c} + \int_s^t K(t, \tau)B(\tau)\tilde{x}_n(\tau)d\tau. \end{aligned} \tag{2.2}$$

Clearly, $\tilde{x}_0(t)$ is strongly continuous, and by induction $\tilde{x}_n(t)$ is strongly continuous for each n. We shall show that $\{\tilde{x}_n(t)\}$ converges to a solution of (2.1) uniformly on compact subsets of $\mathbf{R}_s$.

First, we have from (2.2) that

$$\tilde{x}_1(t) = \tilde{x}_0(t) + \int_s^t K(t, \tau)B(\tau)\tilde{x}_0(\tau)d\tau. \tag{2.3}$$

Since $||B(t)|| \leqq \mu(t)$ by condition (A_3) and also

$$||K(t, s)|| \leqq \exp\left(\int_s^t \omega(\tau)d\tau\right)$$

by assumption (A_2), (2.3) gives

$$||\tilde{x}_1(t) - \tilde{x}_0(t)|| \leqq ||\tilde{c}||\left(\int_s^t \mu(\tau)d\tau\right) \exp\left(\int_s^t \omega(\tau)d\tau\right).$$

In general, we have

$$||\tilde{x}_{n+1}(t) - \tilde{x}_n(\mathrm{t})|| \leqq \int_s^t \exp\left(\int_\tau^t \omega(\sigma)d\sigma\right)\mu(\tau)||\tilde{x}_n(\tau) - \tilde{x}_{n-1}(\tau)||d\tau,$$

and induction leads to

$$(2.4) \qquad ||\tilde{x}_{n+1}(t) - \tilde{x}_n(t)|| \leqq \frac{||\tilde{c}||}{(n+1)!}\left(\int_s^t \mu(\tau)d\tau\right)^{n+1} \exp\left(\int_s^t \omega(\tau)d\tau\right)$$

for any value of n. Thus the series $\tilde{x}_0(t) + \sum_{n=0}^{\infty}[\tilde{x}_{n+1}(t) - \tilde{x}_n(t)]$ converges uniformly on compact subsets of $\mathbf{R}_s$, and by the usual argument it follows that the sequence $\{\tilde{x}_n(t)\}$ converges uniformly on compact subsets of $\mathbf{R}_s$ to a strongly continuous function $\tilde{x}(t)$. Passing to the limit in (2.2) as $n \to \infty$, we obtain

$$\tilde{x}(t) = K(t, s)\tilde{c} + \int_s^t K(t, \tau)B(\tau)\tilde{x}(\tau)d\tau.$$

Consequently, there is at least one strongly continuous solution of (2.1) or equivalently, of (1.2).

To show the uniqueness, let $\tilde{y}(t)$ be any strongly continuous solution of (2.1). Then

$$||\tilde{x}(t) - \tilde{y}(t)|| \leqq \int_s^t ||K(t, \tau)||\ ||B(\tau)||\ ||\tilde{x}(\tau) - \tilde{y}(\tau)||d\tau.$$

For any $T \in \mathbf{R}_s$, there is a constant $M > 0$ such that $||K(t, \tau)||\ ||B(\tau)|| \leqq M$ uniformly for $s \leqq \tau \leqq t \leqq T$. Thus, for any $\epsilon > 0$ and any $t \in [s, T]$

$$||\tilde{x}(t) - \tilde{y}(t)|| < \epsilon + M \int_s^t ||\tilde{x}(\tau) - \tilde{y}(\tau)||d\tau.$$

Now Gronwall's inequality implies

$$||\tilde{x}(t) - \tilde{y}(t)|| < \epsilon \exp[M(t - s)]$$

for all $t \in [s, T]$. Since $\epsilon > 0$ and $T \in \mathbf{R}_s$ were arbitrary, this shows that $\tilde{x}(t) = \tilde{y}(t)$ for all $t \in \mathbf{R}_s$, and proves the theorem.

For future reference, we point out that from (2.4) an upper bound for the solution is given by

$$(2.5) \qquad ||\tilde{x}(t)|| \leqq \lambda(t, s)||\tilde{c}|| \quad (t \in \mathbf{R}_s),$$

where

$$(2.6) \qquad \lambda(t, s) = \exp\left[\int_s^t (\omega(\tau) + \mu(\tau))d\tau\right].$$

https://doi.org/10.4153/CJM-1976-111-4 Published online by Cambridge University Press

The existence and uniqueness of strongly continuous solutions to (1.2) leads to a natural definition of the fundamental matrix, as follows. For $j = 1, 2, \ldots$, let $\tilde{e}_j = (\delta_{ij} : i = 1, 2, \ldots) \in l^1$, and let $\tilde{u}_j(t, s) = (u_{ij}(t, s) : i = 1, 2, \ldots)$ be the unique strongly continuous solution to (1.2) for the initial value $\tilde{c} = \tilde{e}_j$. Then the *fundamental matrix* for the system (1.2) is defined to be $U(t, s) = [u_{ij}(t, s)]$. Clearly $U(t, s)$ is defined whenever $0 \leqq s \leqq t$, and the following properties are immediate. First,

