18
The real Klein–Gordon field

The relativistic scalar field satisfies the Klein–Gordon equation. This equation can be interpreted as the quantum mechanical analogue of the relativistic energy relation

$$E^2 = p^2c^2 + m^2c^4, \tag{18.1}$$

and is found by making the usual replacement $p_\mu \to -\mathrm{i}\hbar\partial_\mu$, and allowing the equation to operate on a real scalar field $\phi(x)$. The result is

$$\left(-\Box + \frac{m^2c^2}{\hbar^2}\right)\phi(x) = 0. \tag{18.2}$$

18.1 The action

If we generalize the single scalar field above to a set of N real scalar fields $\phi_A(x)$ for $A = 1, \ldots, N$, with a linear perturbation, J_A, then all of the physical information about this system can be derived from the following action:

$$S = \int (\mathrm{d}x)\left\{\frac{1}{2}\hbar^2c^2(\partial^\mu\phi_A)(\partial_\mu\phi_A) + \frac{1}{2}m^2c^4\phi_A\phi_A + V(\phi) - J_A\phi_A\right\}. \tag{18.3}$$

Note that the position of the A indices is immaterial here, since they only label the number of the field components. The repeated indices are summed using a Euclidean metric, for which there is no notion of 'up' or 'down' indices.

Looking at this action, it can be noted that it does not have the familiar form of an integral over $T - V$ (kinetic energy minus potential energy). Instead, it has the form of an integral over $-E^2 + p^2 + m^2 + V$. Although this looks dimensionally incorrect, this is not the case, since the dimensions of the field are simply chosen so that S has the dimensions of action.

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In what follows, the position of an index A is chosen for clarity. The Lagrangian density $\mathcal{L}$ is defined by

$$S = \int (\mathrm{d}x)\mathcal{L}. \tag{18.4}$$

In the usual canonical tradition, we define the conjugate momentum to the field $\phi_a(x)$ by

$$\Pi^A_\sigma = \frac{\delta\mathcal{L}}{\delta(\partial^\sigma\phi_A)} = \hbar^2c^2\partial_\sigma\phi, \tag{18.5}$$

where σ is a specific direction, normal to a spacelike hyper-surface. Usually we do not need to be this general and we can just pick $\sigma = 0$ for the normal, which corresponds to the time direction (normal to space in an observer's rest frame). Then we have, more simply,

$$\Pi_A = \hbar^2c^2\partial_0\phi_A. \tag{18.6}$$

The Hamiltonian density is then obtained straightforwardly from the Legendre transformation

$$\mathcal{H} = \Pi(\partial_0\phi) - \mathcal{L}g_{00}. \tag{18.7}$$

Or, using the fully covariant form,

$$\mathcal{H} = \Pi_\sigma(\partial_\sigma\phi) - \mathcal{L}g_{\sigma\sigma}. \tag{18.8}$$

Note the positions of the indices here and the presence of the metric in the second term of the right hand side. The need for this factor will become apparent later when looking at transformations and the energy–momentum tensor. It makes the relativistic Legendre transformation more subtle than that in Euclidean space, because of the indefinite metric. Eqns. (18.7) and (18.8) evaluate to

$$\mathcal{H} = \frac{1}{2}\hbar^2c^2\left[(\partial_0\phi)^2 + (\partial_i\phi)^2\right] + \frac{1}{2}m^2c^4\phi^2 + V(\phi). \tag{18.9}$$

18.2 Field equations and continuity

The variation of the action (with $V = 0$) leads to

$$\begin{aligned}\delta S = &\int (\mathrm{d}x)\left\{\hbar^2c^2\delta\phi_A(-\Box)\phi_A + m^2c^4\phi_A\delta\phi_A - J_A\delta\phi_A\right\}\\ &+ \frac{1}{c}\int \mathrm{d}\sigma^\mu\left\{\frac{1}{2}\hbar^2c^2\delta\phi_A(\partial_\mu\phi_A)\right\}.\end{aligned} \tag{18.10}$$

