

PART B

THEORETICAL METHODS FOR HANDLING NON-LTE PROBLEMS

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THEORETICAL METHODS OF TREATING LINE FORMATION
PROBLEMS IN STEADY-STATE EXTENDED ATMOSPHERES

by

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ABSTRACT

Theoretical methods applicable to the study of line formation in steady-state extended atmospheres are reviewed. The formal solution of the transfer equation is considered, as well as numerical and analytical methods of determining the source function. Topics discussed include: the local frequency transformation, geometrical effects, and the case of large velocity gradients. A new plane-parallel approximation for spherically symmetric moving atmospheres is given that takes account of transverse velocity gradients.

Key words: spectral line formation, spherical geometry, radiative transfer, stellar atmospheres, moving atmospheres.

INTRODUCTION

Methods of handling line formation problems have advanced markedly in recent years for atmospheres that are static and for which the plane-parallel approximation is reasonably valid. However, two major difficulties may arise when extended atmospheres are considered: first, the geometrical extension may be so large that the plane-parallel approximation is no longer valid; and second, macroscopic velocity fields may need to be taken into account. While only the first of these is strictly implied by the term "extended atmosphere," the second is included because so many of the

astronomical examples of extended atmospheres have such macroscopic velocity fields.

In this paper the theoretical work relating to these two difficulties will be reviewed. In doing so primary consideration has been given to work in the spirit of the modern approach (see, e.g. Jefferies, 1968) to line formation, in which the equations of statistical equilibrium are solved in conjunction with the transfer equations. Many of the results of the early work that depend on the assumption of coherent scattering, for example, must be viewed with suspicion for the problem of line formation. However, many of the methods and techniques developed in these papers are relevant, and it has been the intention here to extract such useful information whenever possible.

After introducing the basic equations, the formal solution of the transfer equation will be discussed. Then analytical and numerical methods of determining the source function will be considered, including Sobolev's theory of moving atmospheres.

One new result presented here is a formulation of the plane-parallel approximation for moving atmospheres having spherical symmetry that takes account of transverse velocity gradients.

BASIC EQUATIONS

The line formation problem is defined by the simultaneous solution of the equations of statistical equilibrium and the transfer equations for the atom or ion under consideration. The assumption of complete redistribution is usually made to account for the noncoherent nature of the scattering. While this assumption has proved of great utility in static, plane-parallel atmospheres, there are reasons to believe that it may not be as good when large velocity gradients are present. This is because the mechanism of trapping radiation, which usually produces isotropic, frequency independent intensities in the line core, will not operate so effectively, since radiation can escape more readily and because of the anisotropy introduced by non-uniform expansion. The only work relating to this point is Magnan's (1968). He found only fairly small errors due to the complete redistribution assumption in an atmosphere with velocities of the same order as

the Doppler velocity. However, the question deserves further investigation under a variety of conditions before any final conclusions can be drawn. In this paper complete redistribution will be assumed, and the modifications necessary to treat Doppler redistribution will merely be indicated.

The effect of extended geometry on the equations of statistical equilibrium is simply to make all quantities depend on a general spatial point \underline{r} , instead of the simple height or optical depth variable of the plane-parallel case. The transfer equations must be similarly modified by considering specific intensities which are functions of position \underline{r} , frequency ν , and direction defined by a unit vector $\underline{\ell}$. In geometries having certain symmetries, such as spherical symmetry, the equations may depend on fewer variables, of course, and in the limit of a thin spherically symmetric shell, the plane-parallel equations are recovered. With few exceptions (e.g., Bappu and Menzel, 1954) all of the work done on the problem of extended geometries has assumed spherical symmetry. In cases where Sobolev's theory of moving atmospheres is applicable it is not strictly bound by any particular assumption on the geometry, but in the actual applications of this theory spherical or plane-parallel geometry have been used.

The primary effect of velocity gradients is the Doppler shift of the radiation field as seen in a local frame of reference moving with the material at any point. This manifests itself in those terms in the equations accounting for the interaction between matter and radiation, namely, in the emission and absorption coefficients. For complete redistribution, where the source function is frequency independent, the modification to the usual equations is simply to replace the absorption profile $\phi(\underline{r}, \nu)$

by $\phi(\underline{r}, \underline{\ell}, \nu) = \phi(\underline{r}, \nu - \frac{\nu_0}{c} \underline{\ell} \cdot \underline{v}(\underline{r}))$, where ν_0 is the line center frequency, $\underline{v}(\underline{r})$ is the velocity of the material at point \underline{r} , and c is the velocity of light. This correctly describes the Doppler effect to lowest order in v/c . For the radiation field this effect is important because it can radically change optical depth relations along a ray, as the absorption at any point depends sensitively on any displacement of the profile. Similarly the matter can be brought to quite different states of excitation by motion which causes it to absorb far more or far less in any transition, as it absorbs in different parts of the highly frequency-dependent radiation field.

A secondary effect of velocity fields is to add streaming terms to the statistical equilibrium equations, which describe the changes to the populations from convective transport of material. Although this might be an important effect in some physical situations, it would not seem that it will be so for any stars discussed here. This may be seen by considering an extreme physical example with velocities on the order of 10^3 km s⁻¹ and a characteristic length of 10^6 km, which are typical for some Wolf-Rayet stars. This leads to an effective rate coefficient or order 10^{-3} s⁻¹ which is very small compared with other typical rate coefficients entering the statistical equilibrium equations. Therefore this effect will be neglected here.

The equations of statistical equilibrium are statements that the populations n_i at a point corresponding to the various relevant levels of excitation energy E_i are independent of time

$$0 = \sum_j (n_j \Gamma_{ji} - n_i \Gamma_{ij}) \quad . \quad (1)$$

Typically i ranges from 1 to N for N bound levels, with the addition of values for adjacent stages of ionization.