(2.7) $$U(s, s) = I, \quad s \in \mathbf{R}_0,$$

where $I = [\delta_{ij}]$ is the infinite identity matrix. Also, $U(t, s)$ defines a bounded operator on l^1; in fact from (2.5), we have

(2.8) $$\|U(t, s)\| \leqq \lambda(t, s), \quad 0 \leqq s \leqq t.$$

Furthermore, $U(t, s)$ is strongly continuous as a function of t, for each s; this follows from the strong continuity of the functions $\tilde{u}_j(t, s)$, which form the columns of $U(t, s)$. The next result gives further properties of $U(t, s)$.

THEOREM 2.2. (i) *For each $\tilde{c} \in l^1$, the strongly continuous solution of* (1.2) *is* $U(t, s)\tilde{c}$.

(ii) *Whenever* $s \leqq \tau \leqq t$,

(2.9) $$U(t, \tau)U(\tau, s) = U(t, s).$$

(iii) $U(t, s)$ *is strongly continuous as a function of two variables on* $\Delta = \{(t, s) : 0 \leqq s \leqq t\}$.

Proof. (i) It suffices to show that (2.1) holds for $\tilde{x}(t) = U(t, s)\tilde{c}$ $(t \in \mathbf{R}_s)$. First note that $U(t, s)\tilde{c} = \sum_{j=1}^{\infty} c_j \tilde{u}_j(t, s)$, and the series converges uniformly for t in a compact subset of $\mathbf{R}_s$. Also, by definition of $\tilde{u}_j(t, s)$, we have

$$\tilde{u}_j(t, s) = K(t, s)\tilde{e}_j + \int_s^t K(t, \tau)B(\tau)\tilde{u}_j(\tau, s)d\tau,$$

for each j and each $t \in \mathbf{R}_s$. Therefore

$$\begin{aligned} U(t, s)\tilde{c} &= \sum_{j=1}^{\infty} c_j \tilde{u}_j(t, s) \\ &= \sum_{j=1}^{\infty} c_j K(t, s)\tilde{e}_j + \sum_{j=1}^{\infty} c_j \int_s^t K(t, \tau)B(\tau)\tilde{u}_j(\tau, s)d\tau \\ &= K(t, s)\tilde{c} + \int_s^t K(t, \tau)B(\tau) \sum_{j=1}^{\infty} c_j \tilde{u}_j(\tau, s)d\tau, \end{aligned}$$

where we have used the local uniform convergence of $\sum c_j \tilde{u}_j(t, s)$ to interchange the integral and the summation in the last term. Thus (i) is proved.

(ii) Suppose $\tau \in \mathbf{R}_s$. From (i), it follows that $\tilde{y}(t) = U(t, \tau)U(\tau, s)\tilde{c}$ $(t \in \mathbf{R}_\tau)$ is the strongly continuous solution of (1.2) on $\mathbf{R}_\tau$ with $\tilde{y}(\tau) = U(\tau, s)\tilde{c}$. On the other hand, if $\tilde{x}(t) = U(t, s)\tilde{c}$, then the differential equations in (1.2) are satisfied for $t \in \mathbf{R}_s$, in particular for $t \in \mathbf{R}_\tau$, and we have $\tilde{x}(\tau) = U(\tau, s)\tilde{c} = \tilde{y}(\tau)$. Thus by uniqueness $\tilde{x}(t) = \tilde{y}(t)$ whenever $t \in \mathbf{R}_\tau$, and that proves (ii).

(iii) To see this, we re-examine the proof of Theorem 2.1. Note that the successive approximations and the solution to (2.1) depend on s as well as on t. Furthermore, the first approximation $\tilde{x}_0$ is clearly jointly continuous, and by induction each $\tilde{x}_n$ is jointly continuous. Since the estimate (2.4) together with (A_2) and (A_3) imply that the approximations $\tilde{x}_n$ converge to the solution $\tilde{x}$ uniformly on compact subsets of the triangle $0 \leqq s \leqq t$, $\tilde{x}$ is also jointly continuous. In particular, each $\tilde{u}_j$ is jointly continuous, and that proves (iii).

To conclude this section, we note the following identity, which follows immediately from Theorem 2.2(i): for each $\tilde{c} \in l^1$ and any s and t such that $0 \leqq s \leqq t$,

$$K(t, s)\tilde{c} = U(t, s)\tilde{c} - \int_s^t K(t, \tau)B(\tau)U(\tau, s)\tilde{c}d\tau. \tag{2.10}$$

We need this identity in the proof of the existence of solutions to nonlinear systems.

3. Approximation via finite truncation. In this section, we shall show that under the assumptions of Theorem 2.1, the solution of the system (1.2) is the limit of solutions to finite systems obtained from (1.2) by truncation. This result is of interest in itself, and also is useful in the study of the exponential stability of solutions in Section 5.

