FIBRE BUNDLES AND YANG-MILLS FIELDS

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1. Introduction

Since 1954, when Yang and Mills [7] presented their idea of isotopic gauge transformation, the method of introducing interactions into field theories by using general gauge invariance has been extensively studied.

A general formalism was presented by Utiyama [6]. Reference [6] also contains the first application of the formalism to the theory of gravitation. A more general approach to the Young-Mills formalism applied to general relativity was described by Kibble [3]. In a special case of interacting Dirac field the gauge invariance group can still be enlarged, leading to the possibility of describing short-range interactions together with gravitation and electromagnetism [5]. It is, therefore important to have a definite formulation of the common geometrical content of such theories.

Differential geometers who came into contact with Young-Mills theory realized that the theory is almost identical with the theory of connections in vector bundles. Certain aspects of such relation were discussed in references [1] and [2]. Correspondence between Yang-Mills theory and differential geometry of fibre bundles is explicitly studied in the next section in a form suitable for applications. In Section 3 the formalism is applied to the Utiyama's construction of gravitational interactions. A relation between space-time metric and Yang-Mills fields is derived without the assumption about vanishing torsion of the linear connection of bundle of frames. Use of this *ad hoc* assumption in Utiyama's paper was criticized by Kibble [3], and it was one of the reasons that lead him to consider a generalized approach.

2. Convariant Derivative

Consider a principal fibre bundle $P(B, G)$ (for basic definitions and theorems of the theory of fibre bundles see e.g., [4]) where B is the base manifold of dimension n, and G is the structure group. π denotes the projection map

(1) $$\pi: x \in P \to u = \pi(x) \in B$$

satisfying

(2) $$\pi(x \cdot a) = \pi(x) \text{ for all } x \in P, \ a \in G.$$

Further consider a differentiable function

(3) $$\xi: x \in P \to \xi(x) \in F,$$

satisfying a condition

(4) $$\xi(x \cdot a) = a^{-1}\xi(x), \qquad a \in G.$$

F is a (real or complex) representation space of G. Together with a cross-section

(5) $$\chi: u \in B \to \chi(u) \in P, \ \pi(\chi(u)) = u$$

function ξ defines an F-valued function ζ on manifold B by

(6) $$\zeta(u) = \xi(\chi(u)).$$

In physics, B is the space-time manifold and $\zeta(u)$ is the "old" field that is to interact with "new" Yang-Mills field. The latter fields are contained in a connection Γ defined in $P(B, G)$.

Any tangent vector X at point $x_0 \in P$ can be written as

$$X = X_h + X_v,$$

where X_h and X_v are the horizontal and vertical components of X defined by connection Γ in P. Applying X to function $\xi(x)$

(7) $$X\xi(x_0) = X_h\xi(x_0) + X_v\xi(x_0).$$

The best way to see explicitly how this decomposition works is to consider a differentiable curve $x(t)$ with $x(0) = x_0$ and tangent X at x_0. Then

(8) $$X\xi(x_0) = \lim_{t \to 0} (1/t)(\xi(x(t)) - \xi(x_0)).$$

If $x_h(t)$ denotes the horizontal projection of $x(t)$ passing through $x_0 (x_h(0) = x_0)$ then $x(t)$ may be written as

(9) $$x(t) = x_h(t) \cdot a(t)$$

where $a(t)$ is a differentiable curve in G, $a(0) = e$ (the identity element).

We have

$$X\xi(x_0) = \lim_{t \to 0} (1/t)(\xi(x_h(t)\, a(t)) - \xi(x_0))$$

$$= \lim_{t\to 0} (1/t)(a^{-1}(t)\cdot\xi(x_n(t)) - \xi(x_h(t))$$

$$+ \lim_{t\to 0} (1/t)(\xi(x_h(t)) - \xi(x_0))$$

$$(10) \qquad = X_v\xi(x_0) + X_h\xi(x_0)$$

where

$$(11) \qquad X_h\xi(x_0) = \lim_{t\to 0} (1/t)(\xi(x_h(t)) - \xi(x_0))$$

and

$$(12) \qquad X_v\xi(x_0) = \left(\lim_{t\to 0} (1/t)(a^{-1}(t) - e)\right)\cdot\xi(x_0).$$

Here

$$(13) \qquad \lim_{t\to 0} (1/t)(a^{-1}(t) - e) = A(X)$$

is an element of the Lie algebra of group G.

$X_h\xi(x_0)$ yields the covariant derivative of function $\zeta(u)$ by

$$(14) \qquad \nabla_{\pi(X)}\zeta(u_0) = X_h\xi(x_0),$$

where ζ and ξ are related through Equation (6). $\pi(X)$ means the projection of vector X onto base manifold B.

Let $u(t)$ be a differentiable curve in B with tangent $\pi(X)$ at $u_0 = \pi(x_0)$ and $x(t)$ a curve obtained by

$$x(t) = \chi(u(t))$$

with tangent vector X at $x(0) = x_0$. Transforming $\zeta(u)$ into

$$\tilde{\zeta}(u) = a(u)\zeta(u),$$

where $a(u)\in G$ is differentiable, we have

$$\tilde{\zeta}(u(t)) = a(u(t))\zeta(u(t)).$$

Denoting $a(u(t)) = a_u(t)$

$$\tilde{\zeta}(u(t)) = a_u(t)\xi(x(t)) = \xi(x(t)\cdot a_u^{-1}(t)).$$

But as

$$X_h\xi(x(t)\cdot a_u^{-1}(t)) = \lim_{t\to 0} (1/t)(\xi([x(t)\cdot a_u^{-1}(t)]_h) - \xi(x_0 a_u^{-1}(0))).$$

and

$$[x(t)\cdot a_u^{-1}(t)]_h = x_h(t)\cdot a_u^{-1}(0)$$

we have

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(15) $[X_h\xi(x(t)\cdot a_u^{-1}(t))]_{t=0} = \lim_{t\to 0}(1/t)a_u(0)(\xi(x_h(t)) - \xi(x_0)) = a_u(0)X_h\xi(x_0)$

or

(16) $$\nabla_{\pi(X)}\tilde{\zeta}(u_0) = a(u_0)\nabla_{\pi(X)}\zeta(u_0).$$

Equation (16) shows that operator $\nabla_{\pi(X)}$ has the required covariant properties. Further