The rate coefficients Γ_{ij} may be separated into rate coefficients due to radiation and to collision with other particles (probably mainly electrons, but perhaps high energy protons or alpha particles may contribute)

$$\Gamma_{ij} = R_{ij} + C_{ij} \quad , \quad (2)$$

For the radiative transitions between bound levels,

$$R_{ij} = A_{ij} + B_{ij} \bar{J}_{ij} \quad ,$$

$$E_i > E_j$$

$$R_{ji} = B_{ji} \bar{J}_{ij} \quad , \quad (3)$$

where the A's and B's are the Milne form of the Einstein coefficients and

$$\bar{J}_{ij} = \frac{1}{4\pi} \int_0^\infty d\nu \int d\ell \phi(\mathbf{r}, \nu - \frac{\nu_{ij}}{c} \ell \cdot \mathbf{v}(\mathbf{r})) I_\nu(\mathbf{r}, \ell) \quad (4)$$

Here $I_\nu(\mathbf{r}, \ell)$ is the specific intensity of radiation at frequency ν at point \mathbf{r} in direction ℓ . The normalized profile function is defined by

$$\phi(\mathbf{r}, \nu) = k_\nu^{(\ell)}(\mathbf{r}) / k_\ell(\mathbf{r}) \quad , \quad (5)$$

where $k_\nu^{(\ell)}$ is the line opacity for the transition $i \rightarrow j$, in the rest frame of the material, and where the integrated line opacity k_ℓ is

$$k_\ell(\mathbf{r}) = \int_0^\infty k_\nu^{(\ell)}(\mathbf{r}) d\nu \quad . \quad (6)$$

Thus

$$\int_0^\infty \phi(\mathbf{r}, \nu) d\nu = 1 \quad . \quad (7)$$

The line center frequency ν_{ij} is computed from

$$h\nu_{ij} = E_i - E_j \quad (8)$$

h being Planck's constant.

The appearance of the velocity field in the profile function in Eq. (4) is responsible for the velocity effects associated with the equations of statistical equilibrium. It should be noted that the profile function cannot now be taken outside of the angular integration, as in the static case.

The equation of transfer for radiation in the line transition $i \rightarrow j$ is

$$\begin{aligned} & \ell \cdot \frac{\partial}{\partial \mathbf{r}} I_{\nu}(\mathbf{r}, \ell) \\ &= k_{\ell}(\mathbf{r}) \phi(\mathbf{r}, \nu - \frac{\nu_{ij}}{c} \ell \cdot \mathbf{v}(\mathbf{r})) \left[- I_{\nu}(\mathbf{r}, \ell) + S(\mathbf{r}) \right] \\ & \quad + k_c(\mathbf{r}) \left[- I_{\nu}(\mathbf{r}, \ell) + B_c(\mathbf{r}) \right], \end{aligned} \quad (9)$$

where

$$S(\mathbf{r}) = \frac{2h\nu_{ij}^3}{c^2} \left(\frac{g_i n_j(\mathbf{r})}{g_j n_i(\mathbf{r})} - 1 \right)^{-1} \quad (10)$$

is the line source function and g_i is the statistical weight of level i . The line opacity k_{ℓ} is given by

$$k_{\ell}(\mathbf{r}) = \frac{h\nu_{ij}}{4\pi} \left(n_j B_{ji} - n_i B_{ij} \right). \quad (11)$$

The boundary conditions are usually given by specifying the incident intensities on all boundaries of the region, or, in some cases, certain linear relations (reflection conditions) relating incident and emergent intensities.

In writing Eq. (9) continuum absorption has been taken into account by means of the continuum source function B_c and the continuum opacity k_c . Also, complete redistribution has been assumed in this equation. In order to account for Doppler redistribution properly, the absorption and emission of a number $n_i(\mathbf{r}, \mathbf{w}) d\mathbf{w}$ of particles in level i in velocity range $d\mathbf{w}$ would have to be considered, rather than simply the total number in each level. This would also necessitate writing a more complete equation of statistical equilibrium for $n_i(\mathbf{r}, \mathbf{w})$. Another way to take Doppler redistribution into account is by use of redistribution functions (Hummer, 1962, 1968) although this method assumes that the velocity distribution of the lower level is known to be Maxwellian, and that stimulated emissions are

negligible; these assumptions will certainly apply to resonance lines of high excitation potential, but perhaps not necessarily for other lines. The relationship between non-Maxwellian velocity distributions of the excited levels and Doppler redistribution has been emphasized by Oxenius (1965).

It is somewhat more convenient to work in terms of a normalized frequency variable defined by

$$x = \frac{v - v_{ij}}{\Delta_{ij}}, \quad (12)$$

where

$$\Delta_{ij} = \frac{v_{ij} \bar{v}_{th}}{c}, \quad (13)$$

and where \bar{v}_{th} is a typical Doppler velocity characterizing the atmosphere, given by

$$\bar{v}_{th} = \left[\frac{2kT_{kin}}{m} \right]^{1/2}. \quad (14)$$

Here \bar{T}_{kin} is a typical kinetic temperature, which may include contributions from "microturbulence." For Doppler profiles the profile function then becomes

$$\phi(\underline{r}, x) = \frac{1}{\sqrt{\pi} \delta(\underline{r})} e^{-x^2/\delta^2(\underline{r})}, \quad (15)$$

where

$$\delta(\underline{r}) = \frac{v_{th}(\underline{r})}{\bar{v}_{th}}$$

and

$$v_{th}(\underline{r}) = \left[\frac{2kT_{kin}(\underline{r})}{m} \right]^{1/2} \quad (17)$$

is the Doppler velocity characterizing the profile at the point \underline{r} . A dimensionless variable $u(\underline{r})$ is also

defined in units of \bar{v}_{th} by

$$u(\mathbf{r}) = \frac{v(\mathbf{r})}{\bar{v}_{th}} \quad (18)$$

With these variables Eqs. (4) and (9) become

$$\bar{J}_{ij}(\mathbf{r}) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dx \int d\ell \phi(\mathbf{r}, \mathbf{x} - \ell \cdot \mathbf{u}(\mathbf{r})) I_{\mathbf{x}}(\mathbf{r}, \ell) \quad (19)$$

$$\begin{aligned} \ell \cdot \frac{\partial}{\partial \mathbf{r}} I_{\mathbf{x}}(\mathbf{r}, \ell) &= k_{\ell}(\mathbf{r}) \phi(\mathbf{r}, \mathbf{x} - \ell \cdot \mathbf{u}(\mathbf{r})) \left[-I_{\mathbf{x}} + S \right] \\ &+ k_c(\mathbf{r}) \left[-I_{\mathbf{x}} + B_c \right] \end{aligned} \quad (20)$$

while in Eqs. (5), (6), and (7) the variable v is replaced by x . Finally in Eq. (11) an additional factor of Δ_{ij}^{-1} appears on the right-hand side. The range of the variable x may be taken as $-\infty$ to $+\infty$ with inconsequential error.

The advantage of these dimensionless variables is that material velocities of the order of the typical thermal velocity correspond to $|u| \sim 1$, and it is in these cases that velocity effects begin to be important.

