On Rational Solutions of $x^3 + y^3 + z^3 = R$.

By H. W. RICHMOND.

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1. The earliest proof that every rational number (R) can be expressed as a sum of cubes of three rational numbers (x, y, z), not necessarily positive, was published in 1825 by S. Ryley, a schoolmaster of Leeds[1]: formulae were given for x, y, z in terms of a parameter, such that every value of the parameter led to a system of values of x, y, z satisfying the above relation, and every rational value of the parameter led to a system of rational values of x, y, z. The later solutions referred to by Dickson are found to give the same results as Ryley's formula, as does another method, quoted in a modified form by Landau[2] from a paper by the present writer.[3] Thus it might almost be believed that Ryley's century-old result embodies all that is known with regard to the resolution of a number into three cubes, and that his formula is unique. I propose to examine the rationale of his method and the causes of its success; it will then appear that an infinity of similar formulae exist, and that one of them is at least as simple as his. It is convenient to state Ryley's formula, and the modification made by Landau, in section 2; and to generalize the method in section 3.

Ryley's Formula.

2. In order to obtain this special solution of

$$x^3 + y^3 + z^3 = R, \qquad \text{(i)}$$

we write

$$u = x + y + z; \qquad v = y + z;$$

so that

$$R = (u - v)^3 + (v - z)^3 + z^3 = u^3 - 3v(u^2 - z^2) + 3v^2(u - z). \qquad \text{(ii)}$$

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[1] L. E. Dickson, History of the Theory of Numbers, Vol. II, p. 726. Some further information which I have discovered is to be found in the *Messenger of Mathematics*, 51 (1922), 172.

[2] Vorlesungen über Zahlentheorie, Band III, p. 216.

[3] *Proceedings of the London Mathematical Society* (2) 21 (1922), 404-409.

At this stage we restrict the values of x, y, z, or of u, v, z by imposing the condition that

$$u^3 - 3v(u^2 - z^2) = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

a condition which enables us, by substituting θu for z, to express the ratios $u : v : z$ in terms of θ. It is best to avoid the symbol of proportionality by introducing an undefined multiplier h; so we write

$$u = 3h(1 - \theta^2), \qquad v = h, \qquad z = 3h\theta(1 - \theta^2);$$

and with these values of u, v, z we find that

$$R = 3v^2(u - z) = 9h^3(1 - \theta)^2(1 + \theta).$$

Replace the multiplier h by a new multiplier $\rho = 3h(1 - \theta)$, and we arrive at the final algebraical result that, if x, y, z are determined from

$$u = \rho(1 + \theta); \qquad v = \frac{1}{3}\rho\frac{1}{1 - \theta}; \qquad z = \rho\theta(1 + \theta);$$

then

$$x^3 + y^3 + z^3 = u^3 - 3v(u^2 - z^2) + 3v^2(u - z) = \frac{1}{3}\rho^3\frac{1 + \theta}{1 - \theta}.$$

With these values of u, v, z, and the derived values of x and y, expressed in terms of θ and ρ, the problem of making the sum of the cubes of x, y, z equal to R is reduced to finding the relation between ρ and θ which makes

$$3R = \rho^3(1 + \theta)/(1 - \theta);$$

we therefore substitute

$$\theta = (3R - \rho^3)/(3R + \rho^3).$$

Landau modifies this by using a parameter $\phi = (1 + \theta)/(1 - \theta)$. His equations are

$$u = 12k\phi(\phi + 1); \qquad v = k(\phi + 1)^3; \qquad z = 12k\phi(\phi - 1);$$
$$R = 72k^3\phi(\phi + 1)^6.$$

Replace the multiplier k by $\sigma = 6k(\phi + 1)^2$, and we find that, if

$$u = 2\sigma\frac{\phi}{\phi + 1}; \qquad v = \frac{1}{6}\sigma(\phi + 1); \qquad z = 2\sigma\frac{\phi(\phi - 1)}{(\phi + 1)^2};$$

then

$$x^3 + y^3 + z^3 = \frac{1}{3}\sigma^3\phi.$$

In order to make the sum of the cubes of x, y, z equal to R, we now take

$$\phi = 3R/\sigma^3.$$

On substituting this value of ϕ we obtain values of u, v, z and deduce values of x, y as functions of σ : the result is the same as before, with σ in place of ρ. Rational values of ρ and σ lead to rational values of x, y, z which satisfy the given equation (i).