Given an infinite matrix function $A(t) = [a_{ij}(t)]$, we define $A_n(t) = [a_{ij}^{(n)}(t)]$ as follows:

$$a_{ij}^{(n)}(t) = \begin{cases} a_{ij}(t) & \text{if } 0 \leqq i, j \leqq n, \text{ or if } i = j; \\ 0 & \text{otherwise,} \end{cases}$$

and set $B_n(t) = A_n(t) - D(t)$. If $\tilde{c} = (c_i)$ is a sequence, we write $\tilde{c}_n$ for the truncation of $\tilde{c}$ after n terms: $\tilde{c}_n = (c_1, \ldots, c_n, 0, \ldots)$. Finally, we denote by $\tilde{x}^{(n)}(t)$ the solution to the integral equation

$$\tilde{x}(t) = K(t, s)\tilde{c}_n + \int_s^t K(t, \tau)B_n(\tau)\tilde{x}(\tau)d\tau \quad (t \in \mathbf{R}_s); \tag{3.1}$$

equivalently, $\tilde{x}^{(n)}(t)$ is the solution to the finite system of differential equations

$$\begin{cases} \dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}(t)x_j(t), & t \in \mathbf{R}_s, i = 1, \ldots, n \\ x_i(s) = c_i, & i = 1, \ldots, n, \end{cases} \tag{3.2}$$

augmented by the terms $x_i(t) \equiv 0$ for $i > n$.

THEOREM 3.1. *Assume that* (A_1), (A_2) *and* (A_3) *hold. Then the functions* $\tilde{x}^{(n)}(t)$ *converge to the solution* $\tilde{x}(t)$ *of the system* (1.2), *uniformly on compact subsets of* $\mathbf{R}_s$, *as* $n \to \infty$.

Proof. Fix $T \in \mathbf{R}_s$; we will show that $\tilde{x}^{(n)}(t) \to \tilde{x}(t)$ uniformly on $[s, T]$ as $n \to \infty$. First note that

$$\begin{aligned} ||\tilde{x}(t) - \tilde{x}^{(n)}(t)|| \leqq\ & ||K(t, s)||\, ||\tilde{c} - \tilde{c}_n|| \\ & + \int_s^t ||K(t, \tau)||\, ||[B(\tau) - B_n(\tau)]\tilde{x}(\tau)||d\tau \\ & + \int_s^t ||K(t, \tau)||\, ||B_n(\tau)||\, ||\tilde{x}(\tau) - \tilde{x}^{(n)}(\tau)||d\tau \\ & \equiv I_1 + I_2 + I_3. \end{aligned}$$

Since $K(t, s)$ is uniformly bounded on $[s, T]$, $I_1 \to 0$ as $n \to \infty$ uniformly on $[s, T]$. To estimate I_2, we observe that for any $\tilde{y} \in l^1$.

$$||[B(\tau) - B_n(\tau)]\tilde{y}|| \leqq ||B(\tau)(\tilde{y} - \tilde{y}_n)|| + ||B(\tau)\tilde{y} - (B(\tau)\tilde{y})_n||.$$

The first of these terms tends to 0 as $n \to \infty$, uniformly on $[s, T]$, since $B(\tau)$ is uniformly bounded there and $\tilde{y}_n \to \tilde{y}$ in l^1. The second term tends to 0 as $n \to \infty$ uniformly on $[s, T]$, since $\{B(\tau)\tilde{y} : \tau \in [s, T]\}$ is compact in l^1, and compact sets in l^1 can be characterized as those sets which are closed, bounded and have elements with uniformly converging sums (see the remarks following the proof). Hence, $B_n(\tau) \to B(\tau)$ as $n \to \infty$, strongly and uniformly on $[s, T]$. From this it follows that $||[B(\tau) - B_n(\tau)]\tilde{y}|| \to 0$ as $n \to \infty$, uniformly for $\tau \in [s, T]$ and $\tilde{y}$ is a compact subset of l^1. This is because whenever $W(\tau)$ is a strongly continuous operator, the map $(\tau, \tilde{y}) \mapsto W(\tau)\tilde{y} : \mathbf{R}_s \times l^1 \to l^1$ is continuous, so if $W(\tau_0)\tilde{y}_0$ is small, then $W(\tau)\tilde{y}$ is small uniformly in a neighborhood of $(\tau_0, \tilde{y}_0)$. Now, $\tilde{x}(t)$ being strongly continuous, we have $||[B(\tau) - B_n(\tau)]\tilde{x}(\tau)|| \to 0$, uniformly on $[s, T]$, as $n \to \infty$. Since $K(t, \tau)$ is uniformly bounded for $s \leqq \tau \leqq t \leqq T$, we have $I_2 \to 0$, uniformly for $t \in [s, T]$, as $n \to \infty$.

