THE LATTICE OF INVERSE SUBSEMIGROUPS OF A REDUCED INVERSE SEMIGROUP†

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An inverse semigroup R is said to be *reduced* (or proper) if $\mathscr{R} \cap \sigma = \iota$ (where σ is the minimum group congruence on R). McAlister has shown ([3], [4]) that every reduced inverse semigroup is isomorphic with a " P-semigroup " $P(G, \mathscr{X}, \mathscr{Y})$, for some semilattice $\mathscr{Y}$, poset $\mathscr{X}$ containing $\mathscr{Y}$ as an ideal, and group G acting on $\mathscr{X}$ by order-automorphisms; (and, conversely, every P-semigroup is reduced). In [4], he also found the morphisms between P-semigroups, in terms of morphisms between the respective groups, and between the respective posets.

In this paper, we begin by finding which inverse semigroups can be isomorphic with an inverse subsemigroup of a P-semigroup. (Any such semigroup must be reduced, and hence isomorphic with another P-semigroup. The problem is therefore reduced to finding injective morphisms between P-semigroups.) We then show that any inverse subsemigroup of $P(G, \mathscr{X}, \mathscr{Y})$ is determined by a subgroup of G, a subsemilattice of $\mathscr{Y}$, and a pair $(\mathscr{X}', \theta)$ consisting of a poset $\mathscr{X}'$ and a mapping θ of $\mathscr{X}'$ into $\mathscr{X}$ satisfying certain simple conditions. The obvious question to ask is then: for which inverse subsemigroups can $\mathscr{X}'$ be chosen as a *subposet* of $\mathscr{X}$? In Section 2 we answer this, and some related questions, and investigate a special case which leads to a construction for the congruences on $P(G, \mathscr{X}, \mathscr{Y})$ in terms of subgroups of G and subsemilattices of $\mathscr{Y}$ (that is, independently of $\mathscr{X}$).

Finally, we show that the set of all inverse subsemigroups of $P(G, \mathscr{X}, \mathscr{Y})$ with given semilattice of idempotents and maximal group homomorphic image is a sublattice of the lattice of inverse subsemigroups.

1. Inverse subsemigroups of P-semigroups. Let $\mathscr{Y}$ be a semilattice, $\mathscr{X}$ a poset containing $\mathscr{Y}$ as an ideal and G a group acting on $\mathscr{X}$ (on the left) by order-automorphisms. The P-semigroup $P(G, \mathscr{X}, \mathscr{Y})$ is defined as $P(G, \mathscr{X}, \mathscr{Y}) = \{(A, g) \in \mathscr{Y} \times G \mid g^{-1}A \in \mathscr{Y}\}$, with product $(A, g)(B, h) = (A \wedge gB, gh)$.

These semigroups were introduced by McAlister and McFadden [5], who showed that each P-semigroup R is a reduced inverse semigroup, that is $\mathscr{R} \cap \sigma(R) = \iota$, where $\sigma(R) = \{(x, y) \in R \mid ex = ey \text{ for some } e = e^2 \in R\}$, the minimum group congruence on R (Munn [6]). In a recent paper, McAlister proved that every reduced inverse semigroup is isomorphic with a P-semigroup $P(G, \mathscr{X}, \mathscr{Y})$, for some G, $\mathscr{X}$ and $\mathscr{Y}$ as above ([4; Theorem 2.6]), and further, G, $\mathscr{X}$ and $\mathscr{Y}$ can be chosen so that:

(P1) $G\mathscr{Y} = \mathscr{X}$;

and (P2) for all $g \in G$, $g\mathscr{Y} \cap \mathscr{Y} = \varnothing$.

For the remainder of the paper, when we write $P(G, \mathscr{X}, \mathscr{Y})$ we will assume G, $\mathscr{X}$ and $\mathscr{Y}$

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satisfy (P1) and (P2). (Note from (P2) that for every g in G, $(A,g)\in P(G,\mathscr{X},\mathscr{Y})$ for some $A\in\mathscr{Y}$.)

If $R=P(G,\mathscr{X},\mathscr{Y})$, define $\sigma^\natural(R)$ (or just $\sigma^\natural$) by $(A,g)\sigma^\natural(R)=g$, for all $(A,g)\in R$. Then $\sigma(R)=\sigma^\natural(\sigma^\natural)^{-1}$, and $R\sigma^\natural=G\cong R/\sigma(R)$. (For all relevant properties of P-semigroups, the reader is referred to [**3**], and for semigroups in general to [**1**].)

At this point some notation is needed. Let T be an inverse semigroup. We will denote by $E(T)$, or just E, its semilattice of idempotents. If S is an inverse subsemigroup of T, we will write $S\leqq T$. (If S and T are groups [semilattices], $S\leqq T$ will mean S is a subgroup [subsemilattice] of T.) A useful result in this context is the following, due to O'Carroll.

LEMMA 1.1 ([**7**, Proposition 1.1]). *If R is a reduced inverse semigroup and $U\leqq R$, let $\sigma(R)$ and $\sigma(U)$ be their respective minimum group congruences. Then $\sigma(U)=\sigma(R)\cap(U\times U)$.*

Hence if $U\leqq R=P(G,\mathscr{X},\mathscr{Y})$, $U\sigma^\natural(R)\cong U/\sigma(U)$, and $U\sigma^\natural(R)\leqq R\sigma^\natural(R)=G$. Define $\mathscr{Y}(U)=\{A\in\mathscr{Y}\,|\,(A,1)\in U\}$. Then $\mathscr{Y}(U)\cong E(U)$ and $\mathscr{Y}(U)\leqq\mathscr{Y}(R)=\mathscr{Y}$.

A mapping θ of a poset $\mathscr{X}'$ into another poset $\mathscr{X}$ will be called *monotone* if $X\leqq Y$ implies $X\theta\leqq Y\theta$, and an *order-isomorphism* of $\mathscr{X}'$ into $\mathscr{X}$ if $\mathscr{X}\leqq\mathscr{Y}$ iff $X\theta\leqq Y\theta$ (in which case θ must be injective).

Our first theorem is just a statement of McAlister's construction of the morphisms between P-semigroups. Its proof may be found in Theorems 1.3, 6.1 of [**4**].

THEOREM 1.2. *Let $R'=P(G',\mathscr{X}',\mathscr{Y}')$, $R=P(G,\mathscr{X},\mathscr{Y})$. Suppose there exist*

(i) *a monotone map $\theta:\mathscr{X}'\to\mathscr{X}$ whose restriction to $\mathscr{Y}'$ is a (semilattice) morphism into $\mathscr{Y}$,*

(ii) *a (group) morphism $\phi:G'\to G$, such that if $g\in G'$ and $X\in\mathscr{X}'$, $(gX)\theta=(g\phi)(X\theta)$.*

Define $(A,g)\psi=(A\theta,g\phi)$, for all $(A,g)\in R'$. Then ψ is a morphism of R' into R.

Conversely, suppose $\psi:R'\to R$ is a morphism. For each $g\in G'$, put $g\phi=(A,g)\psi\sigma^\natural(R)$ (where, by (P2) *for R', there exists $A\in\mathscr{Y}'$ such that $(A,g)\in R'$). For each $A\in\mathscr{Y}'$, put $A\theta=\bar{A}$, where $(\bar{A},1)=(A,1)\psi$, and for each $X\in\mathscr{X}'$, put $X\theta=(g\phi)(A\theta)$ (where, by* (P1) *for R', $X=gA$ for some $g\in G'$, $A\in\mathscr{Y}'$).*

Then ϕ,θ satisfy (i) *and* (ii) *above, and if $(A,g)\in R'$, then $(A,g)\psi=(A\theta,g\phi)$.*

Suppose now R is reduced, and R' is an inverse semigroup which is isomorphic with an inverse subsemigroup of R (under $\psi:R'\to R$, say). Then R' must be reduced, and is thus isomorphic to a P-semigroup. This motivates the next theorem.