THE FORMAL SOLUTION

Since the pioneering work of Beals (1931) many investigators have assumed various structures for extended atmospheres, including the specification of the source function, and then have computed the emergent flux using the formal solution of the transfer equation. Such calculations avoid the difficult problems of actually determining the source function as a consistent result of the specification of the fundamental atmospheric parameters. Despite their possible inconsistencies, however, the importance of these calculations for the development of the subject should not be underestimated. It is probably fair to say that the present-day conception of extended stellar atmo-

spheres is largely based upon such calculations, or upon intuition based upon knowledge of the formal solution. It is not yet possible to solve for the source function including all those effects that are known or suspected to be important, so the calculations based on the formal solution are still likely to be important for some time. In any case such calculations will be necessary as a final step in a more detailed calculation and in some cases the primary effects of velocity fields are to be found in this step. For example, the redshift found by Hummer and Rybicki (1968) for an expanding atmosphere is, as pointed out by them, solely due to the change in optical depth relations in various parts of the line, and not to any substantial change in the source function from the static case.

The formal solution of the transfer equation is well-known (see, e.g., Chandrasekhar (1934)). Write Eq. (20) as

$$\frac{\partial I_x(\xi)}{\partial \xi} = \kappa_x(\xi) [-I_x + \mathcal{J}_x] \quad (21)$$

where ξ measures path length along a ray from some convenient point, and where

$$\begin{aligned} \kappa_x(\xi) &= k_\ell(\xi) \phi(\xi, x) + k_c(\xi) \\ \mathcal{J}_x(\xi) &= \frac{k_\ell(\xi) \phi(\xi, x) S(\xi) + k_c(\xi) B_c(\xi)}{k_\ell(\xi) \phi(\xi, x) + K_c(\xi)} \end{aligned} \quad (22)$$

All quantities are expressed as functions of ξ along the ray. Then the formal solution is

$$\begin{aligned} I_x(\xi) &= I_x(0) e^{-\int_0^\xi \kappa_x(\xi') d\xi'} \\ &+ \int_0^\xi \kappa_x(\xi') x(\xi') e^{-\int_{\xi'}^\xi \kappa_x(\xi'') d\xi''} d\xi' \end{aligned} \quad (23)$$

The intensity at a given point is then expressed as a quadrature involving the total opacity κ_x , the total source function \mathcal{J}_x , and a value of intensity at another point on the ray $I_x(0)$, which is known if $\xi = 0$ is a boundary point and $I_x(0)$ is known incident boundary condition. In some cases the zero point can be taken so that the exponential factor multiplying $I_x(0)$ is essentially zero, and the value of $I_x(0)$ is not needed.

For a given set of values for the function κ_x and \mathcal{J}_x the emergent intensity along any ray can be computed. A convenient measure of the emergent flux which would be received by a distant observer over the area of a projected disk of the atmosphere. This quantity might be called the specific luminosity of the star in that direction and at the given frequency; in cases where the star radiates isotropically it is simply the total luminosity per unit frequency range divided by 4π . The actual flux at the point of observation is proportional to the specific luminosity, the factor being the inverse square of the observation distance. The observed shape of the spectrum at any distance from the star is correctly given by the specific luminosity as a function of frequency.

The introduction of any symmetries, such as spherical symmetry, will simplify the problem considerably. In plane-parallel geometry the formal solution is given by Eq. (23) except all quantities depend on the depth variable z alone, so that ξ becomes z and $d\xi$ becomes dz/μ , and similarly for the various primed variables. In spherically symmetric geometry all quantities in Eq. (23) depend on the radius r alone, but the relationship between ξ and r is much more complex. It is sometimes convenient in this case to use a new variable p which represents the distance of closest approach of the ray to the center of spherical symmetry, and to express all quantities in terms of r and p rather than r and μ . This latter procedure was first introduced by Chandrasekhar (1934). It is also possible to use ξ and p as the basic variables and this has essentially been done by Castor and Van Blerkom (1970). These two choices of variables are useful because the transfer equation can be written in terms of a single partial derivative, rather than the two of the r, μ representation. This is already clear from Eq. (21).

Another kind of simplification arises for stars with large velocity gradients. For each

frequency the radiation in a spectral line originates in relatively localized regions that lie very close to the so-called surfaces of constant radial velocity defined by $\mathbf{x} = \underline{\hat{l}} \cdot \underline{u} = u_{\hat{l}}$ where $u_{\hat{l}}$ is the component of velocity along the line of sight. For any given velocity field these surfaces can be determined; then assuming that the continuum opacity is small, the only region where any emission in the line can take place at frequency x is very near the appropriate surface. The line source function S can be assumed to be constant over this region and the intensity along the ray is equal to

$$I_x(\xi) = I_x(0) e^{-\tau_x} + S(1 - e^{-\tau_x}) \quad , \quad (24)$$

where τ_x is the optical thickness of the region along the ray. The computation of τ_x will be postponed until the following section when local frequency variables are introduced. When more than one such surface is cut by a single ray corresponding formulas can be similarly derived. For example, when two surfaces are involved,

$$I_x(\xi) = I_x(0) e^{-(\tau_x + \tau'_x)} + S e^{-\tau'_x} (1 - e^{-\tau_x}) + S' (1 - e^{-\tau'_x}) \quad . \quad (25)$$

Here the ray starts at $\xi = 0$, first cuts the surface corresponding to the unprimed quantities, and then cuts the surface with the primed quantities. For further details of such calculations see Rublev (1961, 1964), Lyong (1967), and Castor (1970).

ANALYTICAL METHODS FOR DETERMINING THE SOURCE FUNCTION

While most methods for actually determining the source function are numerical ones, there are some topics of an analytical nature that are relevant to the line formation problem beyond the simple consid-

erations of the formal solution. Among these are the local frequency transformation, special geometries, the infinite medium with constant velocity gradient, and Sobolev's theory of moving atmospheres. These will be discussed in this section and the numerical methods in the next.

a. Local Frequency Transformation

Milne (1930) introduced a transformation to a new frequency variable x' related to x by

$$x' = x - \int \mathbf{u}(\mathbf{r}) \cdot \mathbf{l} \quad (26)$$

Originally this transformation was formulated in the plane-parallel case; Eq. (26) is the generalization to arbitrary geometry. Clearly x' is the frequency seen in a frame of reference moving with the local material at the point \mathbf{r} when radiation travelling in direction \mathbf{l} has frequency x in the stationary frame.