Thus, there are constants ϵ_n such that $I_1 + I_2 \leqq \epsilon_n$ uniformly on $[s, T]$, and $\epsilon_n \to 0$ as $n \to \infty$. To obtain a final estimate for I_3, we simply note that $||B_n(\tau)|| \leqq ||B(\tau)||$ for all n and all τ, and that $B(\tau)$ is uniformly bounded on $[s, T]$. So, there is a constant M such that

$$I_3 \leqq M \int_s^t ||\tilde{x}(\tau) - \tilde{x}^{(n)}(\tau)||d\tau$$

for all $t \in [s, T]$. Combining these results, we have

$$||\tilde{x}(t) - \tilde{x}^{(n)}(t)|| \leqq \epsilon_n + M \int_s^t ||\tilde{x}(\tau) - \tilde{x}^{(n)}(\tau)||d\tau$$

for all $t \in [s, T]$. Applying Gronwall's inequality yields

$$||\tilde{x}(t) - \tilde{x}^{(n)}(t)|| \leqq \epsilon_n \exp [M(t - s)] \leqq \epsilon_n \exp [M(T - s)]$$

for all $t \in [s, T]$, and the theorem is proved.

The characterization of compact sets in l^1 will be used again in Section 4. Of course, a set is compact if and only if it is complete and totally bounded, and a set S in l^1 is totally bounded if and only if it is bounded, and for every $\epsilon > 0$ there is a positive integer N such that $\sum_{i>N}|y_i| < \epsilon$ for every $\tilde{y} = (y_i) \in S$.

4. The nonlinear system. In this section, we begin the study of the nonlinear system (1.1):

$$\begin{cases} \dot{x}_i(t) = \sum_{j=1}^{\infty} a_{ij}(t)x_j(t) + f_i(t, \tilde{x}(t)), \quad t \in \mathbf{R}_s, \\ x_i(s) = c_i. \end{cases}$$

In addition to the assumptions (A_1), (A_2) and (A_3), we now impose the following conditions on $\tilde{f}(t, \tilde{x}) = (f_i(t, \tilde{x}))$:

(F_1)$f_i : \mathbf{R}_0 \times l^1 \to \mathbf{C}$ *is continuous for each* i;

(F_2) $\|\tilde{f}(t, \tilde{x})\| = \sum_{i=1}^{\infty}|f_i(t, \tilde{x})|$ *converges uniformly on bounded subsets of* $\mathbf{R}_0 \times l^1$.

These assumptions imply that $\tilde{f} : \mathbf{R}_0 \times l^1 \to l^1$ is continuous, and moreover that $\tilde{f}(B)$ is totally bounded in l^1 whenever B is bounded in $\mathbf{R}_0 \times l^1$ (see the remarks at the end of Section 3). We obtain a solution to (1.1) by again converting to a strong integral equation.

THEOREM 4.1. *Let $\tilde{x}(t)$ be a strongly continuous function on $\mathbf{R}_s$. Then* (1.1) *holds if and only if $\tilde{x}(t)$ satisfies the strong integral equation*

$$\tilde{x}(t) = U(t, s)\tilde{c} + \int_s^t U(t, \tau)\tilde{f}(\tau, \tilde{x}(\tau))d\tau. \tag{4.1}$$

Proof. (We refer to Section 2 for the definition and properties of $U(t, s)$.) First assume that $\tilde{x}(t)$ satisfies (1.1). Integrating these equations, we obtain

$$\begin{aligned} x_i(t) = {} & \exp\left(\int_s^t a_{ii}(\tau)d\tau\right)c_i \\ & + \int_s^t \exp\left(\int_\tau^t a_{ii}(\sigma)d\sigma\right) \sum_{j\neq i} a_{ij}(\tau)x_j(\tau)d\tau \\ & + \int_s^t \exp\left(\int_\tau^t a_{ii}(\sigma)d\sigma\right)f_i(\tau, \tilde{x}(\tau))d\tau. \end{aligned}$$

Because of the strong continuity of $\tilde{x}(t)$ and the various boundedness assumptions, this gives

$$\tilde{x}(t) = K(t, s)\tilde{c} + \int_s^t K(t, \tau)B(\tau)\tilde{x}(\tau)d\tau + \int_s^t K(t, \tau)\tilde{f}(\tau, \tilde{x}(\tau))d\tau, \tag{4.2}$$

where all the integrals are strong integrals. Substituting (2.10) in the first and third terms of (4.2), and changing the order of the integration in the resulting double integral, we obtain

$$(4.3) \qquad \tilde{x}(t) = (T_1\tilde{x})(t) + [T_2(I - T_1)\tilde{x}](t),$$

where we define

$$(T_1\tilde{y})(t) = U(t, s)\tilde{c} + \int_s^t U(t, \tau)\tilde{f}(\tau, \tilde{y}(\tau))d\tau$$

$$(T_2\tilde{y})(t) = \int_s^t K(t, \tau)B(\tau)\tilde{y}(\tau)\mathrm{d}\tau$$

for $\tilde{y}(t)$ strongly continuous on $\mathbf{R}_s$. From (4.3) we see that $(I - T_1)\tilde{x}$ is a fixed point for T_2. But $(I - T_1)\tilde{x}$ is strongly continuous, and by Theorem 2.1 (with $\tilde{c} = \tilde{0}$), T_2 has the unique strongly continuous fixed point $\tilde{0}$. So $(I - T_1)\tilde{x} = \tilde{0}$, and $\tilde{x}$ satisfies (4.1).