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Appealing to the action principle (see chapter 4), we surmise that the field equations are

$$\left(-\Box + \frac{m^2c^2}{\hbar^2}\right)\phi_A = (h^2c^2)^{-1}J_A, \tag{18.11}$$

and that the condition for continuity of the field through any n-dimensional surface is

$$\Delta\Pi^A_\sigma = 0. \tag{18.12}$$

If a delta-function source $\Delta J_A = \delta j_A \delta(x)$ is added to J_a exactly on the surface σ, then this continuity equation is modified, and the new condition is given by

$$\Delta\Pi^A_\sigma = \Delta j^A\, n_\sigma, \tag{18.13}$$

where n^μ is the unit normal vector to σ. This equation tells us that a sudden change in the momentum of the field can only be caused by an impulsive force (source) Δj.

18.3 Free-field solutions

The field $\phi(x)$ may be expanded as a linear combination of a complete set of plane wavefunctions satisfying the equation of motion,

$$\phi(x) = \int \frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}}\phi(k)\mathrm{e}^{\mathrm{i}kx}\delta\left(\hbar^2c^2k^2 + m^2c^4\right), \tag{18.14}$$

where $\phi(k)$ are arbitrary coefficients, independent of x. The reality of the field requires that

$$\Phi^*(k) = \Phi(-k). \tag{18.15}$$

The integral ranges over all energies, but one can separate the positive and negative energy solutions by writing

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x), \tag{18.16}$$

where

$$\begin{aligned}\phi^{(+)}(x) &= \int \frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}}\phi(k)\mathrm{e}^{\mathrm{i}kx}\theta(-k_0)\delta\left(\hbar^2c^2k^2 + m^2c^4\right)\\ \phi^{(-)}(x) &= \int \frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}}\phi(k)\mathrm{e}^{\mathrm{i}kx}\theta(k_0)\delta\left(\hbar^2c^2k^2 + m^2c^4\right).\end{aligned} \tag{18.17}$$

The symmetry of the energy relation then implies that

$$\phi^{(+)}(x) = \left(\phi^{(-)}(x)\right)^*. \tag{18.18}$$

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The positive and negative energy solutions to the free relativistic field equations form independently complete sets, with respect to the scalar product,

$$\begin{aligned}(\phi^{(+)}(x), \phi^{(+)}(x)) &= \text{const.}\\ (\phi^{(-)}(x), \phi^{(-)}(x)) &= \text{const.}\\ (\phi^{(+)}(x), \phi^{(-)}(x)) &= 0.\end{aligned} \tag{18.19}$$

18.4 Reality of solutions

It should be noted that the uses of the real scalar field are somewhat limited. The boundary conditions one can apply to a real scalar field are only the retarded or advanced ones. The solution

$$\phi(x) = \int (\mathrm{d}x')\ G(x, x')J(x') \tag{18.20}$$

is only real if the Green function itself is real. This excludes the use of the time-ordered (Feynman) Green function.

18.5 Conserved norm and probability

Since the real scalar field has no complex phase symmetry, Noether's theorem leads to no conserved quantities corresponding to a conserved inner product. It is possible to define an invariant inner product on the manifold of positive energy solutions, however. This is what introduces the complex symmetry in the non-relativistic limit;

$$\phi\partial_0\phi \tag{18.21}$$

has no definite sign.

Since the relativistic energy equation $E^2 = p^2c^2 + m^2c^4$ admits both possibilities, we do this by writing the real field as a sum of two parts,

$$\phi = \phi^{(+)} + \phi^{(-)}, \tag{18.22}$$

where $\phi^{(+)*} = \phi^{(-)}$. $\phi^{(+)}$ is a complex quantity, but the sum $\phi^{(+)} + \phi^{(-)}$ is clearly real. What this means is that it is possible to define a conserved current and therefore an inner product on the manifold of positive energy solutions $\phi^{(+)}$,

$$(\phi_1^{(+)}, \phi_2^{(+)}) = \mathrm{i}\hbar c \int \mathrm{d}\sigma^\mu(\phi_1^{(+)*}\partial_\mu\phi_2^{(+)} - (\partial_\mu\phi_1^{(+)})^*\phi_2^{(+)}) \tag{18.23}$$

and another on the manifold of negative energy solutions $\phi^{(-)}$. Thus there is local conservation of probability (though charge still does not make any sense) of particles and anti-particles separately.