If intensities and other variables are written as functions of x' instead of x then the transfer equation becomes

$$\begin{aligned} \mathbf{l} \cdot \frac{\partial I_{\mathbf{x}'}}{\partial \mathbf{r}} - Q(\mathbf{r}, \mathbf{l}) \frac{\partial I_{\mathbf{x}'}}{\partial x'} = \\ = k_{\mathbf{l}}(\mathbf{r}) \phi(\mathbf{r}, \mathbf{x}') \left[-I_{\mathbf{x}'} + S \right] + \left[k_{\mathbf{c}}(\mathbf{r}) - I_{\mathbf{x}'} + B_{\mathbf{c}} \right] \end{aligned} \quad (27)$$

where Q is the following quadratic form in the components of the vector \mathbf{l} :

$$Q(\mathbf{r}, \mathbf{l}) = \sum_{\alpha\beta} \frac{\partial u_{\alpha}}{\partial r_{\beta}} l_{\alpha} l_{\beta} = \sum_{\alpha\beta} G_{\alpha\beta} l_{\alpha} l_{\beta} \quad (28)$$

Here α and β are tensor indices ranging from 1 to 3, which label the components of the corresponding vectors. Since the antisymmetric part of $G_{\alpha\beta}$ does not contribute to Q , it may be defined as

$$G_{\alpha\beta}(\mathbf{r}) = \frac{1}{2} \left(\frac{\partial u_\alpha}{\partial r_\beta} + \frac{\partial u_\beta}{\partial r_\alpha} \right) = G_{\beta\alpha}(\mathbf{r}) \quad . \quad (29)$$

This may be recognized as the rate of strain tensor of fluid dynamics. There are several special forms for \underline{u} , $G_{\alpha\beta}$, and Q corresponding to cases of various symmetries:

1. Plane-parallel symmetry

Let $\alpha=3$ be the index for the direction normal to the planes of symmetry. Then

$$\begin{aligned} u_1 = u_2 = 0 \quad ; \quad u_3 \equiv u(z) \\ G_{33} = u'(z) \\ Q = \mu^2 u'(z) \quad . \end{aligned} \quad (30)$$

2. Spherical symmetry

Let the indices take the values r , θ , and ϕ , corresponding to the radial, polar, and azimuthal directions at the given point. Then

$$\begin{aligned} u_\theta = u_\phi = 0 \quad ; \quad u_r \equiv u(r) \\ G_{\theta\theta} = G_{\phi\phi} = u(r)/r \quad ; \quad G_{rr} = u'(r) \\ Q = \mu^2 u'(r) + (1-\mu^2)u(r)/r \quad , \end{aligned} \quad (31)$$

Note that here and in the preceding the values of $G_{\alpha\beta}$ not given are zero, and that μ is the usual direction cosine of the ray.

There are two special cases that frequently occur in the spherical case. When $u(r)$ is constant there results $Q = (1-\mu^2)u/r$, and when $u(r)$ is proportional to r (uniform expansion) there results $Q = u'$.

One advantage of such a transformation is that absorption and emission in the local frame are usually close to being isotropic, so that the vari-

ation of intensity with angle can be expected to be smoother at fixed x' than at fixed x . For example, consider a rapidly moving region, $|u| \gg 1$. As the direction of a ray is varied slightly the absorption in the region can vary tremendously when the right condition of line of sight velocity is met for the particular frequency x . This leads to rapidly varying functions of angle, which are hard to handle numerically.

The local frequency transformation makes obvious the fact that only velocity gradients affect the line formation problem, since only gradients appear in Q . This is also quite obvious on physical grounds, but some of the implications are quite deep. If the opacities and profile function are independent of position \underline{r} , and if $Q(\underline{r}, \ell)$ is likewise independent of \underline{r} , then Eq. (27) is an integro-differential equation with spatial translational invariance. It is therefore susceptible to a host of techniques of solution more or less familiar in the static case, but now referring to a constant velocity gradient. This fact forms the basis of Sobolev's analytical solution, to be discussed later in this section.

The problem left in the last section, that of computing the value of τ_x in Eq. (25), will now be completed. It is given by

$$\tau_x = \int_0^{\xi} k_{\ell}(\underline{r}) \phi(\underline{r}, x - u(\underline{r})) d\xi' \quad ,$$

where $\underline{r} = \underline{r}(\xi')$ along the ray and where k_c is assumed negligible. Changing the variable of integration to the local frequency variable gives

$$\tau_x = \frac{k_{\ell}(\underline{r}_0)}{|Q(\underline{r}_0, \ell)|} \int \phi(\underline{r}_0, x') dx' \quad ,$$

where the value $\underline{r} = \underline{r}_0$, the position of the surface of constant velocity, has been inserted in those functions that are slowly varying over the range in which the integrand is not negligible. These functions have further been taken from under the integral. The origin of the factor in the denominator is the transformation of differentials,

$$\frac{d\xi'}{dx'} = \left(\frac{dx'}{d\xi'} \right)^{-1} \quad , \quad (32)$$

and

$$\frac{dx'}{d\xi'} = - \sum_{\alpha\beta} \frac{\partial u_{\ell}(\underline{r})}{\partial r_{\beta}} \frac{\partial r_{\beta}}{\partial \xi'} \ell_{\alpha} \quad , \quad (33)$$

noting that

$$\ell_{\beta} = \frac{\partial r_{\beta}}{\partial \xi'} \quad . \quad (34)$$

The absolute value is to account for the correct ordering of the limits of integration, making the lower limit smaller than the upper limit. From the normalization of ϕ it follows that

$$\tau_{\underline{x}} = \frac{k_{\ell}(\underline{r}_0)}{|Q(\underline{r}_0, \ell)|} \quad , \quad (35)$$

since the range of integration for large velocity gradients is essentially over the entire line.

There are further uses of the local frequency transformation, one of which will appear presently. It will merely be pointed out here that, in such a local frequency description, redistribution functions can be used in their static form, rather than having to take the macroscopic motions directly into account in the redistribution functions themselves.

b. Special Geometries

The two special geometries of interest are the plane-parallel (pp) and spherically symmetric (ss) ones. The directional derivatives for these two cases are

$$\underline{\ell} \cdot \frac{\partial I_{\underline{x}}}{\partial \underline{r}} = \mu \frac{\partial I_{\underline{x}}}{\partial r} \quad (\text{pp}) \quad (36)$$

and

$$\underline{\ell} \cdot \frac{\partial I_{\underline{x}}}{\partial \underline{r}} = \mu \frac{\partial I_{\underline{x}}}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial I_{\underline{x}}}{\partial \mu} \quad (\text{ss}) \quad (37)$$

in terms of the frequency x . In terms of the local (1) frequency variable $x' = x - \mu u(r)$ they are

$$\underline{\lambda} \cdot \frac{\partial I_x}{\partial r} = \mu \frac{\partial I_{x'}}{\partial r} - \mu^2 u'(r) \frac{\partial I_{x'}}{\partial x'} \quad (1pp) \quad (38)$$

$$\underline{\lambda} \cdot \frac{\partial I_x}{\partial r} = \mu \frac{\partial I_{x'}}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial I_{x'}}{\partial \mu} +$$

$$- \left[\mu^2 u'(r) + (1-\mu^2) \frac{u(r)}{r} \right] \frac{\partial I_{x'}}{\partial x'} \quad (1ss) \quad (39)$$