Conversely, suppose $\tilde{x}$ is strongly continuous and (4.1) holds. Then for each i and each $t \in \mathbf{R}_s$,

$$(4.4) \qquad x_i(t) = \sum_{j=1}^{\infty} u_{ij}(t, s)c_j + \int_s^t \sum_{j=1}^{\infty} u_{ij}(t, \tau)f_j(\tau, \tilde{x}(\tau))d\tau.$$

By Theorem 2.2, we have

$$(4.5) \qquad \frac{d}{dt}\left[\sum_{j=1}^{\infty} u_{ij}(t, s)c_j\right] = \sum_{k=1}^{\infty} a_{ik}(t) \sum_{j=1}^{\infty} u_{kj}(t, s)c_j.$$

Examining the second term of (4.4), we observe that the integrand is differentiable. In fact, replacing c_j by $f_j(\tau, \tilde{x}(\tau))$ and s by τ in (4.5), we get

$$\frac{d}{dt}\left[\sum_{j=1}^{\infty} u_{ij}(t, \tau)f_j(\tau, \tilde{x}(\tau))\right] = \sum_{k=1}^{\infty} a_{ik}(t) \sum_{j=1}^{\infty} u_{kj}(t, \tau)f_j(\tau, \tilde{x}(\tau))$$

whenever $t \geqq \tau$, and the sum with respect to k on the right converges absolutely and uniformly for t and τ in bounded sets, by the assumptions on $A(t)$, the continuity of $U(t, s)$ and the assumptions on $\tilde{f}(t, \tilde{x})$. Therefore

$$(4.6) \qquad \begin{aligned} &\frac{d}{dt}\int_s^t \sum_{j=1}^{\infty} u_{ij}(t, \tau)f_j(\tau, \tilde{x}(\tau))d\tau \\ &\qquad = f_i(t, \tilde{x}(t)) + \int_s^t \sum_{k=1}^{\infty} a_{ik}(t) \sum_{j=1}^{\infty} u_{kj}(t, \tau)f_j(\tau, \tilde{x}(\tau))d\tau. \end{aligned}$$

By uniform convergence, we may interchange the summation over k and the integration in the last term. Combining (4.5) and (4.6), we have from (4.4)

$$\dot{x}_i(t) = \sum_{k=1}^{\infty} a_{ik}(t)x_k(t) + f_i(t, \tilde{x}(t)).$$

Since $\tilde{x}(s) = \tilde{c}$ is immediate from (4.1), we have shown that $\tilde{x}(t)$ satisfies (1.1), as required.

THEOREM 4.2. *Assume that conditions* (A_1), (A_2) *and* (A_3) *hold, and assume that* $\tilde{f}(t, \tilde{x})$ *satisfies, in addition to* (F_1) *and* (F_2), *the following condition*:

(F_3) *there is a continuous function* $g : \mathbf{R}_0 \to \mathbf{R}_0$ *such that*

$$||\tilde{f}(t, \tilde{x})|| \leqq g(t)||\tilde{x}||$$

for all t *and all* $\tilde{x}$. *Then the system* (1.1) *has a strongly continuous solution.*

Proof. Let X_s denote the set of strongly continuous functions from $\mathbf{R}_s$ into l^1. With the operations of pointwise addition and scalar multiplication, and the the topology of uniform convergence on compact sets, X_s is a Fréchet space. Define a mapping $T : X_s \to X_s$ as follows: for $\tilde{x} \in X_s$,

$$(T\tilde{x})(t) = U(t, s)\tilde{c} + \int_s^t U(t, \tau)\tilde{f}(\tau, \tilde{x}(\tau))d\tau.$$

Clearly a solution of (4.1) is precisely a fixed point of T.

Now recall that $||U(t, s)|| \leqq \lambda(t, s)$, and that $\lambda(t, s)$ is continuous in t. Therefore, for any positive number r, the equation

$$u(t) = \lambda(t, s)r + \int_s^t \lambda(t, \tau)g(\tau)u(\tau)d\tau$$

has a continuous solution $u(t)$ on $\mathbf{R}_s$. Take $r \geqq ||\tilde{c}||$, fix such a solution $u(t)$, and define

$$B = \{\tilde{x} \in X_s : ||\tilde{x}(t)|| \leqq u(t) \quad \text{for all } t \in \mathbf{R}_s\}.$$

Then B is easily seen to be non-empty, convex, closed, and bounded in X_s. The theorem will follow from the fixed point theorem of Tychonoff (see [**6**, p. 45] or [**3**, p. 163]), if we can show that T is continuous, $T(B) \subseteq B$ and $T(B)$ is totally bounded in X_s. These results are proved in the following lemmas.