THEOREM 1.3. *Let $R=P(G,\mathscr{X},\mathscr{Y})$. Suppose $R'=P(G',\mathscr{X}',\mathscr{Y}')$ is such that there exist θ, ϕ satisfying:*

(i) *θ is a monotone map of $\mathscr{X}'$ into $\mathscr{X}$, whose restriction to $\mathscr{Y}'$ is a monomorphism into $\mathscr{Y}$;*

(ii) *ϕ is a monomorphism of G' into G;*

(iii) *if $g\in G'$, $X\in\mathscr{X}'$, $(gX)\theta=(g\phi)(X\theta)$.*

Then R' is isomorphic with an inverse subsemigroup of R, under the monomorphism ψ, defined by $(A,g)\psi=(A\theta,g\phi)$, for all $(A,g)\in R'$.

Conversely, if R' is isomorphic with an inverse subsemigroup of R, there exist G', $\mathscr{X}'$, $\mathscr{Y}'$, θ and ϕ as above, such that $R'\cong P(G',\mathscr{X}',\mathscr{Y}')$.

Proof. The direct half is clear from the previous theorem, since if $\theta\,|\,\mathscr{Y}'$ and ϕ are injective, so is ψ.

Conversely, if $\alpha : R' \to R$ is a monomorphism, R' is reduced, so there is an isomorphism β, say, of $P' = P(G', \mathscr{X}', \mathscr{Y}')$ onto R', for some G', $\mathscr{X}'$, $\mathscr{Y}'$. Then $\psi = \beta\alpha : P(G', \mathscr{X}', \mathscr{Y}') \to P(G, \mathscr{X}, \mathscr{Y})$ is an (injective) morphism, so the previous theorem can be applied, yielding $\theta : \mathscr{X}' \to \mathscr{X}$ and $\phi : G' \to G$. It remains to show that $\theta | \mathscr{Y}'$ and ϕ are injective.

If A, $B \in \mathscr{Y}'$, then $(A\theta, 1) = (A, 1)\psi$ and $(B\theta, 1) = (B, 1)\psi$, so $A\theta = B\theta$ implies $A = B$, since ψ is injective. Now let $r, s \in P'$ and suppose $r\psi\sigma^\natural(R) = s\psi\sigma^\natural(R)$. If we can show that $r\sigma^\natural(P') = s\sigma^\natural(P')$ then injectiveness of ϕ will follow (from its definition). Now $(r\psi, s\psi) \in \sigma(R) \cap (P'\psi \times P'\psi) = \sigma(P'\psi)$, using Lemma 1.1. Therefore $f(r\psi) = f(s\psi)$ for some idempotent f of $P'\psi$. But from a result of Preston [8], $f = e\psi$ for some idempotent e of P', so $(er)\psi = (es)\psi$; thus since ψ is injective, $r\sigma^\natural(P') = s\sigma^\natural(P')$. Hence ϕ is injective.

Corollary 1.4 ([4, Theorem 1.3]). *Let $R' = P(G', \mathscr{X}', \mathscr{Y}')$ and $R = P(G, \mathscr{X}, \mathscr{Y})$. Suppose there exist*

(i) *an order isomorphism θ of $\mathscr{X}'$ onto $\mathscr{X}$ such that $\mathscr{Y}'\theta = \mathscr{Y}$,*
(ii) *an isomorphism ϕ of G' onto G, such that if $g \in G'$ and $X \in \mathscr{X}'$, then $(gX)\theta = (g\phi)(X\theta)$.*

Then $\psi : R' \to R$, defined by $(A, g)\psi = (A\theta, g\phi)$, is an isomorphism of R' onto R, and conversely, any isomorphism of R' onto R is found in this way.

Using Theorem 1.3, we can actually find the inverse subsemigroups of a P-semigroup directly (rather than up to isomorphism), for if S is a subset of $R = P(G, \mathscr{X}, \mathscr{Y})$, then $S \leqq R$ iff the injection $S \to R$ is a monomorphism.

Theorem 1.5. *Let $R = P(G, \mathscr{X}, \mathscr{Y})$. Suppose G' is a subgroup of G and $\mathscr{Y}'$ is a subsemilattice of $\mathscr{Y}$. If there exists a poset $\mathscr{X}'$ (containing $\mathscr{Y}'$ as an ideal), a group action $*$ of G' on $\mathscr{X}'$ (such that* (P1), (P2) *are satisfied) and a monotone map $\theta : \mathscr{X}' \to \mathscr{X}$ such that*

(i) *$\theta | \mathscr{Y}'$ is the identity,*
and (ii) *if $g \in G'$ and $A \in \mathscr{Y}'$, $(g * A)\theta = gA$,*

then $R' = P(G', \mathscr{X}', \mathscr{Y}')$ is an inverse subsemigroup of R.

Conversely, any inverse subsemigroup R' of R can be found in this way (with $G' = R'\sigma^\natural(R)$, $\mathscr{Y}' = \mathscr{Y}(R')$).

Proof. Given G', $\mathscr{X}'$, $\mathscr{Y}'$, $*$ and θ as above, we see that if $g \in G'$ and $X \in \mathscr{X}'$, then since $X = h * A$ for some $h \in G'$, $A \in \mathscr{Y}'$, $(g * X)\theta = ((gh) * A)\theta = (gh)A = g(h * A)\theta = g(X\theta)$. Thus, using Theorem 1.3, the injection $(A, g) \to (A, g)$ of R' into R is a monomorphism.

Conversely, if $R' \leqq R$, there is an isomorphism β of $P(G'', \mathscr{X}'', \mathscr{Y}'')$ onto R', for some G'', $\mathscr{X}''$, $\mathscr{Y}''$. By Theorem 1.3, there exists $\theta : \mathscr{X}'' \to \mathscr{X}$, monotone, whose restriction to $\mathscr{Y}''$ is a monomorphism, and $\phi : G'' \to G$, a monomorphism, such that $(gX)\theta = (g\phi)(X\theta)$ for all $g \in G''$, $X \in \mathscr{X}''$ and such that $(A, g)\beta = (A\theta, g\phi)$ for all $(A, g) \in P(G'', \mathscr{X}'', \mathscr{Y}'')$.

Put $G' = G''\phi \leqq G$, and $\mathscr{Y}' = \mathscr{Y}''\theta \leqq \mathscr{Y}$. From the definition of ϕ and θ in Theorem 1.2, we see that $R'\sigma^\natural(R) = G''\phi = G'$, and $\mathscr{Y}(R') = \mathscr{Y}'$. Put $\mathscr{X}' = \mathscr{Y}' \cup (\mathscr{X}'' \backslash \mathscr{Y}'')$ and define $\bar{\theta} : \mathscr{X}'' \to \mathscr{X}'$ by

$$X\bar{\theta} = \begin{cases} X\theta & \text{if } X \in \mathscr{Y}'' \\ X & \text{if } X \in \mathscr{X}'' \backslash \mathscr{Y}''. \end{cases}$$

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Since $\theta|\mathscr{Y}''$ is injective, $\bar{\theta}$ is a bijection of $\mathscr{X}''$ upon $\mathscr{X}'$. Now $\bar{\theta}$ induces a natural partial order on $\mathscr{X}'$ from $\mathscr{X}''$: if $X, Y \in \mathscr{X}'$, define $X \leqq Y$ iff $X\bar{\theta}^{-1} \leqq Y\bar{\theta}^{-1}$ (in $\mathscr{X}''$); and $\bar{\theta}$ induces a natural action $*$ of G' on $\mathscr{X}'$: if $h \in G'$, $Y \in \mathscr{X}'$, define $h * Y = ((h\phi^{-1})(Y\bar{\theta}^{-1}))\bar{\theta}$. Then $\bar{\theta}$ is an order-isomorphism of $\mathscr{X}''$ into $\mathscr{X}'$ such that $\mathscr{Y}''\bar{\theta} = \mathscr{Y}'$, and for all $g \in G''$, $X \in \mathscr{X}''$, $(gX)\bar{\theta} = (g\phi) * (X\bar{\theta})$.

