TRANSLATES OF L^∞ FUNCTIONS AND OF BOUNDED MEASURES

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1

D. A. Edwards has shown [1] that if X is a locally compact Abelian group and $f \in L^\infty$, then the translate f_a of f varies continuously with a if and only if f is (equal l.a.e. to) a bounded, uniformly continuous function. He remarks that this is a sort of dual to part of a result due to Plessner and Raikov which asserts that an element μ of the space M_b of bounded Radon measures on X belongs to L^1 (i.e., is absolutely continuous relative to Haar measure) if and only its translates vary continuously with the group element, the relevant topology on M_b being that defined by the natural norm of M_b as the dual of the space of continuous functions vanishing at infinity. The proof he uses (ascribed to Reiter) applies equally well in both cases, and also to the case in which X is non-Abelian. A brief examination shows that in the latter case it is ultimately immaterial whether left- or right-translates are considered; since the extra complexities of this case are principally terminological, we shall direct no further attention to it.

In this note it is shown that continuity can be sensibly weakened, and that the method extends to other situations as well.

2

In this section we formulate a result for distributions on $X = R^n$ which contains the specialisation to R^n of the preceding statements about bounded functions and bounded measures as special cases.

First a definition. If $1 \leqq p \leqq \infty$, a distribution T is said to satisfy condition (C_p) if there exists an integrable set K of strictly positive measure (which may without loss of generality be assumed to be compact) and a distribution T^* such that the set

$$(1) \qquad \{T_a - T^* : a \in K\}$$

is relatively compact in M_b if $p = 1$, or in L^p if $1 < p \leqq \infty$. Evidently, if one knows a priori that T lies in M_b if $p = 1$, or in L^p if $1 < p \leqq \infty$,

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one may replace the set (1) by the set

$$(1') \qquad \{T_a : a \in K\}.$$

In what follows we denote by C_c^∞ the space of functions on R^n which are indefinitely differentiable and have compact supports. In addition, p' denotes the exponent conjugate to p, so that $1/p+1/p' = 1$.

The Hahn-Banach theorem, in conjunction with the known density of C_c^∞ in $L^{p'}$ and in C_0, and with the known form of the duals of these latter spaces, shows that the set of distributions S for which the number

$$(2) \qquad N_p(S) = \operatorname{Sup} \{|\langle f, S\rangle| : f \in C_c^\infty, \|f\|_{L^{p'}} \leqq 1\}$$

is finite can be identified with M_b if $p = 1$, or with L^p if $1 < p \leqq \infty$, the norm N_p being at the same time identifiable with the customary norm on these spaces. Thus the condition (C_p) amounts to requiring that the set (1) is totally bounded relative to the metric defined by N_p.

These remarks having been made, we state and prove the main theorem. In it we denote by L_c^1 the set of integrable functions with compact supports.

THEOREM 1. *Let T be any distribution.*

(I) *Suppose that $1 \leqq p < \infty$, that T satisfies (C_p), and that*

$$(3) \qquad f * T \in L^p \quad \textit{for each } f \in L_c^1.$$

Then $T \in L^p$.

(II) *Suppose $p = \infty$, that T satisfies (C_∞), and that*

$$(4) \qquad \begin{cases} f * T \textit{ is a bounded, continuous function} \\ \textit{for each } f \in L_c^1. \end{cases}$$

Then T is a bounded, continuous function. The term "bounded" can be removed from (4) and from the conclusion.

If the appropriate condition (C_p) is fulfilled with some K having interior points, (3) and (4) can be replaced by the single assumption that $T \in \mathscr{D}'_{L^p}$; see [2], p. 57.

PROOF. If (1) is relatively compact, and if we choose $a_0 \in K$ such that $m(U \cap K) > 0$ for each neighbourhood U of a_0, then the set of distributions $T_a - T_{a_0}$, a ranging over K, is also relatively compact. (Compactness of K is easily seen to ensure the existence of such points a_0.) Moreover, $T_a - T_{a_0} \to 0$ distributionally as $a \to a_0$. Using the general principle, that a separated topology weaker than a compact topology is necessarily identical with the latter, it follows that $N_p(T_a - T_{a_0}) \to 0$ as $a \in K$, $a \to a_0$. Putting $K' = K - a_0$, it is equivalent to assert that $N_p(T_x - T) \to 0$ as $x \in K'$, $x \to 0$. The set K' is such that $m(K' \cap U) > 0$ for each neighbourhood U of 0. If we choose for each compact neighbourhood U of 0 an integrable function f_U such that $f_U \geqq 0$, $f_U = 0$ outside $K' \cap U$, $\int f_U dx = 1$, it follows that

$$N_p(f_U * T - T) \to 0 \tag{5}$$

as U shrinks. This is most easily seen by expressing $f_U * T$ as the vector-valued integral $\int f_U(x) T_x dx$.