Reasons why the Method is successful.

3. It will simplify matters if we regard homogeneous equations in x, y, z as representing plane curves: when we have to deal with equations such as (i) which are not homogeneous, we shall regard x, y, z as cartesian coordinates in space; a homogeneous equation will then represent a cone with vertex at the origin. There need be no confusion.

Repeating the steps of the last section, we assume that x, y, z are proportional to rational algebraic functions of a parameter, and write

$$x = h\,X(t); \qquad y = h\,Y(t); \qquad z = h\,Z(t); \quad \ldots\ldots\ldots\text{(iv)}$$

where at least one of the functions $X(t)$, $Y(t)$, $Z(t)$ is of order three and no one is of higher order. This implies that x, y, z are connected by a homogeneous relation of order three which represents a cubic curve having a double point. Substituting these values we find that

$$x^3 + y^3 + z^3 = h^3\,F(t). \qquad \ldots\ldots\ldots\ldots\ldots\text{(v)}$$

The cubic curve

$$x^3 + y^3 + z^3 = 0 \qquad \ldots\ldots\ldots\ldots\ldots\text{(vi)}$$

meets the former cubic curve in nine points which correspond to the values of t for which $F(t)$ vanishes. It is true that in section 2 the equation was of a lower order than nine, viz. three when expressed in terms of the parameter θ, and seven in terms of ϕ; but as usual this means that the missing intersections lie at the point at which the parameter is infinite: two cubics must have nine common points. The form of F is the first crucial feature of the method, viz.

$$F(t) \equiv K\,(t-a)^6\,(t-b)^2\,(t-c); \qquad \ldots\ldots\ldots\ldots\text{(vii)}$$

because of this we are able first to replace h by a new multiplier, s,

$$s = h\,(t-a)^2\,(t-b), \qquad \ldots\ldots\ldots\ldots\text{(viii)}$$

so that

$$x^3 + y^3 + z^3 = Ks^3\,(t-c)\,/\,(t-b); \quad \ldots\ldots\ldots\ldots\text{(ix)}$$

we then make this expression equal to R by solving a linear equation in t. The geometrical interpretation of all this is that we have discovered a rational curve of order nine on the surface (i), the intersection of (i) with a cubic cone whose vertex is the origin, and have obtained expressions for the coordinates x, y, z of points on the curve as rational algebraic functions of a parameter s. [What we have done is to repeat the work of section 2 in terms of a generalized parameter t in place of θ,

$$t=(P\theta+Q)/(R\theta+S)].$$

But, for the numerical applications we wish to make, something more than this is demanded of our formulae. It is, if not essential, at all events highly desirable that the coefficients in the functions $X(t)$, $Y(t)$, $Z(t)$ should be rational numbers,[1] because then a rational value of t leads to rational values of x, y, z. If the coefficients in X, Y, Z are rational numbers, so also are those of $F(t)$. The forms of the factors of F show that under these conditions a, b, c must be rational numbers; therefore the points of the first cubic at which t has these values have rational coordinates. These points however lie also on the second cubic (vi), and it is well known that only three points on this curve have rational coordinates, viz. the three real inflexions, whose coordinates are 1, -1, 0, in various orders. Of the nine intersections of (iv) and (vi) six lie on (iv) where t has the value a, two where t is b and one where t is c; and each point is one of the real inflexions of (vi). Now it is not possible to draw through nine arbitrary points of (vi) a second cubic; through eight arbitrary points of (vi) there pass a pencil of cubic curves, all of them having the same ninth point in common with (vi). If the eight points are made up of the inflexion at which t is a taken six times and that at which t is b taken twice, the tangent at the former taken twice with that at the latter taken once form one of the cubics of the pencil; the ninth point at which t is c therefore coincides with the latter inflexion: and this inference is confirmed if we express the coordinates of points on (vi) by elliptic functions. That two different values of t should give the same point of (iv) is only to be accounted for by this point being the double point which the cubic (iv) is bound to possess: as t passes