It is easily checked that $P(G', \mathscr{X}', \mathscr{Y}')$ is well-defined. Hence from Corollary 1.4, the mapping $\bar{\psi} : P(G'', \mathscr{X}'', \mathscr{Y}'') \to P(G', \mathscr{X}', \mathscr{Y}')$ defined by $(A, g)\bar{\psi} = (A\bar{\theta}, g\phi)$ is an isomorphism. Therefore the map $\bar{\psi}^{-1}\beta : P(G', \mathscr{X}', \mathscr{Y}') \to R'$ is an isomorphism. But if $(B, h) \in P(G', \mathscr{X}', \mathscr{Y}')$, $(B, h)\bar{\psi}^{-1}\beta = (B\bar{\theta}^{-1}, h\phi^{-1})\beta = (B\theta^{-1}, h\phi^{-1})\beta = (B, h)$, so $\bar{\psi}^{-1}\beta$ is the identity. Hence $R' = P(G', \mathscr{X}', \mathscr{Y}')$. The map " θ " of the statement of the theorem is then $\bar{\theta}^{-1}\theta : \mathscr{X}' \to \mathscr{X}$, whence $\bar{\theta}^{-1}\theta\,|\,\mathscr{Y}'$ is the identity. If $h \in G'$, $B \in \mathscr{Y}'$, and $g = h\phi^{-1}$, $A = B\theta^{-1}$ then $(h * B)(\bar{\theta}^{-1}\theta) = (gA)\theta = (g\phi)(A\theta) = hB$. Thus (i) and (ii) are satisfied.

2. Unitary-like conditions. The question asked in the introduction can now be more formally posed: if $R' \leqq R = P(G, \mathscr{X}, \mathscr{Y})$, when can we assume $\mathscr{X}'$ of Theorem 1.5 to be a subposet of $\mathscr{X}$? Equivalently, by Corollary 1.4, when is $\theta : \mathscr{X}' \to \mathscr{X}$ an order-isomorphism of $\mathscr{X}'$ into $\mathscr{X}$?

In this section we answer this and some related questions. Firstly, however, we will discuss unitariness and similar properties of inverse semigroups.

DEFINITION ([**1**, Vol. 2, p. 55]). Let T be any semigroup, and A a subset of T. A is *left* [*right*] *unitary* (in T) if $s \in A$, $st \in A$ [$ts \in A$] imply $t \in A$. A is *unitary* (in T) if it is both left and right unitary.

LEMMA 2.1. *If S is an inverse subsemigroup of an inverse semigroup T, the following are equivalent* (*and hence equivalent to unitariness of S*):

(i) *S is left unitary;*
(ii) *S is right unitary;*
(iii) *$et \in S$, for some $e \in E(T)$, implies $t \in S$.*

Proof. Suppose $et \in S$, for some $e \in E(T)$. Then $(et)(et)^{-1}t = et \in S$, and $(et)(et)^{-1} \in S$. Similarly, $t(et)^{-1}(et) = et \in S$, and $(et)^{-1}(et) \in S$. Thus if S is either left or right unitary, $t \in S$, that is (iii) is satisfied.

Conversely, suppose s, $st \in S$. Then $(s^{-1}s)t = s^{-1}(st) \in S$, and $s^{-1}s \in E(T)$, so (iii) implies $t \in S$. Hence (iii) implies (i); similarly (iii) implies (ii).

Keeping (iii) of this lemma in mind, we define some weaker " unitary-like " conditions in inverse semigroups.

DEFINITION. Let T be an inverse semigroup and $S \leqq T$. We say S is *partially unitary* if $tt^{-1} \in S$, $et \in S$ for some $e \in E(T)$ imply $t \in S$. We say S is *weakly unitary* if $tt^{-1} \in S$, $et \in S$, $t^{-1}t \leqq f$, for some $e \in E(T)$, $f \in E(S)$ together imply $t \in S$.

Clearly unitary implies partially unitary, which in turn implies weakly unitary. These definitions are easily shown to be equivalent to their obvious duals. In reduced inverse semigroups, we can obtain more readily usable conditions (equivalent to those just defined), by means of the next lemma.

LEMMA 2.2. *Let R be a reduced inverse semigroup and $S \leqq R$. If $pp^{-1} \in S$ and $p\sigma^\natural \in S\sigma^\natural$, then $ep \in S$ for some $e \in E(R)$.*

Proof. Since $p\sigma^\natural \in S\sigma^\natural$, $p\sigma s$ for some $s \in S$. Therefore $(ss^{-1})p\sigma(pp^{-1})s$; also $(ss^{-1})p\mathcal{R}(pp^{-1})s$. But R is reduced, so $\mathcal{R} \cap \sigma = \iota$. Thus $(ss^{-1})p = (pp^{-1})s \in S$.

COROLLARY 2.3. *Let R be reduced and $S \leqq R$. S is partially unitary iff $pp^{-1} \in S$, $p\sigma^\natural \in S\sigma^\natural$ imply $p \in S$; S is weakly unitary iff the conditions*

$$pp^{-1} \in S, \quad p\sigma^\natural \in S\sigma^\natural, \quad p^{-1}p \leqq f \quad \textit{for some } f \in E(S)$$

together imply $p \in S$.

Proof. If S is partially unitary, the condition is satisfied, by the previous lemma. Conversely, if $pp^{-1} \in S$ and $ep \in S$, for some $e \in E(R)$, then $p\sigma ep$, so $p\sigma^\natural \in S\sigma^\natural$, and from the condition, we infer that $p \in S$. Thus S is partially unitary. The weakly unitary case is similar.

Throughout this section, we will see that the results are simpler to state in the case of *full* inverse subsemigroups (those inverse subsemigroups containing all the idempotents of the larger semigroup). Note firstly that unitariness, partial unitariness and weak unitariness are equivalent for full inverse subsemigroups. The previous corollary is then simplified considerably.

COROLLARY 2.4. *If R is reduced and S is a full inverse subsemigroup of R, then S is unitary iff $p\sigma^\natural \in S\sigma^\natural$ implies $p \in S$, that is, iff S is a union of σ-classes of R.*

With these definitions we now turn our attention to the question posed earlier.

THEOREM 2.5. *In the notation of Theorem 1.5, suppose θ is a mapping satisfying the conditions of that theorem. Then θ is an order-isomorphism of $\mathcal{X}'$ into $\mathcal{X}$ iff $R'(=P(G',\mathcal{X}',\mathcal{Y}'))$ is weakly unitary in R.*

Proof. Given G', $\mathcal{X}'$, $\mathcal{Y}'$, $*$ and θ as in Theorem 1.5, with θ an order-isomorphism, we have $R' = P(G',\mathcal{X}',\mathcal{Y}') \leqq R = P(G,\mathcal{X},\mathcal{Y})$. Suppose $p = (V,v) \in R$ and $pp^{-1} \in R'$, $p\sigma^\natural \in R'\sigma^\natural$ and $p^{-1}p \leqq f$, for some $f \in E(R')$. Then $(V,1) = pp^{-1} \in E(R')$, so $V \in \mathcal{Y}(R') = \mathcal{Y}'$; also $v = p\sigma^\natural \in R'\sigma^\natural = G'$. Let $f = (A,1)$ for some $A \in \mathcal{Y}'$. Then $(v^{-1}V,1) = p^{-1}p \leqq (A,1)$, so $v^{-1}V \leqq A$ (in $\mathcal{X}$). From (ii) of Theorem 1.5, $(v^{-1} * V)\theta = v^{-1}V \leqq A = A\theta$, and since θ is an order-isomorphism, $v^{-1} * V \leqq A$ (in $\mathcal{X}'$). Further, $\mathcal{Y}'$ is an ideal of $\mathcal{X}' = G' * \mathcal{Y}'$. Hence $v^{-1} * V \in \mathcal{Y}'$. Thus $p = (V,v) \in P(G',\mathcal{X}',\mathcal{Y}') = R'$. So R' is weakly unitary.