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18.6 Normalization

The scalar product is only defined for normalizable wave-packet solutions, i.e. those for which $(\phi, \phi) < \infty$. A plane wave is a limiting case, which can only be defined by box normalization. It does not belong to the Hilbert space. However, adopting an invariant normalization in momentum space, one can express plane waves simply. Noting that the following construction is both invariant and 'on shell', i.e. satisfies the Klein–Gordon equation,

$$\begin{aligned}\phi &= \int \frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}} \mathrm{e}^{\mathrm{i}kx} \theta(\pm k_0) \delta(p^2c^2 + m^2c^4) \\ &= \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{\mathrm{e}^{\mathrm{i}kx}}{2p_0}.\end{aligned} \tag{18.24}$$

Adopting the normalization

$$(\phi(p), \phi(p)) = 2p_0 \, \delta(\mathbf{p} - \mathbf{p}')(2\pi)^n, \tag{18.25}$$

a positive energy solution takes the form

$$\phi^+(p) = \mathrm{e}^{\mathrm{i}kx} \quad \left(p_0 = \sqrt{\mathbf{p}^2 + m^2}\right). \tag{18.26}$$

18.7 Formal solution by Green functions

The formal solution of the equations of motion can be written down in terms of Green functions. The essence of the procedure is to find the inverse of the differential operator on the left hand side of eqn. (18.11). Formally, we may write

$$\phi_A(x) = \left(-\Box + \frac{m^2c^2}{\hbar^2}\right)^{-1} (\hbar^2c^2)^{-1} J_A, \tag{18.27}$$

where this is given meaning by comparing it with the expression involving the Green function or 'kernel' $G_{AB}(x, x')$:

$$\phi_A(x) = (\hbar^2c^2)^{-1} \int (\mathrm{d}x') G_{AB}(x, x') J_B(x'). \tag{18.28}$$

Comparing eqns. (18.27) and (18.28), we see that $G(x, x')$ must satisfy the equation

$$\left(-\Box + \frac{m^2c^2}{\hbar^2}\right) G_{AB}(x, x') = \delta_{AB}\delta(x, x'), \tag{18.29}$$

and thus we see that $G_{AB}(x, x')$ is the inverse of the differential operator, insofar as $\delta_{AB}\delta(x, x')$ can be regarded as the 'identity' operator.

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In this case, the indices A, B on the Green function are superfluous, since

$$G_{AB}(x, x') = \delta_{AB} G(x, x'), \tag{18.30}$$

but non-diagonal terms in A, B might be important when the components of the field interact. This is the case in a gauge theory, for example.

The Green function $G(x, x')$ is not unique: there is still a freedom to choose the boundary conditions. By this we mean a specification of how the field is affected by changes in the source J_a both in the past and in the future. The 'causal' Green function, also referred to as the retarded Green function, is such that $\phi(x)$ is only affected by a change in $J(x')$ if $x > x'$.

18.8 All the Green functions

The symmetry of the Green functions is as follows:

$$\begin{aligned} \overline{G}_{AB}(x, x') &= \overline{G}_{BA}(x', x) \\ \tilde{G}_{AB}(x, x') &= -\tilde{G}_{BA}(x', x) \\ G_{\mathrm{F}AB}(x, x') &= G_{\mathrm{F}BA}(x', x). \end{aligned} \tag{18.31}$$