Now if (3) (or (4)) is satisfied, each $f_U * T \in L^p$ [or belongs to the space of bounded, continuous functions]. On the other hand if K has interior points, a_0 may be chosen to be one of them, in which case K' is a neighbourhood of 0 and each f_U may be chosen to be continuous, and accordingly $f_U * T \in L^p$ [or is bounded and continuous] provided $f \in \mathscr{D}'_{L^p}$. Regarding this, see Schwartz [2], p. 57.

In any case (5) shows that the $f_U * T$ form a Cauchy family in L^p [or a family of bounded, continuous functions which is Cauchy for the structure of uniform convergence]. Completeness of the latter space shows that this family is convergent in L^p [or uniformly in the space of bounded, continuous functions], and (5) then shows that $T \in L^p$ [or is a bounded, continuous function]. Only verbal changes are required to cover the case in which the $f_U * T$ are continuous but not necessarily bounded.

COROLLARY 1. *Suppose that T is a distribution and that either $1 \leqq p < \infty$ and T satisfies* (3), *or $p = \infty$ and T satisfies* (4). *Suppose further that, for the appropriate value of p, one or other of the following conditions is fulfilled:*

(*a*) *to each $\alpha > 0$ corresponds a distribution T^α such that the set*

$$S_\alpha = \{a \in X : N_p(T_a - T^\alpha) \leqq \alpha\}$$

has strictly positive interior measure;

(*b*) *there exists a set $A \subset X$ with strictly positive interior measure such that the set*

$$\sum = \{T_a : a \in A\}$$

is separable relative to the norm N_p.

The conclusions are: if $1 \leqq p < \infty$, then $T \in L^p$; if $p = \infty$, then T is a bounded, continuous function.

PROOF. In either case it suffices to show that the appropriate condition (C_p) is fulfilled.

(a) Since the norm N_p is translation-invariant, if a_0 be chosen from S_α, one obtains for all $a \in S_\alpha$ the inequality

$$N_p(T_{a-a_0} - T) = N_p(T_a - T_{a_0}) \leqq N_p(T_a - T^\alpha) + N_p(T^\alpha - T_{a_0}) \leqq 2\alpha.$$

Thus we may as well assume from the outset that $T^\alpha = T$ and that S_α contains the origin. Then, if a and b belong to S_α, we have

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$$N_p(T_{a-b}-T) = N_p(T_a-T_b) \leqq N_p(T_a-T)+N_p(T-T_b) \leqq 2\alpha.$$

On the other hand, since S_α has strictly positive interior measure, the difference set $W_\alpha = S_\alpha - S_\alpha$ is a neighbourhood of 0 ([3], p. 50). The relation $N_p(T_x-T) \leqq 2\alpha$ for $x \in W_\alpha$ then implies at once that (C_p) is satisfied.

(b) This is reducible to case (a), as follows. The set A may be assumed to be a compact set with strictly positive measure. By hypothesis, there exists a sequence $(a_n)_{n=1}^\infty$ of points of A such that, for any $\alpha > 0$,

$$\begin{aligned} A &= \bigcup_{n=1}^{\infty} \{a \in A : N_p(T_a - T_{a_n}) \leqq \alpha\} \\ &= \bigcup_{u=1}^{\infty} A_n, \end{aligned}$$

say. From (2) it appears that the function $x \to N_p(T_x)$ is lower semicontinuous, so that each set A_n is compact. Since A has strictly positive measure, the same is true of A_n for at least one integer $n = n(\alpha)$. But, if we define $T^\alpha = T_{a_{n(\alpha)}}$, it is evident that $S_\alpha \supset A_{n(\alpha)}$, so that condition (a) is fulfilled.

Remarks. (1) Only minor modifications in the proof of Theorem 1 are needed to show that if T is a distribution with the property that, for some compact set K satisfying $m(K) > 0$, the set

$$\left\{f * T : f \in L^1,\ f \geqq 0,\ f = 0 \text{ off } K,\ \int f dx = 1\right\} \tag{6}$$

is relatively compact in L^p (where $1 \leqq p < \infty$), then $T \in L^p$. If also K has interior points, one may in (6) impose the condition that $f \in C_c^\infty$.

Similarly, if (6) is relatively compact in L, and if $f * T$ is, for each f of the type specified in (6), a bounded continuous function, then T is (equal a.e. to) a bounded continuous function.

(2) The methods used in [4] combine with Theorem 1 to show that if $T \in L^\infty$ is such that $f * T$ satisfies a Lipschitz condition at the origin for each $f \in L^1$ (or for each f in a nonmeagre subset of L^1), then T itself satisfies a Lipschitz condition.

3

The case of bounded measures and bounded functions.

The main theorem, taken in conjunction with the remarks immediately preceding it, lead directly to the following conclusion.