Conversely, suppose R' is weakly unitary. We know $R' = P(G',\mathcal{X}',\mathcal{Y}')$ for some G', $\mathcal{X}'$, $\mathcal{Y}'$, $*$ and θ as in Theorem 1.5. Let $X, Y \in \mathcal{X}'$ and suppose $X\theta \leqq Y\theta$ (in $\mathcal{X}$). We must show $X \leqq Y$. Since $\mathcal{X}' = G' * \mathcal{Y}'$, there exist $g, h \in G'$, $A, B \in \mathcal{Y}'$ such that $X = g * A$, $Y = h * B$. Using (ii) of Theorem 1.5, $gA = (g*A)\theta \leqq (h*B)\theta = hB$. Thus $(h^{-1}g)A \leqq B$ (in $\mathcal{X}$). Since $\mathcal{Y}$ is an ideal of $\mathcal{X}$, $(h^{-1}g)A \in \mathcal{Y}$. Therefore $p = (A, g^{-1}h) \in P(G,\mathcal{X},\mathcal{Y})$. Further $pp^{-1} = (A,1) \in R'$, $p\sigma^\natural = g^{-1}h \in G' = R'\sigma^\natural$, and $p^{-1}p = ((h^{-1}g)A,1) \leqq (B,1) \in E(R')$. Hence $p \in R'$, and so $(h^{-1}g) * A \in \mathcal{Y}'$. But $\theta|\mathcal{Y}'$ is the identity, so $h^{-1}g * A = (h^{-1}g * A)\theta = (h^{-1}g)A \leqq B$ (in $\mathcal{Y}'$, and therefore in $\mathcal{X}'$). Hence $g * A \leqq h * B$, that is $X \leqq Y$.

COROLLARY 2.6 (to the proof). *In the notation of Theorem 1.5, suppose θ is a mapping* *satisfying the conditions of that theorem. Then θ is injective iff R' satisfies:*

$$pp^{-1},\ p^{-1}p \in R',\ p\sigma^\natural \in R'\sigma^\natural \Rightarrow p \in R'$$

Proof. By replacing inequalities by equalities in the relevant places of the proof of the theorem, this follows immediately.

If we restrict ourselves to full inverse subsemigroups of $P(G, \mathscr{X}, \mathscr{Y})$, the situation is again simplified (by using Corollary 2.4).

COROLLARY 2.7. *In the notation of Theorem* 1.5, *suppose θ is a mapping satisfying the conditions of that theorem, and further, that $\mathscr{Y}' = \mathscr{Y}$. Then θ is an order-isomorphism iff R' is unitary.*

Using Corollary 1.4, we can now give some alternative interpretations of Theorem 2.5. If θ of that theorem is an order-isomorphism of $\mathscr{X}'$ onto $\mathscr{X}'\theta \subseteq \mathscr{X}$, it is easily checked that $\mathscr{X}'\theta = G'\mathscr{Y}'$ (in $\mathscr{X}$), and that $P(G', G'\mathscr{Y}', \mathscr{Y}')$ is well-defined. Then by Corollary 1.4, $P(G', \mathscr{X}', \mathscr{Y}') = P(G', G'\mathscr{Y}', \mathscr{Y}')$ (not, of course, as triples but as inverse semigroups). Conversely, if $R' = P(G', G'\mathscr{Y}', \mathscr{Y}')$, the injection $G'\mathscr{Y}' \to \mathscr{X}$ is an order-isomorphism satisfying the properties of Theorem 1.5; hence $R' \leqq R$.

Therefore the inverse subsemigroups $P(G', \mathscr{X}', \mathscr{Y}')$ for which $\mathscr{X}'$ can be taken as a subposet of $\mathscr{X}$ (and thus as $G'\mathscr{Y}'$) are just the weakly unitary ones. These results are summarized in the next theorem, which from another viewpoint therefore can also be considered as a structure theorem for the weakly unitary inverse subsemigroups (in terms of subgroups of G and subsemilattices of $\mathscr{Y}$). A definition is required first.

DEFINITION. Let $R = P(G, \mathscr{X}, \mathscr{Y})$. If $G' \leqq G$ and $\mathscr{Y}' \leqq \mathscr{Y}$, the pair $(G', \mathscr{Y}')$ is said to be *compatible* (in R) if

(i) $g\mathscr{Y}' \cap \mathscr{Y}' \neq \varnothing$ for each g in G',
(ii) $\mathscr{Y}'$ is an ideal of $G'\mathscr{Y}'$ (in $\mathscr{X}$).

Thus given $G' \leqq G$ and $\mathscr{Y}' \leqq \mathscr{Y}$, $P(G', G'\mathscr{Y}', \mathscr{Y}')$ is well-defined (and hence a (weakly unitary) inverse subsemigroup of R) iff $(G', \mathscr{Y}')$ is a compatible pair in R. Note also that if $(A, g) \in R$, then $(A, g) \in P(G', G'\mathscr{Y}', \mathscr{Y}')$ iff $A, g^{-1}A \in \mathscr{Y}'$ and $g \in G'$. Hence

$$P(G', G'\mathscr{Y}', \mathscr{Y}' = \{(A, g) \in \mathscr{Y}' \times G' \mid g^{-1}A \in \mathscr{Y}'\}.$$

THEOREM 2.8. *Let $R = P(G, \mathscr{X}, \mathscr{Y})$, and $S \leqq R$. Put $G' = S\sigma^\natural$ and $\mathscr{Y}' = \mathscr{Y}(S)$. Then S is weakly unitary iff $S = P(G', G'\mathscr{Y}', \mathscr{Y}')$. Thus there is a one-to-one correspondence $(G', \mathscr{Y}') \to P(G', G'\mathscr{Y}', \mathscr{Y}')$ between the compatible pairs and the weakly unitary inverse subsemigroups of R.*

Proof. This follows from the comments above.

We now specialize to partially unitary inverse subsemigroups of $R = P(G, \mathscr{X}, \mathscr{Y})$ and show (in Theorem 2.9) that these are just the "naturally-occurring" inverse subsemigroups $R \cap (\mathscr{Y}' \times G')$, for certain compatible pairs $(G', \mathscr{Y}')$ in R. These inverse subsemigroups will be used in the next section.

DEFINITION. Let $R = P(G, \mathscr{X}, \mathscr{Y})$. If $G' \leqq G$ and $\mathscr{Y}' \leqq \mathscr{Y}$, the pair $(G', \mathscr{Y}')$ is said to be *fully compatible* (in R) if

(i) $g\mathscr{Y}'\cap\mathscr{Y}' \neq \varnothing$, for each g in G',
(ii) $G'\mathscr{Y}'\cap\mathscr{Y} = \mathscr{Y}'$.