[1] It does not seem possible to make a more precise statement. Certain functions having irrational coefficients can be transformed into functions having rational coefficients, and would therefore serve our purpose. But it is also to be noted that, although coordinates of points of a cubic surface are generally expressible by rational algebraical functions of two parameters, yet in a bipartite cubic surface real points do not correspond to real parameters. It is not impossible that this irregularity in a matter of real and unreal numbers should have a parallel in a matter of rational and irrational numbers. Hence only a guarded statement can be made.

through the value b we have a branch of (iv) which touches (vi); as t passes through the value c we have another branch which crosses it at the same point.

Under the conditions that the functions X, Y, Z of (iv) shall have rational coefficients, and that the function F shall factorize as in (vii), the only solutions are those we have found. Six intersections must lie at one inflexion, say at $(0, -1, 1)$, and three either at this or at another inflexion, say at $(1, -1, 0)$. The pencils of cubics through the nine points are

$$x^3 + y^3 + z^3 = \lambda (y + z)^3;$$
$$x^3 + y^3 + z^3 = \lambda (y + z)^2 (x + y);$$

according as the second point of inflexion does or does not coincide with the first. No member of the former pencil has a double point; but in the latter, if λ is 3, the cubic has a double point at $(1, -1, 0)$. This is the one and only solution under the conditions stated, and is seen to be Ryley's solution. To this extent Ryley's solution is unique.

An Extension of the Method.

4. The method, however, imposes a quite unnecessary restriction upon the functions $X(t)$, $Y(t)$, $Z(t)$ when it is assumed that $F(t)$ must contain[1] a linear factor raised to the sixth power. The whole process is just as effective if

$$F(t) \equiv K (t - a_1)^3 (t - a_2)^3 (t - b)^2 (t - c)$$

and the new multiplier s which replaces h is defined by

$$s = h (t - a_1)(t - a_2)(t - b).$$

With equations (vii) and (viii) thus modified equation (ix) still holds, and so on to the end,—with one difference. It is not necessary for a_1 and a_2 to have rational values. The quadratic of which they are roots must have rational coefficients, the line joining the corresponding points of (iv) must have a rational equation, and the third intersection of this line with (vi) must have rational coordinates, *i.e.* must be one of the three real points of inflexion of (vi). We therefore expect to obtain solutions similar to that of Ryley by

[1] The functions X, Y, Z cannot be of lower order than three. It is supposed here that they are of order three, but they might be of higher order.

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considering cubic curves which have 3-point contacts with (vi) at two points collinear with an inflexion of (vi), and have a node accounting for the remaining three intersections at another inflexion; that is, a double point at which one branch touches (vi).

A simple Solution.

5. We may again suppose that the node is to lie at (1, − 1, 0); the two 3-point contacts will be taken collinear with (0, − 1, 1). Now $x = 0$ is a line through the latter, and may be expected to lead to a specially simple solution, worth investigating separately. The line $x = 0$ meets (vi) in one real and two imaginary inflexions, (0, − 1, 1), (0, $-\omega$, 1), (0, $-\omega^2$, 1); and the tangents at the last two, which have 3-point contacts, have the combined equation