An interesting fact emerges from these expressions. The plane-parallel equations are usually obtained by dropping the term involving the partial derivative with respect to μ , as in Eqs. (36) and (37), the argument being that this term is of order of the ratio of a mean free path to the radius of curvature of a layer. By the same argument the corresponding term in Eq. (39) may be dropped, but this still leaves an extra term involving u/r when compared to Eq. (38). This term cannot be dropped on any reasonable grounds since in extended atmospheres it is on the same order as the term in $u'(r)$, which is clearly not negligible since it accounts for all the velocity gradient effects. Thus a paradox has appeared: Making the plane-parallel approximation first and performing the local frequency transformation second does not give the same results as these operations performed in the reverse order.

We feel that the reason for this paradox is that the derivative $\partial I_x / \partial \mu$ at constant x is not small and cannot be dropped, while the derivative $\partial I_{x'} / \partial \mu$ at constant x' is small and can be dropped. This point was discussed in the preceding subsection. As a consequence Eq. (36) is probably not the best formulation of the plane-parallel approximation for moving spherically symmetric atmospheres. A more correct formulation (c) in local frequency variables would be

$$\underline{\omega} \cdot \frac{\partial I_{\underline{x}'}}{\partial \underline{r}} = \mu \frac{\partial I_{\underline{x}'}}{\partial r} +$$

$$- \left[\mu^2 u'(r) + (1-\mu^2) \frac{u(r)}{r} \right] \frac{\partial I_{\underline{x}'}}{\partial x'} \quad (\text{clpp}) \quad (40)$$

rather than Eq. (38).

In order to understand why this new plane-parallel approximation differs from the usual one, it is useful to distinguish between two types of spherical divergence effects: First, there is the usual divergence of the rays, which may be neglected for cases where the thickness of the layer in question is much smaller than the mean free path. Second, there is the divergence of the velocities, which may be quite important because of the extreme sensitivity of the absorption coefficient to slight shifts in frequency. The terms corresponding to divergence of the velocities have been retained in this new formulation while the terms corresponding to the divergence of the rays have been dropped. It is interesting to note that precisely the same arguments have been used by McCrea and Mitra (1936) for including Doppler shifts in the moving atmosphere equations while dropping aberration effects, even though these are of the same order in v/c .

This idea may be illustrated by some examples giving some further physical insight. Suppose first of all that $u(r)$ is proportional to r , the velocity field for a uniformly expanding atmosphere. The local velocity field as viewed from a local frame of reference moving with the material at any point is isotropic; in fact it appears to be a uniform expansion away from that point. This simple fact is correctly represented by Eq. (40) where the coefficient of $\partial I_{\underline{x}'}/\partial x'$ is independent of μ , while in Eq. (38) it is proportional to μ^2 , implying a strong directional effect. Another case of interest is that of constant outflow, $u(r) = \text{const}$; the term in Eq. (38) now vanishes, which implies the absence of any velocity effects. This is clearly not so, however, since there is a transverse velocity gradient in this case, which, for example, makes it easier for photons to escape the atmosphere along a tangent. These examples show the superiority of Eq. (40) over Eq. (38).

It remains to write the corrected formula in terms of the variable x ,

$$\underline{\ell} \cdot \frac{\partial I_x}{\partial \underline{r}} = \mu \frac{\partial I_x}{\partial r} - (1-\mu^2) \frac{u(r)}{r} \frac{\partial I_x}{\partial x} \quad (\text{cpp}). \quad (41)$$

There would not seem to be any particular advantage in using Eq. (41) over Eq. (40) since a derivative with respect to frequency still appears. However, this derivative can be eliminated by performing a different sort of frequency transformation to a frequency x'' defined by

$$x'' = x + \left(\frac{1}{\mu} - \mu \right) \int \frac{u(r)}{r} dr, \quad (42)$$

where any indefinite integral can be chosen. Then

$$\underline{\ell} \cdot \frac{\partial I_{x''}}{\partial \underline{r}} = \mu \frac{\partial I_{x''}}{\partial r} \quad (\text{ppp}), \quad (43)$$

which is exactly in plane-parallel form, but in terms of the new frequency variable. This will be called the pseudo-plane-parallel formulation (ppp), and may be of particular use for methods, such as those using integral equations, where frequency derivatives cannot be easily handled.

It is interesting to write the (ppp) equation in full for the case of a uniform expansion. Taking

$$\int \frac{u}{r} dr = u'$$

for this case yields

$$\begin{aligned} \mu \frac{\partial I_{x''}}{\partial r} = & k_\ell(r) \phi \left(r, x'' - \frac{u(r)}{\mu} \right) \left[- I_{x''} + S \right] + \\ & + k_c(r) \left[- I_{x''} + B_c \right]. \quad (44) \end{aligned}$$

This would be precisely the usual plane-parallel equation, except that in the argument of the profile function u is divided, rather than multiplied by μ .

c. Infinite Medium With Constant Velocity Gradient

Sobolev (1957) showed that in an infinite plane-parallel medium with a constant velocity gradient the solution for the source function of a two-level atom with no continuum could be reduced to the solution of the integral equation (in a notation which differs from Sobolev's)

$$S(\tau) = (1-\epsilon) \int_{-\infty}^{\infty} K(|\tau-\tau'|) S(\tau') d\tau' + \epsilon B(\tau) \quad . \quad (45)$$

Here ϵ is a constant, the ratio of the collisional de-excitation rate to the total de-excitation rate, and the kernel function K is defined by

$$K(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} dx \int_0^1 \frac{d\mu}{\mu} \phi(x) \phi(x + \gamma\mu\tau) \exp \left\{ - \int_0^{\tau} \phi(x + \gamma\mu z) dz \right\} . \quad (46)$$

The variable τ is an equivalent integrated line optical depth, defined for a medium at rest. The velocity gradient is a constant and is defined by

$$\gamma = \frac{\partial u}{\partial \tau} \quad . \quad (47)$$

The factor A that appears in Sobolev's paper is to be taken as unity here because of the use of integrated line absorption to define the optical depth scale rather than the line center absorption that Sobolev uses. The function $B(\tau)$ is the Planck function at the local electron temperature and at the line frequency.