If $(G',\mathscr{Y}')$ is fully compatible, it is also compatible, for $\mathscr{Y}'$ is an ideal of $G'\mathscr{Y}'$. If $g\in G'$, $A\in\mathscr{Y}'$ and $gA \leqq B$ for some $B\in\mathscr{Y}'$, then since $\mathscr{Y}'\subseteq\mathscr{Y}$, an ideal of $\mathscr{X}$, $gA\in G'\mathscr{Y}'\cap\mathscr{Y} = \mathscr{Y}'$. Hence $P(G', G'\mathscr{Y}', \mathscr{Y}')$ is well-defined. In fact if $g\in G'$ and $A\in\mathscr{Y}'$ then $g^{-1}A\in\mathscr{Y}'$ iff $g^{-1}A\in\mathscr{Y}$ (since $G'\mathscr{Y}'\cap\mathscr{Y} = \mathscr{Y}'$); thus $P(G', G'\mathscr{Y}', \mathscr{Y}') = R\cap(\mathscr{Y}' \times G')$.

THEOREM 2.9. *Let $R = P(G, \mathscr{X}, \mathscr{Y})$. If S is a partially unitary inverse subsemigroup of R, then $(G', \mathscr{Y}') = (S\sigma^\natural, \mathscr{Y}(S))$ is fully compatible and $S = R\cap(\mathscr{Y}' \times G')$. Conversely, if $(G', \mathscr{Y}')$ is a fully compatible pair in R, then $S = R\cap(\mathscr{Y}' \times G')$ is a partially unitary inverse subsemigroup of R, and $(G', \mathscr{Y}') = (S\sigma^\natural, \mathscr{Y}(S))$.*

Proof. If S is partially unitary, then it is weakly unitary. By the previous theorem, therefore, $(G', \mathscr{Y}')$ is compatible and $S = P(G', G'\mathscr{Y}', \mathscr{Y}')$. Suppose $g\in G'$, $A\in\mathscr{Y}'$ and $gA\in\mathscr{Y}$. Then $p = (A, g^{-1})\in R$, $pp^{-1} = (A, 1)\in S$ and $p\sigma^\natural = g^{-1}\in S\sigma^\natural$, so $p\in S$ (using Corollary 2.3), that is $gA\in\mathscr{Y}'$. Therefore $G'\mathscr{Y}'\cap\mathscr{Y} = \mathscr{Y}'$ (since $\mathscr{Y}'\subseteq G'\mathscr{Y}'\cap\mathscr{Y}$); so $(G', \mathscr{Y}')$ is fully compatible, and $S = R\cap(\mathscr{Y}' \times G')$.

Conversely, if $(G', \mathscr{Y}')$ is fully compatible, then, as we have seen, $S = R\cap(\mathscr{Y}' \times G') = P(G', G'\mathscr{Y}', \mathscr{Y}') \leqq R$ (from the previous theorem), and $S\sigma^\natural = G', \mathscr{Y}(S) = \mathscr{Y}'$. To show S is partially unitary, let $p\in R$, with $pp^{-1}\in S$ and $p\sigma^\natural\in S\sigma^\natural$. If $p = (A, g)$, then $A\in\mathscr{Y}(S) = \mathscr{Y}'$ and $g\in S\sigma^\natural = G'$. Thus $p = (A, g)\in R\cap(\mathscr{Y}' \times G') = S$.

Finally, we again consider the case of full inverse subsemigroups. Let $R = P(G, \mathscr{X}, \mathscr{Y})$ and $G' \leqq G$. Then $g\mathscr{Y}\cap\mathscr{Y} \neq \varnothing$ for every g in G', and $G'\mathscr{Y}\cap\mathscr{Y} = \mathscr{Y}$ trivially. Thus we have the following lemma.

LEMMA 2.10. *Let $R = P(G, \mathscr{X}, \mathscr{Y})$. The pair $(G', \mathscr{Y})$ is fully compatible for every subgroup G' of G.*

As a corollary to Theorem 2.9, we then have the following result.

COROLLARY 2.11. *Let $R = P(G, \mathscr{X}, \mathscr{Y})$ and let S be a full inverse subsemigroup of R. Then S is unitary iff $S = R\cap(\mathscr{Y} \times G')$ for some subgroup G' of G.*

Thus there is a one-to-one correspondence $G' \to R\cap(\mathscr{Y} \times G')$ between the subgroups of G and the full unitary inverse subsemigroups of R.

3. **Congruences on *P*-semigroups.**

In this section we use the methods of §2 to give a way of constructing the congruences on a P-semigroup $P(G, \mathscr{X}, \mathscr{Y})$ in terms of subsemilattices of $\mathscr{Y}$ and subgroups of G. Preston [8] has shown that in any inverse semigroup T, the congruences are in one-to-one correspondence with the kernel normal systems of T. We will use this approach to find congruences on a P-semigroup.

DEFINITION Let T be an inverse semigroup. A *kernel normal system* in T is a set $\mathscr{A} = \{A_i \mid i\in I\}$ of disjoint inverse subsemigroups of T, whose union contains $E(T)$, and such that:

(K1) for each $t\in T$ and $i\in I$, $t^{-1}A_it\subseteq A_j$ for some $j\in I$;
(K2) if s, st, $tt^{-1}\in A_i$, then $t\in A_i$.

PROPOSITION 3.1 [**1**, Theorem 7.48]. *If $\mathscr{A} = \{A_i \mid i \in I\}$ is a kernel normal system in an inverse semigroup T, define*

$$\rho(\mathscr{A}) = \{(a,b) \in T \times T \mid aa^{-1}, bb^{-1}, ab^{-1} \in A_i \text{ for some } i \in I\}.$$

Then $\rho(\mathscr{A})$ is a congruence on T. Conversely, if ρ is a congruence on T, then $\mathscr{A} = \{e\rho \mid e \in E(T)\}$ is a kernel normal system in T and $\rho(\mathscr{A}) = \rho$.

LEMMA 3.2. *Let S, T be inverse semigroups, $S \leqq T$. S is partially unitary iff $s, st, tt^{-1} \in S$ imply $t \in S$.*

Proof. This is similar to Lemma 2.1.

Thus $\{A_i \mid i \in I\}$ is a kernel normal system in T iff the A_i's are disjoint partially unitary inverse subsemigroups of T, such that $E(T) \subseteq \cup\{A_i \mid i \in I\}$, and satisfying K1. When T is a P-semigroup, Theorem 2.9 can therefore be applied.

First we consider normal partitions of the idempotents of T: a partition $\mathscr{E} = \{E_i \mid i \in I\}$ of $E(T)$ is *normal* if each E_i is a subsemilattice and for each $t \in T$, $i \in I$, there exists $j \in I$ such that $t^{-1}E_i t \subseteq E_j$.

PROPOSITION 3.3 (Reilly and Scheiblich [**9**]). *Let T be an inverse semigroup, $E = E(T)$. If $\mathscr{E}$ is a normal partition of E, there exist largest and smallest congruences on T whose restrictions to $E \times E$ induce the partition $\mathscr{E}$. Conversely, the partition of E induced by the restriction of any congruence on T to $E \times E$ is normal.*

From now on, we restrict ourselves to P-semigroups. If $R = P(G, \mathscr{X}, \mathscr{Y})$, the isomorphism $(A, 1) \to A$ of $E(R)$ onto $\mathscr{Y}$ induces a one-to-one correspondence between the partitions of $E(R)$ and those of $\mathscr{Y}$. Consequently, we will define a *normal partition of* $\mathscr{Y}$ to be a collection $\{\mathscr{Y}_i \mid i \in I\}$ of subsemilattices of $\mathscr{Y}$ such that $\{\mathscr{Y}_i \times \{1\} \mid i \in I\}$ is a normal partition of $E(R)$.