$$y^2 - yz + z^2 = 0.$$

It follows that all the cubic curves of the pencil

$$x^3 + y^3 + z^3 = \lambda (x + y)(y^2 - yz + z^2)$$

have three 3-point contacts with (vi) at the three required points. If λ has the value 3, the curve has a double point at (1, − 1, 0), one branch touching (vi) there. Give λ the value 3, and substitute

$$z = \theta (x + y);$$

a factor $(x + y)^2$ can be rejected, and we find that

$$x(1 - 3\theta^2 + \theta^3) = y(2 - 3\theta + 3\theta^2 - \theta^3).$$

We simplify this somewhat by writing $1 - \psi$ for θ; if we also introduce a multiplier h, as was done in section 2, we find that the values

$$x = h(1 + \psi^3);\ y = h(3\psi - 1 - \psi^3);\ z = h(3\psi - 3\psi^2) \quad \ldots\ldots(\text{x})$$

make
$$x^3 + y^3 + z^3 = 9h^3\psi(1 - \psi + \psi^2)^3.$$

We follow the procedure of section 2 and introduce in place of h a new multiplier s such that

$$s = 3h(1 - \psi + \psi^2):$$

then
$$x^3 + y^3 + z^3 = \tfrac{1}{3}s^3\psi.$$

In order to express a given rational number R as a sum of cubes of three rational numbers, we take the value $3R/s^3$ for ψ; every rational value of s then gives a rational solution. The formula gives x, y, z

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explicitly and is not more involved than Ryley's. Both are curiously complicated and unsymmetrical in contrast to the given equation (i) of which they are solutions.

A positive rational number is expressed as a sum of three *positive* rational cubes by Ryley's formula if the parameter lies between certain limits. (See the paper referred to in footnote 3). With the present formulae the limits are simpler and more easily determined. The values of ψ for which y vanishes are $2\cos(2\pi/9)$, $2\cos(8\pi/9)$, $2\cos(14\pi/9)$. For x, y, z to be of the same sign, ψ must either lie between 1 and $2\sin 10^\circ$ or $0{\cdot}34730$, or between -1 and $-2\cos 20^\circ$ or $-1{\cdot}87938$; these are the limits required.

A two-parameter solution of $x^3+y^3+z^3=R$.

6. We shall now obtain the most general solution given by the process sketched in section 4. We have to consider pencils of cubic curves which have three-point contacts with (vi) at two points collinear with one inflexion $(0, -1, 1)$, and another three-point contact at a second inflexion $(1, -1, 0)$. Certain of these curves will have nodes at the last point, and will enable us to express the coordinates x, y, z, in terms of cubic functions of a parameter t which, substituted in (vi), will lead to an equation in t of order nine of the form

$$F(t)=K(t^2+dt+e)^3(t-b)^2(t-c).$$

It is convenient to use a letter Q to denote $x^3+y^3+z^3$, and to work with

$$u\equiv x+y+z;\quad v=y+z;\quad w=x+y;$$

rather than with x, y, z. In terms of u, v, w, we find that

$$Q\equiv x^3+y^3+z^3\equiv u^3+3vw(v+w-2u). \qquad \text{(xi)}$$

Suppose that the two points of osculation collinear with $(0, -1, 1)$ lie on the line $u=pv$; then the equation

$$Q-(u-pv)^3=0$$

contains v as a factor; the remaining quadratic factor represents a conic having three-point contact with $Q=0$ at the two points required. Multiplying this by w, the tangent at the second inflexion $(1, -1, 0)$, we see that

$$[Q-(u-pv)^3]\frac{w}{v}=0$$

is a cubic curve (made up of a line and a conic) osculating $Q = 0$ at the three required points, so that, as k varies, the equation

$$[Q - (u - pv)^3]\frac{w}{v} + kQ = 0$$

gives us one of the pencils we have to consider. Expanding, we find

$$3w^2(v+w-2u)+3pu^2w-3puvw+p^3v^2w+ku^3+3kvw(v+w-2u)=0,$$

and from this it is clear that the curve will have a node at the second inflexion $(1, -1, 0)$ (at which u and w vanish) if $3k = -p^3$.