The translational invariance of the problem is clear, since the kernel function of this integral equation is a function of the difference $|\tau - \tau'|$ alone. Such integral equations have been extensively studied, but usually in the static case. Much is known about the solutions of such equations and how they depend on ϵ and on $B(\tau)$.

A very useful physical picture is obtained by regarding the quantity $S(\tau)$ in this equation as the probability of emission of a single photon, rather than as the average number of such emissions. Then $\epsilon B(\tau)$ gives the probability density of creation of a photon at the point τ , $K(\tau)$ gives the single-step distribution function for the free propagation distance of the photon before absorption, and ϵ is the probability that an absorption will be followed by the destruction of the photon. A very useful concept associated with this probabilistic viewpoint is that of thermalization length, an average distance traveled by the photon between its creation and its destruction (see Rybicki and Hummer, 1969). By use of the concept of thermalization length much can be said about the solution $S(\tau)$ without actually solving the entire problem, and it is instructive to do so in the present simple case of a moving atmosphere.

There is a very important physical distinction between the usual static case of thermalization and the present one, however, which is due to a change in the normalization of the kernel function. Ordinarily the normalization would be

$$\int_{-\infty}^{\infty} K(|\tau|) d\tau = 1 \quad (\text{static case}) \quad (48)$$

since any photon eventually is absorbed somewhere. But in the present case this is to be replaced by

$$\int_{-\infty}^{\infty} K(|\tau|) d\tau = 1 - \beta \quad (49)$$

with $0 < \beta < 1$. This implies that a photon need not be absorbed, but can escape the medium entirely with escape probability β . This is an entirely new phenomenon which might be called intrinsic escape,

since it does not depend on the presence of boundaries, but rather on the rapidly decreasing opacity seen by a photon as it becomes further and further removed into the line wings by virtue of the Doppler shift of the material. This opacity decreases rapidly enough so that the optical thickness in the direction of a velocity gradient is bounded no matter how long the geometrical path length is.

From Eq. (46) it follows that

$$\beta = |\gamma| \int_0^1 \left[1 - \exp\left(-\frac{1}{|\gamma| \mu^2}\right) \right] \mu^2 d\mu \quad (50)$$

For details of this reduction the reader is referred to Sobolev (1957).

In spite of this normalization the equations can again be brought to the form

$$S(\tau) = (1 - \bar{\epsilon}) \int_{-\infty}^{\infty} \bar{K}(|\tau - \tau'|) S(\tau') d\tau' + \bar{\epsilon} \bar{B}(\tau) \quad (51)$$

where the kernel \bar{K} is now normalized,

$$\int_{-\infty}^{\infty} \bar{K}(|\tau|) d\tau = 1 \quad (52)$$

This is accomplished by the definitions

$$\begin{aligned} \bar{K}(\tau) &= \frac{1}{1 - \beta} K(\tau) \\ \bar{\epsilon} &= 1 - (1 - \beta)(1 - \epsilon) = \epsilon + \beta - \epsilon\beta \\ \bar{B}(\tau) &= \frac{\epsilon}{\bar{\epsilon}} B(\tau) \end{aligned} \quad (53)$$

Since Eq. (51) is now in the usual form, the thermalization length may be discussed. First of all, it should be noted that when $\beta \gg \epsilon$ the value of $\bar{\epsilon}$ is insensitive to ϵ , and in fact $\bar{\epsilon} \sim \beta$. This means that the mechanism of loss of photons by

intrinsic escape dominates the loss by collisional de-excitation, and therefore the thermalization process will be determined by β rather than ϵ when $\beta \gg \epsilon$. Secondly, it should be noted that the kernel $\bar{K}(\tau)$ is very much more sharply cut off for large values of τ than in the static case, at least for Doppler profiles. This is because it is precisely those photons which would have traveled a long distance in the static case that now escape and do not appear in the distribution $\bar{K}(\tau)$ at all. The actual cutoff of the kernel can be roughly estimated to be at distances of the order of $\tau \sim 1/\gamma$, since this is roughly the scale over which the profile shifts through its own width, and photons that travel to larger distances will escape completely.

Rybicki and Hummer (1969) have shown that when the distribution of single flights is sharply cut off, the thermalization length Λ is of the coherent type, and is of the order of the width of the kernel times $(\bar{\epsilon})^{-1/2}$, that is, $\Lambda \sim \gamma^{-1}(\bar{\epsilon})^{-1/2}$. When properties vary slowly on the scale of the thermalization length an approximate solution to Eq. (51) may be obtained by removing $S(\tau)$ from under the integral. Then

$$S(\tau) = \bar{B}(\tau) \quad , \quad (54)$$

or, when $\beta \gg \epsilon$,

$$S(\tau) = \frac{\epsilon B(\tau)}{\beta} \quad . \quad (55)$$

The source function is in this case simply determined by the creation rate $\epsilon B(\tau)$ and the escape probability β .

It is important to note that for Eq. (55) to be valid it is not necessary to have very large gradients, $\gamma \gg 1$. For example, a fairly small γ can still lead to a β satisfying $\beta \gg \epsilon$, thus the sole requirement is one concerning thermalization length and the scale of variation of properties L . For small γ it follows that $\beta \sim \gamma/3$ (see Sobolev 1957), and $\beta \gg \epsilon$ implies $\bar{\epsilon} \sim \beta$, so that the thermalization length is $\Lambda \sim \gamma^{-3/2}$. The condition of validity is simply $\Lambda \gg L$, and if L is sufficiently large this can be met for arbitrarily small γ .

The reason that so much space has been given here to the discussion of Eqs. (45) and (51) from

the point of view of thermalization theory is that, within the framework of the simplifying assumptions made, it has been possible to derive solution (55), which is identical with the one found in Sobolev's theory of moving atmospheres. Furthermore (and most importantly) it has been possible to fix conditions of validity of this solution, which relate the magnitude of the velocity gradient to scales of variation of the physical parameters. In particular, it has been pointed out that this solution may be valid for "small" gradients, whereas Sobolev's theory of moving atmospheres has often been regarded only as a "large" gradient theory. These considerations will be useful in the following discussion.