COROLLARY 2. *Let T be a bounded measure [or a bounded, measurable function] such that, for some integrable set K having strictly positive measure, the set (1′) is relatively compact in M_b [or in L^∞]. Then $T \in L^1$ [or is a continuous function, necessarily bounded and uniformly continuous].*

It is to be noted that, although the main theorem is formulated for the special groups $X = R^n$, the proof goes over almost without change when X is a general locally compact Abelian group and T is restricted a priori as in Corollary 2. This leads to the generalization of D. A. Edwards' result spoken of in § 1.

COROLLARY. 3. *The conclusions of Corollary* 1 *are valid whenever* $T \in M_b$ [*or* L^∞] *is such that the function* $x \to T_x$ *is measurable* (*relative to the normed topology on* M_b *or* L^∞).

PROOF. Measurability entails (Lusin's Theorem) that there exists a compact set K with strictly positive measure such that the said function has a restriction to K which is continuous, in which case the set (1′) is compact.

4

Another criterion for absolute continuity of measures.

THEOREM 2. *Let X be a first countable locally compact Abelian group and μ a* (*Radon*) *measure on X such that, for each relatively compact Borel subset A of X, the function $x \to \mu(xA)$ is* (*equal l.a.e. to*) *a continuous function on X. Then $\mu \in L^1_{\mathrm{loc}}$* (*i.e., is absolutely continuous relative to Haar measure on X*).

(b) *If X is R^n or a finite dimensional torus group, and if T is a distribution on X such that $f * T$ is* (*equal distributionally to*) *a continuous function for each bounded Borel function f having a compact support, then $T \in L^1_{\mathrm{loc}}$.*

PROOF. (a) The main hypothesis signifies that the convolution $\mu * c_A$ is (equal l.a.e. to) a continuous function, c_A denoting the characteristic function of the relatively compact Borel set A. If f is any bounded Borel function vanishing outside a compact set, then f is the uniform limit of finite linear combinations of functions c_A, where A is a Borel subset of K. Accordingly, $\mu * f$ is the locally uniform limit of finite linear combinations of functions $\mu * c_A$, and is therefore continuous.

Take now a neighbourhood base $(U_n)_{n=1}^{\infty}$ at 0 in X. For each n choose a positive continuous function g_n with a compact support within U_n and such that $\int g_n dx = 1$. If f is as in the statement of the theorem,

$$\int f(g_n * \mu)dx = \int \check{g}_n(\mu * \check{f})dx,$$

where $\check{h}(x) = h(-x)$. The right-hand side here converges as $n \to \infty$ to the value at 0 of the continuous function which is equal l.a.e. to $\mu * \check{f}$. Since every Haar measurable function is equal l.a.e. to a Borel function, this

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shows that the sequence $(g_n * \mu)_{n=1}^{\infty}$ is a weak Cauchy sequence in L^1_{loc}, and is therefore convergent in that space. At the same time, $(g_n * \mu)_{n=1}^{\infty}$ converges vaguely to μ, whence it follows that $\mu \in L^1_{\text{loc}}$. This proves (a).

(b) This follows from the preceding argument by virtue of the known fact ([2], p. 48) that, for the groups X specified, a distribution T such that $f * T$ is continuous whenever f is continuous and has a compact support is necessarily a measure.

5

The corollary stated in [1] asserts some interesting consequences concerning continuous linear operators A mapping translation-invariant subspaces of L^p $(1 \leqq p < \infty)$ into L^∞. The results of the present paper afford numerous results of this type. We record only one, and this in a somewhat specialised form.

Suppose that X is a finite-dimensional torus group, V a vector subspace of L^p where $p \geqq 2$ and A a linear operator from V into $\mathscr{D}'$ (the space of distributions on X). Suppose further that

(i) If $v \in V$ and $f \in L^\infty$, then $v * f \in V$ and $A(v * f) = (Av) * f$;

(ii) if $v \in V$ has an absolutely convergent Fourier series, then Av is a continuous function.

The conclusion is that A maps V into L^1.

In fact, if $v \in V \subset L^p$ and $f \in L^\infty$, then (by Bessel's inequality) $v * f$ has an absolutely convergent Fourier series, so that (i) and (ii) show that $(Av) * f$ is continuous. Theorem 2 implies then that $Av \in L^1$, as asserted.

Notice that we have not assumed a priori that A is continuous in any sense. On the other hand, if V is topologised as a locally convex space, and is translation-invariant, and if A is continuous and commutes with translations, then mild conditions on V will ensure that (i) is satisfied. Sufficient such conditions are that the function $x \to v_x$ is [weakly] continuous from X into V for each $v \in V$; and that, in V, the closed convex envelope of any [weakly] compact set is weakly compact. This last condition is fulfilled whenever V is quasicomplete.

Amongst the linear operators A satisfying (i) appear all the "multiplier operators" A_k, where k is a complex-valued function on the charater group and $A_k v$ is defined by the requirement that $(A_k v)^\wedge = k \cdot \hat{v}$, the hat denoting passage to the Fourier transform. The result we have obtained includes assertions about such operators defined on (not necessarily closed) subspaces of L^p.

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