The symmetrical parts of the Wightman functions may be constructed explicitly. For example

$$\begin{aligned} \frac{1}{2}\left[G^{(+)}_{AB}(x, x') + G^{(+)}_{BA}(x', x)\right] &= \frac{1}{2}\left[G^{(+)}_{AB}(x, x') - G^{(-)}_{AB}(x, x')\right] \\ &= \frac{1}{2}\left[G^{(+)}_{AB}(x, x') - \left(G^{(+)}_{AB}(x, x')\right)^*\right] \\ &= \mathrm{i}\mathrm{Im} G^{(+)}_{AB}(x, x'). \end{aligned} \tag{18.32}$$

The retarded, advanced and Feynman Green functions are all constructed from causally selective combinations of the Wightman functions.

$$\begin{aligned} G^{(+)}_{AB}(x, x') &= -G^{(-)}_{BA}(x', x) \\ \left(G^{(+)}_{AB}(x, x')\right)^* &= G^{(-)}_{AB}(x, x'). \end{aligned} \tag{18.33}$$

The properties of the step function lead to a number of linear relations:

$$\begin{aligned} G_{\mathrm{r}}(x, x') &= -\theta(t - t')\tilde{G}(x, x') \\ G_{\mathrm{a}}(x, x') &= \theta(t' - t)\tilde{G}(x, x') \\ G_{\mathrm{r}}(x, x') &= G_{\mathrm{F}}(x, x') - G^{(-)}(x, x') \\ G_{\mathrm{a}}(x, x') &= G_{\mathrm{F}}(x, x') + G^{(+)}(x, x') \\ G_{\mathrm{F}}(x, x') &= -\theta(t - t')G^{(+)}(x, x') + \theta(t' - t)G^{(-)}(x, x'). \end{aligned} \tag{18.34}$$

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Some caution is needed in interpreting the latter two relations, which should be considered formal. The causal properties of the Green functions distinguish $G^{(\pm)}(x, x')$, which satisfy the homogeneous eqn. (5.65), from $G_{\rm r}, G_{\rm F}$, which pose as right-inverses for a differential operator and satisfy an equation such as eqn. (5.62). We can investigate this by calculating time derivatives. Starting with the definition in eqn. (18.34), we obtain the time derivatives using the relations in sections A.1 and A.2 of Appendix A:

$$\begin{aligned}\partial_t G_{\rm F}(x, x') = &-\delta(t, t')\tilde{G}(x, x') - \theta(t - t')\partial_t G^{(+)}(x, x') \\ &+\theta(t' - t)\partial_t G^{(-)}(x, x'),\end{aligned} \tag{18.35}$$

where eqn. (5.71) was used. The second derivative is thus

$$\begin{aligned}\partial_t^2 G_{\rm F}(x, x') = &- \partial_t\delta(t - t')\tilde{G}(x, x') - \delta(t - t')\partial_t\tilde{G}(x, x') \\ &- \delta(t - t')\partial_t\tilde{G}(x, x') - \theta(t - t')\partial_t^2 G^{(+)}(x, x') \\ &+ \theta(t' - t)\partial_t^2 G^{(-)}(x, x').\end{aligned} \tag{18.36}$$

The property in eqn. (A.14) was used here. Thus using eqn. (5.73) we may write

$$\begin{aligned}\partial_t^2 G_{\rm F}(x, x') = &\delta(t - t')\delta(\mathbf{x} - \mathbf{x}') - \theta(t - t')\partial_t^2\tilde{G}(x, x') \\ &+\theta(t' - t)\partial_t^2 G^{(-)}(x, x').\end{aligned} \tag{18.37}$$

From this it should be clear that

$$\begin{aligned}(-\Box + M^2)G_{\rm F}(x, x') &= \delta(t - t')\delta(\mathbf{x} - \mathbf{x}') \\ &\quad - \theta(t - t')(-\Box + M^2)\tilde{G}(x, x') \\ &\quad + \theta(t' - t)(-\Box + M^2)G^{(-)}(x, x') \\ &= c\delta(x, x').\end{aligned} \tag{18.38}$$

The Green function for the scalar field is directly related to that for the electromagnetic field in the Lorentz–Feynman gauge, up to factors of $\hbar$ and μ_0.