Castor (1970) also discusses the validity of Sobolev's theory by deriving diffusion-like corrections and estimating their magnitude. This is roughly analogous to the above procedure, since the use of the coherent type of thermalization length is essentially a diffusion theory result. However, the criterion Castor obtains is that the typical mean thermal velocities must be small in comparison to typical macroscopic velocities. By estimating the macroscopic velocity as the gradient times a scale length we may put this into the form $L \gg \gamma^{-1}$, which is a less restrictive condition than the one obtained here. This point needs further investigation.

d. Sobolev's Theory of Moving Atmospheres

In general, velocity gradients present great difficulties in the problem of line formation. However, the remarkable theory of moving atmospheres developed by Sobolev (1947, 1957) demonstrates that velocity gradients are actually a simplifying feature when they are of sufficient magnitude. In favorable cases an entire coupled multi-level transfer problem, a formidable problem even in the static case, can be reduced to a set of algebraic equations for the populations at each point in the atmosphere. A brief discussion of Sobolev's theory will now be given to show how the concept of escape probability can be extended to multi-level problems. The continuum opacity k_c will be assumed negligible in the neighborhood of each line.

The basic idea of the theory is as follows: consider a region of the atmosphere in which the properties are more or less homogeneous, and which is sufficiently large in the sense to be discussed below. In the static case the radiative transition rate between any two levels, say i to j , is exactly the same as the rate in the reverse transition, j to i . This is because any transition leading to the emission of a photon will in turn produce the reverse transition when that photon is absorbed somewhere else in the same region. The two transition rates averaged over the region must then be equal, and if the conditions are homogeneous the equality applies also at a single point. The condition on the size of the homogeneous region can be seen to be that every photon emitted in the region must also be absorbed there, to a good approximation.

When there is a velocity gradient a photon can escape the atmosphere entirely, and in this case the downward radiative transition rate will exceed the upward rate. The difference in these two rates is simply $n_i A_i \beta_{ji}$, where β_{ji} is the probability of escape of the photon due to the Doppler shift of the profile.

The escape probability β_{ji} can be easily found by assuming a constant velocity gradient over the region of interest. Suppose a photon is emitted at frequency x and direction ℓ at a certain point. It is convenient to use a frame of reference such that this point is at rest at the origin. The velocity field is then

$$u_\alpha(\underline{r}) = \sum_\beta \frac{\partial u_\alpha}{\partial r_\beta} r_\beta \quad (56)$$

The total optical thickness of the medium from the origin in the direction ℓ at frequency x is

$$\tau_x(\underline{\ell}) = k_{ij} \int_0^\infty \phi(x - \underline{u}(\xi \underline{\ell}) \cdot \underline{\ell}) d\xi \quad (57)$$

Setting $x' = x - \underline{u}(\xi \underline{\ell}) \cdot \underline{\ell} = x - \xi Q(\underline{\ell})$, there results

$$\tau_x(\underline{\ell}) = \frac{k_{ij}}{|Q(\underline{\ell})|} \int_{-\infty}^x \phi(x') dx' \quad (58)$$

The integrated line opacity for the transition i to j is denoted by k_{ij} .

The probability of escape in a frequency range dx about x and in an angular range $d\ell$ about ℓ is

$$\frac{1}{4\pi} d\ell \phi(x) dx \exp \left[-\tau_x(\ell) \right]. \quad (59)$$

The net probability of escape is thus

$$\beta_{ji} = \frac{1}{4\pi} \int d\ell \int_{-\infty}^{\infty} dx \phi(x) \exp \left[-\frac{k_{ij}}{|Q|} \int_{-\infty}^x \phi(x') dx' \right].$$

Changing the variable of integration from x to

$$t = \int_{-\infty}^x \phi(x') dx'$$

yields the final result

$$\beta_{ji} = \frac{1}{4\pi} \int d\ell \frac{|Q(\ell)|}{k_{ij}} \left[1 - \exp \left(-\frac{k_{ij}}{Q(\ell)} \right) \right]. \quad (60)$$

The escape probability is therefore independent of the profile function under the assumption of complete redistribution. The restriction to rectangular profiles in the original work of Sobolev (1947) is unnecessary, as pointed out by Sobolev (1957).

The expression (60) is the generalization of the escape probability formula for arbitrary velocity gradients. Substituting in the appropriate Q for plane-parallel geometry, namely Eq. (30), the expression (50) is again obtained, since $u'(z)/k = \partial u / \partial \tau = \gamma$. Similarly Eq. (31) for spherical geometry

leads to the formula

$$\beta_{ji}(r) = \frac{1}{k_{ij}} \int_0^1 \left| \mu^2 u'(r) + (1-\mu^2) u(r)/r \right| \cdot \left[1 - \exp\left(\frac{-k_{ij}}{\left| \mu^2 u'(r) + (1-\mu^2) u(r)/r \right|} \right) \right] d\mu \quad (61)$$

An equivalent formula was derived by Castor (1970). For the special case of uniform expansion, $u' = u/r = \text{const.}$, and the corresponding formula is

$$\beta_{ji} = \frac{u'}{k_{ij}} \left[1 - \exp\left(-\frac{k_{ij}}{u'} \right) \right] \quad (62)$$

The determination of the escape probability for each transition depends on local parameters in the atmosphere, namely the integrated line opacity and the velocity gradient. The equations of statistical equilibrium can be written in terms of net radiative rates, which are simply related to the escape probabilities, as shown above. It follows that an entire multi-level problem can be reduced to an algebraic set of equations, which can be solved locally at each point in the atmosphere, without regard for conditions at other points. This uncoupling of the various parts of the atmosphere is due to the Doppler shift between material at separated points, which causes them to absorb and emit in quite different parts of the spectrum.

As an example of such a formulation of a multi-level problem, consider the case of an atom in the dilute radiation field ρ_{ic} of a star. The statistical equilibrium equations are

$$n_i \left(\sum_{k=1}^{i-1} A_{ik} \beta_{ki} + B_{ic} \rho_{ic} \right) = \sum_{k=i+1}^{\infty} n_k A_{ki} \beta_{ik} + n_e n^+ C_i(T_e) \quad (63)$$

where all collisional processes except recombination have been neglected. The local physical parameters fix all the constants in this equation, which may then be solved for the n_i . For details see Sobolev (1947).

A more recent and more sophisticated example of the formulation and solution of multi-level problems using Sobolev's method is given by Castor and Van Blerkom (1970), who solve a 30-level He atom in spherical geometry. Collision processes are included and the effects of continuous absorption are taken into account approximately.