In [**4**, Proposition 6.4] McAlister characterized normal congruences on $E(R)$ (that is, those congruences on $E(R)$ which induce normal partitions of $E(R)$) for P-semigroups $R = P(G, \mathscr{X}, \mathscr{Y})$. His characterization was in terms of partitions of $\mathscr{X}$; however, as our main preliminary result shows, we need in fact only consider partitions of $\mathscr{Y}$.

PROPOSITION 3.4. *Let $R = P(G, \mathscr{X}, \mathscr{Y})$. Let $\mathscr{E} = \{\mathscr{Y}_i \mid i \in I\}$ be a partition of $\mathscr{Y}$ (into subsemilattices) such that*

(i) *if $g \in G$ and $i \in I$, there exists $j \in I$ (call it gi) such that $g\mathscr{Y}_i \cap \mathscr{Y} \subseteq \mathscr{Y}_{gi}$,*
(ii) *if $C \in \mathscr{Y}$ and $i \in I$, there exists $k \in I$ (call it iC) such that $C \wedge \mathscr{Y}_i \subseteq \mathscr{Y}_{iC}$.*

Then $\mathscr{E}$ is a normal partition of $\mathscr{Y}$, and conversely, every normal partition of $\mathscr{Y}$ has this form.

Proof. We show that if $\mathscr{E}$ satisfies (i) and (ii), it is a normal partition of $\mathscr{Y}$, by showing that $\{E_i \mid i \in I\}$ is a normal partition of $E(R)$, where $E_i = \{(A, 1) \mid A \in \mathscr{Y}_i\}$ for each $i \in I$.

Let $(V, v) \in R$, and $i \in I$. We must show that for some $j \in I$, $(V, v)^{-1}E_i(V, v) \subseteq E_j$. Let $(A, 1) \in E_i$. Then

$$\begin{aligned}(V, v)^{-1}(A, 1)(V, v) &= (v^{-1}V \wedge v^{-1}A, v^{-1})(V, v)\\ &= (v^{-1}(V \wedge A), 1).\end{aligned}$$

Now $V\wedge A\in V\wedge\mathscr{Y}_i\subseteq\mathscr{Y}_{iV}$, and so

$$v^{-1}(V\wedge A)\in v^{-1}\mathscr{Y}_{iV}\cap\mathscr{Y}\subseteq\mathscr{Y}_{v^{-1}(iV)}.$$

Thus since $v^{-1}(iV)$ is independent of A,

$$(V,v)^{-1}E_i(V,v)\subseteq E_{v^{-1}(iV)}.$$

Conversely, let $\mathscr{E}=\{\mathscr{Y}_i\,|\,i\in I\}$ be a normal partition of $\mathscr{Y}$ (and $\{E_i\,|\,i\in I\}$ the corresponding normal partition of $E(R)$). Let $g\in G$, $i\in I$ and suppose $A\in\mathscr{Y}_i$, with $gA\in\mathscr{Y}$. Then $(A,g^{-1})\in R$, so $(A,g^{-1})^{-1}E_i(A,g^{-1})\subseteq E_j$ for some $j\in I$. Therefore $(gA,1)=(A,g^{-1})^{-1}(A,1)(A,g^{-1})\in E_j$, that is $gA\in\mathscr{Y}_j$. If B is any other element of $\mathscr{Y}_i$ such that $gB\in\mathscr{Y}$, then $(B,g^{-1})^{-1}E_i(B,g^{-1})\subseteq E_k$ for some $k\in I$, and $gB\in\mathscr{Y}_k$ similarly. But $(g(A\wedge B),1)=(A,g^{-1})^{-1}(B,1)(A,g^{-1})=(B,g^{-1})^{-1}(A,1)(B,g^{-1})\in E_j\cap E_k$. Hence $j=k$ and $gB\in\mathscr{Y}_j$. Therefore $g\mathscr{Y}_i\cap\mathscr{Y}\subseteq\mathscr{Y}_j$. If $C\in\mathscr{Y}$ and $i\in I$, then $(C,1)^{-1}E_i(C,1)\subseteq E_k$ for some $k\in I$, that is $C\wedge\mathscr{Y}_i\subseteq\mathscr{Y}_k$.

Thus (i) and (ii) are satisfied.

THEOREM 3.5. *Let $R=P(G,\mathscr{X},\mathscr{Y})$. Let $\mathscr{E}=\{\mathscr{Y}_i\,|\,i\in I\}$ be a normal partition of $\mathscr{Y}$ (as in Proposition 3.4). For each $i\in I$, let N_i be a subgroup of G such that*

(i) *$g\mathscr{Y}_i\cap\mathscr{Y}_i\neq\varnothing$, for all $g\in N_i$,*
(ii) *$v^{-1}N_iv\subseteq N_{v^{-1}(iV)}$ for all $(V,v)\in R$.*

Let $A_i=R\cap(\mathscr{Y}_i\times N_i)$, $i\in I$. Then $\mathscr{A}=\{A_i\,|\,i\in I\}$ is a kernel normal system in R, and the relation $\rho(\mathscr{A})$ defined by $((A,g),(B,h))\in\rho(\mathscr{A})$ iff for some $i\in I$, $gh^{-1}\in N_i$ and $A,B,A\wedge(gh^{-1})B\in\mathscr{Y}_i$, is a congruence on R.

Conversely if ρ is a congruence on R, let $\mathscr{A}=\{e\rho\,|\,e\in E(R)\}=\{A_i\,|\,i\in I\}$. Then $\mathscr{A}$ is a kernel normal system in R, obtained in the above way from $\{(A_i\sigma^\natural,\mathscr{Y}(A_i))\,|\,i\in I\}$.

Proof. Suppose $\mathscr{E}=\{\mathscr{Y}_i\,|\,i\in I\}$ and $\{N_i\,|\,i\in I\}$ satisfy the above conditions. Note firstly from (i) that for all $g\in N_i$, $\mathscr{Y}_{gi}\cap\mathscr{Y}_i\neq\varnothing$, so $gi=i$. Thus $N_i\mathscr{Y}_i\cap\mathscr{Y}\subseteq\mathscr{Y}_i$. Hence $(N_i,\mathscr{Y}_i)$ is fully compatible for each $i\in I$, and so each $A_i=R\cap(\mathscr{Y}_i\times N_i)$ is a partially unitary inverse subsemigroup of R, by Theorem 2.9. Further, the A_i's are disjoint, and $E(T)\subseteq\cup\{A_i\,|\,i\in I\}$. Thus, to show $\mathscr{A}=\{A_i\,|\,i\in I\}$ is a kernel normal system in R it remains only to show that if $a\in R$ and $i\in I$, $a^{-1}A_ia\subseteq A_j$ for some $j\in I$.

Let $a=(V,v)\in R$, and suppose $x=(A,g)\in A_i$ (so that $g^{-1}A\in\mathscr{Y}_i$). Then $a^{-1}xa=(v^{-1}(V\wedge A\wedge gV),v^{-1}gv)$. Firstly, $v^{-1}gv\in v^{-1}N_iv\subseteq N_{v^{-1}(iV)}$ (using (ii)). Note now that $N_i\subseteq N_{iV}$ for all $V\in\mathscr{Y}$ (putting $v=1$, in (ii)). Thus $g\in N_{iV}$, and so $g(iV)=iV$. Since $g^{-1}A\wedge V\in\mathscr{Y}_{iV}$, $A\wedge gV=g(g^{-1}A\wedge V)\in g\mathscr{Y}_{iV}\cap\mathscr{Y}\subseteq\mathscr{Y}_{g(iV)}=\mathscr{Y}_{iV}$. Therefore $v^{-1}(V\wedge A\wedge gV)=v^{-1}((V\wedge A)\wedge(A\wedge gV))\subseteq v^{-1}\mathscr{Y}_{iV}\subseteq\mathscr{Y}_{v^{-1}(iV)}$. By definition, then, $a^{-1}xa\in A_{v^{-1}(iV)}$, and since $v^{-1}(iV)$ is independent of x, $a^{-1}A_ia\subseteq A_{v^{-1}(iV)}$.