LEMMA 4.1. *T is continuous.*

Proof. Fix $\tilde{x} \in X_s$. Then for any $\tilde{y} \in X_s$ and any $t \in \mathbf{R}_s$,

$$\begin{aligned} ||(T\tilde{y})(t) - (T\tilde{x})(t)|| &\leqq \int_s^t ||U(t, \tau)|| \, ||\tilde{f}(\tau, \tilde{y}(\tau)) - \tilde{f}(\tau, \tilde{x}(\tau))||d\tau \\ &\leqq M_{t,s} \cdot \sup_{s \leqq \tau \leqq t} ||\tilde{f}(\tau, \tilde{y}(\tau)) - \tilde{f}(\tau, \tilde{x}(\tau))|| \end{aligned}$$

where $M_{t,s} = (t - s) \sup \{||U(t, \tau)|| : s \leqq \tau \leqq t\}$. This shows that it is sufficient to prove that for any $T \geqq s$, $||\tilde{f}(\tau, \tilde{y}(\tau)) - \tilde{f}(\tau, \tilde{x}(\tau))||$ can be made small uniformly on $[s, T]$ by making $||\tilde{y}(\tau) - \tilde{x}(\tau)||$ small uniformly on $[s, T]$.

Now fix $T \geqq s$ and $\epsilon > 0$. If $||\tilde{y}(\tau) - \tilde{x}(\tau)|| < 1$ for $\tau \in [s, T]$, then $||\tilde{y}(\tau)|| \leqq 1 + \sup_{s \leqq \tau \leqq T}||\tilde{x}(\tau)||$. By ($F_2$), there exists a positive integer N such that $\sum_{i>N}|f_i(\tau, \tilde{y}(\tau))| < \epsilon$ for all $\tau \in [s, T]$ and all such $\tilde{y}$. This implies

$$||\tilde{f}(\tau, \tilde{y}(\tau)) - \tilde{f}(\tau, \tilde{x}(\tau))|| \leqq \sum_{i=1}^{N} |f_i(\tau, \tilde{y}(\tau)) - f_i(\tau, \tilde{x}(\tau))| + 2\epsilon.$$

Now a fairly straightforward argument involving the continuity of the f_i, and the continuity of $\tilde{x}$ shows that there is a number $\delta > 0$ such that $||\tilde{y}(\tau) - \tilde{x}(\tau)|| < \delta$ on $[s, T]$ implies

$$\sum_{i=1}^{N} |f_i(\tau, \tilde{y}(\tau)) - f_i(\tau, \tilde{x}(\tau))| < \epsilon$$

on $[s, T]$. This completes the proof of the lemma.

LEMMA 4.2. $T(B) \subseteq B$.

Proof. If $\tilde{x} \in B$, then for any $t \in \mathbf{R}_s$,

$$\begin{aligned} ||(T\tilde{x})(t)|| &\leqq ||U(t, s)\tilde{c}|| + \int_s^t ||U(t, \tau)|| \, ||\tilde{f}(\tau, \tilde{x}(\tau))d\tau \\ &\leqq \lambda(t, s)||\tilde{c}|| + \int_s^t \lambda(t, \tau)g(\tau)||\tilde{x}(\tau)||d\tau \\ &\leqq \lambda(t, s)r + \int_s^t \lambda(t, \tau)g(\tau)u(\tau)d\tau \\ &= u(t). \end{aligned}$$

Thus $T\tilde{x} \in B$, as required.

LEMMA 4.3. *$T(B)$ is totally bounded in X_s.*

Proof. By the Ascoli-Arzéla theorem [**3**, p. 34], it suffices to show that, for each $t \in \mathbf{R}_s$, the set $V_t = \{(T\tilde{x})(t) : \tilde{x} \in B\}$ is totally bounded in l^1, and that $T(B)$ is equicontinuous.

Since $U(t, \tau)$ is a strongly continuous operator-valued function, the mapping $(\tau, \tilde{v}) \mapsto U(t, \tau)\tilde{v}$ from $[s, t] \times l^1$ into l^1 is continuous. Thus, if S is a compact set in l^1, the set $\{U(t, \tau)\tilde{v} : s \leqq \tau \leqq t, \tilde{v} \in S\}$ is compact. In particular, $V = \{U(t, \tau)\tilde{f}(\tau, \tilde{x}(\tau)) : s \leqq \tau \leqq t, \tilde{x} \in B\}$ is totally bounded, in view of (F_2). Therefore co $(V)^-$, the closed convex hull of V, is compact (see [**3**, pp. 163-4]). Since

$$(T\tilde{x})(t) = U(t, s)\tilde{c} + \int_s^t U(t, \tau)\tilde{f}(\tau, \tilde{x}(\tau))d\tau,$$

and if $\tilde{x} \in B$, the integral has the value $(t - s)\tilde{v}$ for some $\tilde{v} \in \mathrm{co}(V)^-$, the set V_t is totally bounded.