$$D_{\mu\nu}(x, x')\Big|_{\alpha=1} = \mu_0\hbar^2\ G(x, x')\Big|_{m=0} g_{\mu\nu}. \tag{18.39}$$

18.9 The energy–momentum tensor

The application of Noether's theorem for spacetime translations leads to a symmetrical energy–momentum tensor. Although the sign of the energy is ambiguous for the Klein–Gordon field, we can define a Hamiltonian with the interpretation of an energy density which is positive definite, from the zero–zero component of the energy–momentum tensor. Using the action and the formula

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(11.44), we have

$$\begin{aligned}\theta_{00} &= \frac{\partial \mathcal{L}}{\partial(\partial^0\phi)_A}(\partial_0\phi_A) - \mathcal{L}g_{00} \\ &= \mathcal{H} = \frac{1}{2}\hbar^2c^2\left[(\partial_0\phi_A)^2 + (\partial_i\phi_A)^2\right] + \frac{1}{2}m^2c^4\phi_A^2 + V(\phi). \quad (18.40)\end{aligned}$$

This quantity has the interpretation of a Hamiltonian density. The integral over all space provides a definition of the Hamiltonian:

$$H = \int \mathrm{d}\sigma\, \mathcal{H}. \quad (18.41)$$

The explicit use of zero instead of a general timelike direction here makes this definition of the Hamiltonian explicitly non-covariant. Note that this is not a differential Hamiltonian operator analogous to that in eqn. (17.3), but more like an expectation value. In the quantum theory (in which the fields are operator-valued) this becomes the Hamiltonian operator.

The off-diagonal spacetime components give

$$\begin{aligned}\theta_{0i} = \theta_{i0} &= \frac{\partial \mathcal{L}}{\partial(\partial^0\phi)_A}(\partial_i\phi)_A \\ &= \hbar^2c^2(\partial_0\phi_A)(\partial_i\phi_A). \quad (18.42)\end{aligned}$$

Since there is no invariant inner product for the real scalar field, it is awkward to define this as a field momentum. However, on the manifold of positive energy solutions $\phi^{(+)}$, the integral over all space may be written

$$\begin{aligned}\int \mathrm{d}\sigma\, \theta_{0i} &= c\int \mathrm{d}\sigma\, (\phi^{(-)} \overset{\leftrightarrow}{\partial_0} \partial_i\phi^{(+)}) \\ &\equiv -(\phi^{(+)},\, p_i c\phi^{(+)}), \quad (18.43)\end{aligned}$$

where $p_i = -\mathrm{i}\hbar\partial_i$. The diagonal spatial components are

$$\begin{aligned}\theta_{ii} &= \frac{\partial \mathcal{L}}{\partial(\partial^i\phi_A)}(\partial_i\phi_A) - \mathcal{L} \\ &= \hbar^2c^2(\partial_i\phi_A)^2 - \frac{1}{2}\hbar^2c^2(\partial^\mu\phi_A)(\partial_\mu\phi_A) - \frac{1}{2}m^2c^4\phi_A^2 - V(\phi), \quad (18.44)\end{aligned}$$

where the repeated i index is not summed. The off-diagonal 'stress' tensor is

$$\begin{aligned}\theta_{ij} &= \frac{\partial \mathcal{L}}{\partial(\partial^i\phi_A)}(\partial_j\phi_A) \\ &= (\partial_i\phi_A)(\partial_j\phi_A), \quad (18.45)\end{aligned}$$

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where $i \neq j$. Notice that the trace of the space parts in $n+1$ dimensions gives

$$\sum_i \theta_{ii} = \mathcal{H} - m^2c^4\phi_A^2 - 2V(\phi) + (n-1)\mathcal{L} \tag{18.46}$$

so that the full trace is

$$\theta^\mu_{\ \mu} = g^{\mu\nu}\theta_{\nu\mu} = -m^2c^4\phi_A^2 - 2V(\phi) + (n-1)\mathcal{L}, \tag{18.47}$$

which vanishes in $1+1$ dimensions in the massless, potential-less theory.

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