An important area for future investigation is the determination of criteria for the validity of Sobolev's theory. The crucial question seems to be, as in the simplified two-level case, how small a scale of variation of the physical parameters is allowable while still maintaining the local homogeneity that is necessary for the use of the escape probability concept. Possible approaches are to extend the two-level thermalization arguments or Castor's diffusion correction terms to the multi-level case.

NUMERICAL METHODS

Most numerical calculations have treated the effects of velocity gradients for two-level atoms in plane-parallel geometries. These calculations have been exploratory in nature, to discover effects rather than to obtain accurate solutions to a physical problem. This seems to be appropriate at the present stage of development of the subject.

Kulander (1964, 1968) assumed an atmosphere which consisted of several layers in which all physical properties were constant. This allowed a semi-analytic approach to be taken, since the solution for the discrete ordinate intensities in each layer was a linear combination of elementary exponential functions. Boundary conditions at the interface between two layers were simply that all intensity components must be continuous. In this way Kulander was able to solve for the source function and the emergent intensities for many different cases. This work showed that the source function was generally discontinuous across the boundary between layers and sometimes large increases in source function near the surface could be obtained in this way. These increases simply

mean that the radiation field near the boundary is quite weak in the line core for material at rest, but as the material moves it can absorb the more intense radiation in the line wings. Examples were shown where emission features could actually result from these increases in source function. In general, line profiles are now much more complex and, of course, asymmetrical.

Kulander (1967) also developed a numerical method of solution using differential equations, which is directly applicable to media with continuous variation of properties. However, it would seem that this method must be unstable for large optical thicknesses for reasons given by Hummer and Rybicki (1967) in their discussion of the fundamental matrix method. The method of slabs, on the other hand, can be used for arbitrarily thick atmospheres, including semi-infinite ones, if not too many slabs are taken. When large numbers of slabs are taken in order to model a continuous distribution of properties then the choice of a stable method of solving the relevant equations becomes important. The number of frequency and angular components that can be treated poses another limitation of the method. Kulander used a single-point angle quadrature, which is roughly equivalent to the Eddington approximation. Numerical problems would probably limit the total number of frequency components to perhaps 100. Up to 13 components were used by Kulander (1967).

It should perhaps be mentioned here that the methods used by Abyankhar (1964a,b; 1965) are closely related to the above slab method, but they have been formulated in terms of coherent scattering so that the details are not directly relevant here.

Hummer and Rybicki (1968) have used a differential equation method based on an extension of the Riccati method (Rybicki and Hummer, 1967), which is applicable to continuous variation of properties. Using this method they show how a uniformly expanding atmosphere can produce a red-shifted emission line. The reason for this shift is simply an optical depth effect, the actual change in the source function due to the motion being irrelevant. The emission and absorption of the material closest to the observer is shifted to the violet and therefore optical depth unity on the violet side of the line occurs much closer to the surface, where the excitation is smaller. This reduces the intensity of the violet emission, and the line appears red-shifted.

The small step size which must be used in the numerical integration to avoid multiscale instabil-

ities limits the use of the Riccati method to atmospheres having optical depths perhaps of a few hundred (Rybicki and Hummer, 1967). There is a way of treating semi-infinite atmospheres if variation of properties is confined to a layer near the surface, again of no more than a few hundred optical depths in thickness. The Riccati method is also limited by the number of discrete ordinates in angle and frequency that can be reasonably handled, no more than perhaps 60 components in one hemisphere.

Another method which no doubt can be employed to advantage in this problem is that of Feautrier (1964) which should avoid the difficulties of the Riccati method as far as the multiscale instability is concerned.

One modification of the method is necessary if the usual form of the equations in the method is not to be changed, namely, the quantities J and F must now be defined by

$$J(\tau, \mu, x) = \frac{1}{2} [I(\tau, \mu, x) + I(\tau, -\mu, -x)]$$

$$F(\tau, \mu, x) = \frac{1}{2} [I(\tau, \mu, x) - I(\tau, -\mu, -x)] \quad . \quad (64)$$

Changing the sign of x , as well as that of μ , takes advantage of the symmetry of the profile function $\phi(\tau, \mu, x)$ under the joint interchanges $\mu \rightarrow -\mu$ and $x \rightarrow -x$.

The methods employing integral equations that have been used in the static case can be modified to include velocity gradients, as Kalkofen (1970) has shown. One advantage of this method is that a much larger number of angle and frequency points can be taken than in the above methods, since the computation time increases linearly rather than quadratically or cubically with the number of components chosen. This advantage may be very important for cases of large velocity gradients.

Calculations of line formation have been performed by Mathis (1968) and Magnan (1968) in spherical geometries. Mathis used an iteration scheme to solve for the source function and emergent intensities for a uniformly expanding, spherically symmetric atmosphere, and he found a red-shifted emission line of the same type as the plane-parallel calculation of Hummer and Rybicki (1968). Since his method appears to be equivalent to Λ - iteration

in the plane-parallel case, its rate of convergence will be slow when the mean number of scatterings is large.

Magnan (1968) used a Monte Carlo method for a spherically symmetric atmosphere with a constant velocity of outflow. This macroscopic velocity was the same order of magnitude as the thermal velocity. He treated several cases, which included effects of dipole scattering and of various boundary conditions. A very interesting and important comparison is that between the calculations made with and without the assumption of complete redistribution. The difference between the emergent profiles for these two cases is that the complete redistribution profile seems more smoothed-out than the exact calculation, but the differences are quite small. It would be unwise, however, to generalize about the adequacy of the assumption of complete redistribution from this calculation, since the physical case is not a severe one, the macroscopic velocity not being very large in comparison to thermal velocities.

An outstanding advantage of the Monte Carlo method is the ease of formulation in complex situations, which derives from the functional matching of a probabilistic numerical method to a basically probabilistic physical process. However, as is well known, there is a heavy penalty in that the accuracy of the calculation grows as the square root of the computation time so that the method is only useful when the time to calculate one "event" is very small. This requires that the mean number of scatterings be small. In a calculation such as Magnan's the advantages are clear, since the case is not severe and the aims are limited. The range of problems for which the method is useful has yet to be determined.

It might be pointed out that the effects found by Magnan are entirely due to the transverse velocity gradient, which no plane-parallel calculation, as presently formulated, accounts for properly. Therefore, there is no plane-parallel calculation that can be compared with this one in order to determine the effect of spherical geometry. By use of the improved plane-parallel approximation given in this paper such a comparison would be possible.

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DISCUSSION

The manuscript of Rybicki arrived very late and consequently it was not possible to prepare a condensed version of the lively discussion following this paper. One aspect of the problem of methods was discussed by C. Magnan who has submitted the following summary of his remarks.