To prove that $T(B)$ is equicontinuous, fix $t \in \mathbf{R}_s$; then for $\tilde{x} \in B$ and $h > 0$

$$\begin{aligned} &||(T\tilde{x})(t + h) - (T\tilde{x})(t)|| \\ &\quad \leqq ||[U(t + h, s) - U(t, s)]\tilde{c}|| \\ &\qquad + ||\int_t^{t+h} U(t + h, \tau)\tilde{f}(\tau, \tilde{x}(\tau))d\tau|| \\ &\qquad + ||\int_s^t [U(t + h, \tau) - U(t, \tau)]\tilde{f}(\tau, \tilde{x}(\tau))d\tau|| \\ &\quad = E_1 + E_2 + E_3. \end{aligned}$$

Clearly $E_1 \to 0$ as $h \to 0^+$, and E_1 is independent of $\tilde{x}$. To estimate E_2, fix any $T > t$; then for h such that $t < t + h < T$,

$$E_2 \leqq \int_t^{t+h} ||U(t + h, \tau)|| \, ||\tilde{f}(\tau, x(\tau))||d\tau \leqq hPQ,$$

where $P = \sup \{||U(\sigma, \tau)|| : i \leqq \tau \leqq \sigma \leqq T\}$ and

$$Q = \sup \{||\tilde{f}(\tau, \tilde{x}(\tau))|| : t \leqq \tau \leqq T, \tilde{x} \in B\}.$$

The last estimate is $O(h)$ as $h \to 0^+$ and is independent of $\tilde{x} \in B$. Finally, we have from (2.9) that

$$E_3 \leqq ||[U(t + h, t) - I] \int_s^t U(t, \tau)\tilde{f}(\tau, \tilde{x}(\tau))d\tau||.$$

Now $U(t + h, t) - I \to 0$ strongly, hence uniformly on compact sets, as $h \to 0^+$. For any $\tilde{x} \in B$, the integral in the last expression has its value in $(t - s) \operatorname{co}(V)^-$. Since $\operatorname{co}(V)^-$ is compact, $E_3 \to 0$ as $h \to 0^+$, uniformly with respect to $\tilde{x} \in B$. This shows that $T(B)$ is equicontinuous to the right at t. Equicontinuity to the left at t is proven similarly, and that proves the lemma.

Remarks. (a) With regard to Lemma 4.3, we should point out that in proving left equicontinuity of $T(B)$, the estimation of the term corresponding to E_3 is more complicated, but not essentially different. Also, there is another method by which these particular terms can be estimated: take the norm inside the integral, and apply the Lebesgue dominated convergence theorem.

(b) The main importance of condition (F_3) of Theorem 4.2 is to guarantee the existence of the scalar function $u(t)$ which is used to define the set B. This condition can be weakened as follows: there is a continuous function $g : \mathbf{R}_s \times \mathbf{R}_0 \to \mathbf{R}_0$, monotonic non-decreasing in the second variable, such that $||\tilde{f}(t, \tilde{x})|| \leqq g(t, ||\tilde{x}||)$, and such that the scalar equation

$$u(t) = \lambda(t, s)r + \int_s^t \lambda(t, \tau)g(\tau, u(\tau))d\tau$$

has a solution $u(t)$.

5. Exponential stability. In this section, we are concerned with the asymptotic behavior of solutions to (1.1). We say that the system (1.1) is *exponentially stable* if there are positive constants K and δ such that every strongly continuous solution $\tilde{x}(t)$ of (1.1) satisfies

$$(5.1) \qquad ||\tilde{x}(t)|| \leqq K||\tilde{c}||e^{-\delta(t-s)}, \quad t \in \mathbf{R}_s.$$

Regarding (1.2) as a special case of (1.1), we first seek conditions on $A(t)$ which guarantee the exponential stability of (1.2).

We call $A(t)$ *vertically diagonally dominant* if there is a positive number δ such that

$$(5.2) \qquad -\operatorname{Re} a_{jj}(t) \geqq \delta + \sum_{i \neq j} |a_{ij}(t)|$$

for $j = 1, 2, \ldots$ and $t \in \mathbf{R}_0$. Finite systems of differential equations with a diagonally dominant coefficient matrix have been studied by Kahane [**5**] and Fink [**4**]. The following theorem extends their result to the context of the present article.

THEOREM 5.1. *Assume that conditions* (A_1) *and* (A_3) *hold, and that* $A(t)$ *is vertically diagonally dominant. Then, for each* $\tilde{c}$ *in* l^1, *the system* (1.2) *is exponentially stable; in fact, with* δ *as in* (5.2), *we have*

$$(5.3) \qquad ||\tilde{x}(t)|| \leqq ||\tilde{c}||e^{-\delta(t-s)}, \quad t \in \mathbf{R}_s.$$

Proof. First note that (A_2) is satisfied with $\omega(t) = -\delta$ for all t, so (1.2) has a unique strongly continuous solution for each $\tilde{c}$.