The curve of the pencil of cubics which will lead to a solution of the given equation (i) is therefore

$$3[Q - (u - pv)^3]\frac{w}{v} - p^3 Q = 0$$

or

$$Q(3w - p^3 v) = 3w(u - pv)^3, \qquad \text{..........(xii)}$$

an unnecessary factor v occurring in the last. Expanded, the equation is

$$v(9w^2 - 9p^2uw - 3p^3w^2 + 6p^3uw) - p^3u^3 + 9pu^2w - 18uw^2 + 9w^3 = 0.$$

In this we substitute $u = tw$ and so express u, v, w as polynomials in t, of orders not exceeding three, with a multiplier h in each for convenience:

$$\left.\begin{aligned} v &= h(9 - 18t + 9pt^2 - p^3t^3) \\ w &= 3h[(3p^2 - 2p^3)t + (p^3 - 3)] \\ u &= 3h[(3p^2 - 2p^3)t^2 + (p^3 - 3)t] \end{aligned}\right\} \qquad \text{............(xiii)}$$

7. These equations are of the proper form, and contain two parameters t and p; but they are complicated. To calculate Q it is best to use the result (xi), and this suggests certain simplifications. Thus we find

$$(3w - p^3v) = h(p^2t - 3)^3,$$
$$(u - pv) = h(p^2t - 3)(p^2t^2 - 6pt + 3p + 3t),$$
$$Q = 9h^3[(3p^2 - 2p^3)t + (p^3 - 3)](p^2t^2 - 6pt + 3p + 3t)^3.$$

Here Q has assumed a form to which the method used in Ryley's solution (and again in section 5) is applicable, namely a linear function of t multiplied by the cube of another function. We replace h by a new multiplier g defined by

$$g = 3h(p^2t^2 - 6pt + 3p + 3t);$$

u, v, w, and from them x, y, z, are expressed as rational fractions in t and p, each multiplied by the new multiplier g. With these values of x, y, z,

$$x^3 + y^3 + z^3 = \tfrac{1}{3} g^3 [(3p^2 - 2p^3)\, t + (p^3 - 3)].$$

In order to make Q or $x^3 + y^3 + z^3$ have a given rational value R, it is enough to give t the value which makes the expression in square brackets equal to $3R/g^3$. Substituting this value of t throughout we have values of x, y, z whose cubes have the sum R expressed in terms of two parameters, p and g. Rational values of the parameters give rational values of x, y, z.

It is possible to present these equations in a better form. In place of the parameter t we may use $s = p^2 t - 3$; we may write $p^3 h$ for h, in order to avoid fractions, and use S to denote the quantity in square brackets multiplied by three.

Then
$$\left.\begin{aligned} u &= x + y + z = hp\,(s + 3)\, S; \\ v &= \quad y + z \quad = h\,(3S - s^3); \\ w &= \quad x + y \quad = h\, p^3 S; \\ Q &= x^3 + y^3 + z^3 = 3h^3 p^3 S\,(S + s^2)^3; \end{aligned}\right\} \quad \text{(xiv)}$$
where $S = 3\,[(3 - 2p)\, s + (6 - 6p + p^3)]$.

But the interest of this result lies in the fact that a two-parameter solution has been determined, rather than in the form the solution takes. Every small convex area in the finite part of the surface (1) contains points with rational coordinates (x, y, z). If any three positive numbers a, b, c are chosen, values of x, y, z satisfying (1) can be found for which the differences of x/a, y/b, z/c are less than any assigned small number : (the same is almost always true if one or more of a, b, c are negative). But on the other hand it must be noticed that the two-parameter solution is not a complete solution: for all the values of x, y, z obtained are such that equation (xii), or the expanded form of that equation which follows, is satisfied by a rational value of p.