Hence $\mathscr{A}$ is a kernel normal system in R, and by Proposition 3.1, $\rho(\mathscr{A})$ is a congruence.

Conversely, if ρ is a congruence on R, let $\mathscr{A}=\{e\rho\,|\,e\in E(R)\}=\{A_i\,|\,i\in I\}$ (for some set I). By Proposition 3.1, $\mathscr{A}$ is a kernel normal system in R. Each A_i is therefore a partially unitary inverse subsemigroup of R, and, using Theorem 2.9, $A_i=R\cap(\mathscr{Y}_i\times N_i)$, where the pair $(N_i,\mathscr{Y}_i)=(A_i\sigma^\natural,\mathscr{Y}(A_i))$ is fully compatible, for all $i\in I$. Thus (i) is satisfied immediately.

Let $E_i=E(A_i)=\mathscr{Y}_i\times\{1\}$, $i\in I$. Then $\{E_i\,|\,i\in I\}$ is a normal partition of $E=E(R)$, since it is the partition induced by $\rho \mid E \times E$. Hence $\mathscr{E} = \{\mathscr{Y}_i \mid i \in I\}$ is a normal partition of $\mathscr{Y}$. To show (ii) is satisfied, let $i \in I$ and $(V, v) \in R$. Since $\mathscr{A}$ is a kernel normal system, there exists j such that $(V, v)^{-1}A_i(V, v) \subseteq A_j$. Thus $v^{-1}(A_i\sigma^\natural)v \subseteq A_j\sigma^\natural$, that is $v^{-1}N_iv \subseteq N_j$; and $(V, v)^{-1}E_i$ $(V, v) \subseteq E_j$. But from the proof of Proposition 3.4, $(V, v)^{-1}E_i(V, v) \subseteq E_{v^{-1}(iV)}$; so $j = v^{-1}(iV)$, and $v^{-1}N_iv \subseteq N_{v^{-1}(iV)}$, as required.

For idempotent-separating congruences, this reduces to McAlister's characterization, in Proposition 3.2 of [4] (although, as we shall see, his condition (2) is redundant). Recall that if S is an inverse semigroup, with $E(S) = E$, a congruence ρ is *idempotent-separating* if $\rho \mid E \times E = \iota$.

Let $R = P(G, \mathscr{X}, \mathscr{Y})$ and suppose, then, that $\mathscr{E}$ of Theorem 3.5 is just $\{\{A\} \mid A \in \mathscr{Y}\}$. Index $\mathscr{E}$ by $\mathscr{Y}$ (that is, put $\mathscr{Y}_A = \{A\}, A \in \mathscr{Y}$). Let N_A be a subgroup of G, for each $A \in \mathscr{Y}$. Then (i) just says $gA = A$ for all $g \in N_A$; (ii) says $v^{-1}N_Av \subseteq N_{v^{-1}(V \wedge A)}$ for all $(V, v) \in R$, or equivalently, $v^{-1}N_Av \subseteq N_{v^{-1}D}$ for all $(D, v) \in R$ such that $D \leqq A$. Note that this implies $N_A \subseteq N_D$ for all $D \leqq A$. Hence N_A is a subgroup of $C_A = \{g \in G \mid gD = D \text{ for all } D \leqq A\}$.

Theorem 3.5 thus specializes to the following.

Corollary 3.6. *Let $R = P(G, \mathscr{X}, \mathscr{Y})$. For each $A \in \mathscr{Y}$, let N_A be a subgroup of C_A such that if $(D, v) \in R$ and $D \leqq A$, $v^{-1}N_Av \subseteq N_{v^{-1}D}$. Then $\mathscr{N} = \{\{A\} \times N_A \mid A \in \mathscr{Y}\}$ is a group kernel normal system in R (that is, each element of $\mathscr{N}$ is a group) and $\rho(\mathscr{N})$, defined by $((A, g), (B, h)) \in \rho(\mathscr{N})$ iff $A = B$ and $gh^{-1} \in N_A$, is an idempotent-separating congruence on R.*

Conversely, if ρ is an idempotent-separating congruence on R, then ρ has this form (with $N_A = \{g \in G \mid (A, g)\rho(A, 1)\}$, $A \in \mathscr{Y}$).

(As we commented earlier, McAlister includes a second condition: (2) $N_AN_B \subseteq N_{A \wedge B}$. This, however, is a consequence of the fact, noted above, that $D \leqq A$ implies $N_A \subseteq N_D$.)

4. The lattice of inverse subsemigroups. If T is an inverse semigroup, denote by $\mathscr{L}(T)$ (or just $\mathscr{L}$) its lattice of inverse subsemigroups. Let $R = P(G, \mathscr{X}, \mathscr{Y})$; suppose H is a subgroup of G and $\mathscr{Z}$ is a subsemilattice of $\mathscr{Y}$. Define $\mathscr{L}_{(H, \mathscr{Z})}(R)$ (or just $\mathscr{L}_{(H, \mathscr{Z})}$) by $\mathscr{L}_{(H, \mathscr{Z})} = \{S \in \mathscr{L} \mid S\sigma^\natural = H, \mathscr{Y}(S) = \mathscr{Z}\}$.

Now $\mathscr{L}_{(H, \mathscr{Z})}$ may be empty. For example, suppose $(A, 1)$ is an idempotent with trivial $\mathscr{H}$-class in some $P(G, \mathscr{X}, \mathscr{Y})$, where G is non-trivial. Put $\mathscr{Z} = \{A\}$. The only inverse subsemigroup S of $P(G, \mathscr{X}, \mathscr{Y})$ with $\mathscr{Y}(S) = \mathscr{Z}$ is $\{(A, 1)\}$; but $\{(A, 1)\}\sigma^\natural = \{1\}$, so $\mathscr{L}_{(G, \mathscr{Z})}$ is empty.

Theorem 4.1. *Let $R = P(G, \mathscr{X}, \mathscr{Y})$, and $H \leqq G, \mathscr{Z} \leqq \mathscr{Y}$. Then $\mathscr{L}_{(H, \mathscr{Z})}$ is a join-complete sublattice of $\mathscr{L}$. Thus if $\mathscr{L}_{(H, \mathscr{Z})}$ is non-empty, it has a maximum element $1_{(H, \mathscr{Z})} = \bigvee\{S \in \mathscr{L}_{(H, \mathscr{Z})}\}$.*

Proof. If $\mathscr{L}_{(H, \mathscr{Z})}$ is empty, the result is trivially true. So assume $\mathscr{L}_{(H, \mathscr{Z})}$ is non-empty, and suppose $S_i \in \mathscr{L}_{(H, \mathscr{Z})}$, $i \in I$, where $S_i \neq S_j$ for $i \neq j$. We show firstly that $S = \bigvee\{S_i \mid i \in I\} \in \mathscr{L}_{(H, \mathscr{Z})}$, that is, $S\sigma^\natural = H$ and $\mathscr{Y}(S) = \mathscr{Z}$.