Now let $\tilde{x}^{(n)}(t)$ be as in Section 3. Since the first n coordinates of $\tilde{x}^{(n)}(t)$ form the solution to (3.2), and the diagonal dominance for $A(t)$ restricts to $A_n(t)$, we can apply the result of Kahane [**5**] and Fink [**4**] to get

$$(5.4) \qquad ||\tilde{x}^{(n)}(t)|| \leqq ||\tilde{c}_n||e^{-\delta(t-s)}, \quad t \in \mathbf{R}_s$$

for each n. Letting $n \to \infty$ in (5.4) and using Theorem 3.1, we obtain (5.3).

Now we turn to the nonlinear system (1.1), and investigate the asymptotic relationships between solutions of (1.1) and that of (1.2). The following theorem is a natural extension of a result for finite systems (cf. [**2**, pp. 54, 65]).

THEOREM 5.2. *Assume that* $A(t)$ *satisfies conditions* (A_1), (A_2) *and* (A_3), *and that* $\tilde{f}(t, \tilde{x}(t))$ *satisfies conditions* (F_1), (F_2) *and* (F_3) *with* $\int_0^\infty g(t) < \infty$. *Then exponential stability of* (1.2) *implies exponential stability of* (1.1).

Proof. Suppose (1.2) is exponentially stable, and let K and δ be such that (5.1) holds for solutions to (1.2). Then by Theorem 2.2(i), we have

$$||U(t, s)\tilde{c}|| \leqq K||\tilde{c}||e^{-\delta(t-s)}$$

whenever $0 \leqq s \leqq t$ and $\tilde{c} \in l^1$. Applying this to $\tilde{c} = \tilde{e}_j$, and taking the supremum over j, we get

$$(5.5) \qquad ||U(t, s)|| \leqq Ke^{-\delta(t-s)}, \quad 0 \leqq s \leqq t.$$

Let $\tilde{x}(t)$ be a strongly continuous solution to (1.1). Then by Theorem 4.1,

$$\tilde{x}(t) = U(t, s)\tilde{c} + \int_s^t U(t, \tau)\tilde{f}(\tau, \tilde{x}(\tau))d\tau, \quad t \in \mathbf{R}_s,$$

and therefore, using (5.5) and (F_3),

$$||\tilde{x}(t)|| \leqq K||\tilde{c}||e^{-\delta(t-s)} + K\int_s^t e^{-\delta(t-\tau)}g(\tau)||\tilde{x}(\tau)||d\tau.$$

https://doi.org/10.4153/CJM-1976-111-4 Published online by Cambridge University Press

The last inequality is equivalent to

$$||\tilde{x}(t)||e^{\delta(t-s)} \leqq K||\tilde{c}|| + K\int_s^t e^{\delta(\tau-s)}g(\tau)||\tilde{x}(\tau)||d\tau.$$

By Gronwall's inequality, we have

$$||\tilde{x}(t)||e^{\delta(t-s)} \leqq K||\tilde{c}|| \exp\left(K\int_s^t g(\tau)d\tau\right).$$

If we put $K_1 = K \exp\ (K\int_0^\infty g(\tau)d\tau)$, we get

$$||\tilde{x}(t)|| \leqq K_1||\tilde{c}||e^{-\delta(t-s)},$$

and the theorem is proved.

COROLLARY. *If* (A_1) *and* (A_3) *hold, and* $A(t)$ *is vertically diagonally dominant, and if* (F_1), (F_2), *and* (F_3) *hold with* $\int_0^\infty g(\tau)d\tau < \infty$, *then* (1.1) *is exponentially stable.*

REFERENCES

1. R. Bellman, *The boundedness of solutions of infinite systems of linear differential equations*, Duke Math. J. *14* (1947), 695–706.
2. W. A. Coppel, *Stability and asymptotic behavior of differential equations* (Heath Math. Monographs, Boston, 1965).
3. R. E. Edwards, *Functional analysis theory and applications* (Holt, Rinehart, Winston, New York etc., 1965).
4. A. M. Fink, *Almost periodic solutions to forced Lienard equations*, Proc. 6th International Conference on Nonlinear Oscillations, 1974, pp. 95–105.
5. Charles Kahane, *Stability of solutions of linear systems with dominant main diagonal*, Proc. Amer. Math. Soc. *33* (1972), 69–71.
6. V. Lakshmikantham and S. Leela, *Differential and integral inequalities*, vol. 1 (Academic Press, New York, 1969).
7. J. P. McClure and R. Wong, *On infinite systems of linear differential equations*, Can. J. Math. *27* (1975), 691–703.
8. Leonard Shaw, *Solutions for infinite-matrix differential equations*, J. Math. Anal. Appl. *41* (1973), 373–383.

University of Manitoba,
Winnipeg, Manitoba