(17) $$X\xi(x(t)) = \frac{d}{dt}\xi(x(t)) = \frac{d}{dt}\zeta(u(t)) = \pi(X)\,\zeta(u(t))$$

and using Equations (10) to (14)

(18) $$\nabla_{\pi(X)}\zeta(u) = \pi(X)\zeta(u) - A(X)\zeta(u).$$

Choosing $\{\partial/\partial u', \cdots, \partial/\partial u^n\}$ as the basis of the tangent vector space at $u \in B$ and $\{A_1, \cdots, A_d\}$ as the basis of the Lie algebra of group G we can rewrite (18) for

$$\pi(X) = \frac{\partial}{\partial u^k} \text{ as}$$

(19) $$\nabla_{\pi(X)}\zeta(u) = \frac{\partial}{\partial u^k}\,\zeta(u) - \sum_{p=1}^{d} B_k^p(u)A_p\zeta(u).$$

This is the covariant derivative as introduced in Reference [6] with $B_k(u)$ being the "new" Yang-Mills fields.

3. Yang-Mills Fields and Tetrads

We shall now consider a specific case of bundle of frames in the four-dimensional space-time manifold. Structure group G is then the homogeneous Lorentz group, and representation space F is to be considered as a four-dimensional real vector space with a basis $\{e_1, e_2, e_3, e_4\}$. Points of fibre manifold P are local frames defined by four orthogonal vectors

(20) $$X_k^\mu \frac{\partial}{\partial u^\mu}^{(*)} \qquad k = 1, 2, 3, 4.$$

Group G acts on vectors (20) in the same way as on vectors e_k of the basis of space F.

We consider a function

(21) $$\xi : P \to F, \quad \xi(x) = Y_\mu^k e_k,$$

where local frame $x \in P$ is defined by X_k^μ, and

(*) A pair of identical indices will always mean summation.

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(22) $$X^{\mu}_{k} Y^{k}_{\nu} = \delta^{\mu}_{\nu}, \qquad X^{\mu}_{i} Y^{k}_{\mu} = \delta^{k}_{i}.$$

Function ξ satisfies condition (4), and besides that it defines (together with a cross-section in P) function ζ (see Equation (6)).

(23) $$\zeta(u) = h^{k}_{\mu}(u) e_k.$$

Functions $h^{k}_{\mu}(u)$ assign a local orthogonal frame to every point u of the space-time manifold and are usually called tetrads in general relativity.

The question we want to answer is as follows: Suppose we have a connection in the bundle of frames with co-ordinates $\Gamma_{\mu\nu}^{\;\;\rho}$ defined in the usual way [4]. What is the relation between $\Gamma^{\rho}_{\mu\nu}$ and the Yang-Mills fields of Equation (19)?

Vector $X_\mu = \dfrac{\partial}{\partial x^{\mu}}$ has its horizontal lift given by [4]

$$X^{*}_{\mu} = \frac{\partial}{\partial x^{\mu}} - \Gamma^{\rho}_{\mu\nu} X^{\nu}_{k} \frac{\partial}{\partial X_k}.$$

Applied on function ξ this gives

(24) $$X^{*}_{\mu}\xi(x) = \Gamma^{\rho}_{\mu\nu} Y^{k}_{k} e_k.$$

By definition of covariant derivative (14) with

$\pi(X) = X_\mu$ (i.e., $X_h = X^{*}$) and (23) we have

(24) $$\nabla_{X} h^{k}_{\nu}(u) e_k = \Gamma^{\rho}_{\mu\nu} h^{k}_{\rho}(u) e_k.$$

At the same time

(26) $$\nabla_{X} h^{k}_{\nu}(u) e_k = \frac{\partial h^{k}_{\nu}(u)}{\partial u^{\mu}} \cdot e_k - \sum_{q=1}^{6} B^{q}_{\mu}(u) A_q(h^{k}_{\nu}(u) e_k).$$

A_q are 4×4 matrices forming a basis of the Lie algebra of the homogeneous Lorentz group. Normally, index q is replaced by a pair of indices $i,j = 1,2,3,4$ and

$$B^{ij}_{\mu} = -B^{ji}_{\mu},$$

$$A_{14} = -A_{41} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad A_{24} = -A_{42} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A_{34} = -A_{43} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad A_{12} = -A_{21} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$A_{23} = -A_{32} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A_{31} = -A_{13} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Such choice also corresponds to Reference [6].

A direct calculation yields

$$\frac{1}{2} \sum_{i,j=1}^{4} B_\mu^{ij} A_{ij}(h_\nu^k(u)e_k) = B_\mu^{kj} h_{j\nu} e_k, \tag{27}$$

where $h_{4\nu} = h_\nu^4$ and $h_{j\nu} = -h_\nu^j$ if $j \neq 4$.

Comparing (25) and (27)

$$\Gamma_{\mu\nu}^\rho h_\rho^k = \frac{\partial h_\nu^k}{\partial u^\mu} - B_\mu^{kj} h_{j\nu}$$

which is identical to the relation obtained in Reference [6] under assumption that $\Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho$. No such assumption is necessary when using the general approach described above.

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