Clearly $\mathscr{Y}(S) = \mathscr{Z}$ iff $E(S) = E' = \mathscr{Z} \times \{1\}$. But $E' = E(S_i)$, $i \in I$, so $E' \subseteq E(S)$. Now every element of S can be written in the form $u_1 \dots u_n$ for some $u_j \in S_{i_j}$, $i_j \in I$, $1 \leqq j \leqq n$. Thus if $e \in E(S)$, $e = u_1 \dots u_nu_n^{-1} \dots u_1^{-1}$ (since $e = ee^{-1}$). For each j $(1 \leqq j \leqq n)$, put $e_j = u_j \dots u_nu_n^{-1} \dots u_j^{-1}$. Then $e_n = u_nu_n^{-1} \in E(S_{i_n}) = E'$. Assume that $e_{k+1} \in E'$ $(1 \leqq k \leqq n-1)$. Now

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$e_k = u_k e_{k+1} u_k^{-1}$, so $e_k \in E(S_{i_k})$, since $e_{k+1} \in E(S_{i_k})$. Therefore, by induction, $e = e_1 \in E'$. Hence $E(S) = E'$, that is $\mathscr{Y}(S) = \mathscr{Z}$.

Also $H \subseteq S\sigma^\natural$, since $H = S_i\sigma^\natural$, $i \in I$. Conversely, if $u \in S$, then $u = u_1 \dots u_n$ as above, so $u\sigma^\natural = u_1\sigma^\natural \dots u_n\sigma^\natural \in H$, since each $u_j\sigma^\natural \in H$. Hence $S = \bigvee \{S_i \mid i \in I\} \in \mathscr{L}_{(H,\mathscr{Z})}$.

Suppose now that $S_1, S_2 \in \mathscr{L}_{(H,\mathscr{Z})}$. Clearly $E(S_1 \cap S_2) \subseteq E' \subseteq E(S_1 \cap S_2)$, so $\mathscr{Y}(S_1 \cap S_2) = \mathscr{Z}$. Also $(S_1 \cap S_2)\sigma^\natural \subseteq H$. Suppose $h \in H = S_1\sigma^\natural = S_2\sigma^\natural$, so that $s\sigma^\natural = t\sigma^\natural = h$, for some $s \in S_1$, $t \in S_2$. Then $(s,t) \in \sigma(R) \cap ((S_1 \vee S_2) \times (S_1 \vee S_2)) = \sigma(S_1 \vee S_2)$, by Lemma 1.1. Hence $es = et$, for some $e \in E(S_1 \vee S_2) = E' = E(S_1 \cap S_2)$ (using the first half of this proof). But $es = et \in S_1 \cap S_2$, and $(es)\sigma^\natural = s\sigma^\natural = h$. Hence $H \subseteq (S_1 \cap S_2)\sigma^\natural$, so in fact $H = (S_1 \cap S_2)\sigma^\natural$. Thus $S_1 \cap S_2 \in \mathscr{L}_{(H,\mathscr{Z})}$.

Therefore $\mathscr{L}_{(H,\mathscr{Z})}$ is a sublattice of $\mathscr{L}$, join-complete by the first half of the proof.

In general $\mathscr{L}_{(H,\mathscr{Z})}$ is not complete, as is seen from the following example. Let I_1 be the free inverse semigroup on one generator. McAlister and McFadden have shown ([5]) that every free inverse semigroup is reduced; thus I_1 can be represented as $P(G, \mathscr{X}, \mathscr{Y})$, for some G, $\mathscr{X}$ and $\mathscr{Y}$. (The form of $\mathscr{X}$ and $\mathscr{Y}$ is irrelevant to us here.) We prefer however to use Gluskin's description [2]: $I_1 = \{(p,q,r) \in Z^3 \mid p, p+q, r, q+r \geqq 0, p+q+r \geqq 1\}$ with product $(p,q,r)(p',q',r') = (\max(p, p'-q), q+q', \max(r', r-q'))$. It is then easily checked that

(i) $I_1/\sigma \cong (Z, +)$, induced by the map

$$\sigma^\natural : (p,q,r) \to q,\ (p,q,r) \in I_1;$$

(ii) $J_{(p,q,r)} \leqq J_{(p',q',r')}$ iff $p+q+r \geqq p'+q'+r'$;

(iii) $E(I_1) = \{(p,0,r) \in I_1\} = E$, say.

Consider the sublattice $\mathscr{L}_L$ of $\mathscr{L}(I_1)$ consisting of all *large* inverse subsemigroups of I_1, that is ([3]), all those inverse subsemigroups S of I_1 with $S\sigma^\natural = Z$ and $E(S) = E$. (Thus when I_1 is represented as $P(G, \mathscr{Z}, \mathscr{Y})$, (with $G = Z$), $\mathscr{L}_L = \mathscr{L}_{(G,\mathscr{Y})}$.) We show $\cap \mathscr{L}_L = E \notin \mathscr{L}_L$.

Assume otherwise; that is, assume there exists $x = (p,q,r) \in \cap \mathscr{L}_L$ with $q \neq 0$. Put $n = p+q+r$, $y = (n,1,0) \in I_1$, and $S = [E \cup \{y\}] \in \mathscr{L}_L$. Then $x \in S$, so x can be written as a product involving y (or y^{-1}), since x is non-idempotent. Thus $J_x \leqq J_y$, that is, $p+q+r = n \geqq n+1$, a contradiction.

Hence $\cap \mathscr{L}_L \notin \mathscr{L}_L$, and $\mathscr{L}_L$ is not a complete sublattice.

The results of Section 2 have interesting applications in the context of this section. For when the pair $(H, \mathscr{Z})$ is compatible, $\mathscr{L}_{(H,\mathscr{Z})}$ contains a unique weakly unitary element $P(H, H\mathscr{Z}, \mathscr{Z})$, by Theorem 2.8. In fact from the next lemma (which could also have been deduced from an analysis of the proof of Theorem 2.8), $P(H, H\mathscr{Z}, \mathscr{Z})$ is the maximum element of $\mathscr{L}_{(H,\mathscr{Z})}$.

LEMMA 4.2. *Let R be reduced and $S_1, S_2 \leqq R$. Suppose $E(S_1) = E(S_2)$ and $S_1\sigma^\natural = S_2\sigma^\natural$. If S_2 is weakly unitary, then $S_1 \leqq S_2$.*

Proof. Let $p \in S_1$. Then pp^{-1}, $p^{-1}p \in E(S_1) = E(S_2)$, and $p\sigma^\natural \in S_1\sigma^\natural = S_2\sigma^\natural$. By Corollary 2.3, $p \in S_2$.

We summarize these results in the following.

COROLLARY 4.3. *Let* $R = P(G, \mathscr{X}, \mathscr{Y})$. *Let* $H \leqq G$ *and* $\mathscr{Z} \leqq \mathscr{Y}$, *and suppose* $(H, \mathscr{Z})$ *is compatible. Then* $1_{(H, \mathscr{Z})} = P(H, H\mathscr{Z}, \mathscr{Z})$, *the unique weakly unitary element of* $\mathscr{L}_{(H, \mathscr{Z})}$.

Similarly, if $(H, \mathscr{Z})$ is fully compatible in R, $1_{(H, \mathscr{Z})}$ is the partially unitary inverse subsemigroup $R \cap (\mathscr{Z} \times H)$.

Finally, we can again say considerably more for full inverse subsemigroups of $R = P(G, \mathscr{X}, \mathscr{Y})$, by using the last two results of Section 2. By Lemma 2.10, every pair $(H, \mathscr{Y})$ is (fully) compatible, and hence $\mathscr{L}_{(H, \mathscr{Y})}$ is non-empty, for every $H \leqq G$. Using Lemma 4.2 and Corollary 2.11, each full inverse subsemigroup S of R is thus a *large* inverse subsemigroup of a (full) unitary inverse subsemigroup of R. In fact, from Corollary 4.3, $S \leqq R \cap (\mathscr{Y} \times H)$, the unitary maximum element of $\mathscr{L}_{(H, \mathscr{Y})}$, where $H = S\sigma^{\natural}$.

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