QF-3 ENDOMORPHISM RINGS OF Σ-QUASI-PROJECTIVE MODULES†

by JOSÉ L. GÓMEZ PARDO and NIEVES RODRÍGUEZ GONZÁLEZ

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A ring R is called *left* QF-3 if it has a minimal faithful left R-module. The endomorphism ring of such a module has been recently studied in [**7**], where conditions are given for it to be a left PF ring or a QF ring. The object of the present paper is to study, more generally, when the endomorphism ring of a Σ-quasi-projective module over any ring R is left QF-3. Necessary and sufficient conditions for this to happen are given in Theorem 2. An useful concept in this investigation is that of a QF-3 module which has been introduced in [**11**]. If M is a finitely generated quasi-projective module and $\sigma[M]$ denotes the category of all modules isomorphic to submodules of modules generated by M, then we show that $\mathrm{End}({}_RM)$ is a left QF-3 ring if and only if the quotient module of M modulo its torsion submodule (in the torsion theory of $\sigma[M]$ canonically defined by M) is a QF-3 module (Corollary 4). Finally, we apply these results to the study of the endomorphism ring of a minimal faithful R-module over a left QF-3 ring, extending some of the results of [**7**].

Throughout this paper R denotes an associative ring with identity, and R-mod denotes the category of left R-modules. If M is a module, then we will say that a module N is *M-generated* (*M-cogenerated*) if it is a quotient (resp. a submodule) of a direct sum $M^{(I)}$ (resp. direct product M^I) of copies of M. If N is M-cogenerated, then we will also say that N is *M-torsionless* and that M is a *N-cogenerator*. The full subcategory of R-mod consisting of the submodules of M-generated modules will be denoted by $\sigma[M]$; it is a locally finitely generated Grothendieck category [**11**]. We recall that a module N is *M-projective* (*M-injective*) if, for every quotient module (resp. submodule) X of M, the homomorphism $\mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, X)$ (resp. $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(X, N)$) is an epimorphism and, in particular, M is quasi-projective when it is M-projective. M is a projective object of $\sigma[M]$ precisely when it is Σ-quasi-projective, that is, $M^{(I)}$ is quasi-projective for each set I. The largest M-generated submodule of a module X will be denoted by X_M. $E(N)$ will stand for an injective envelope of N in R-mod; if N belongs to $\sigma[M]$, then its injective envelope in this category is precisely $E(N)_M$. A module is called *finitely cogenerated* (*FC* for short) if it has a finitely generated essential socle. When ${}_RR$ is injective and finitely cogenerated, R is said to be a *left* PF ring. The endomorphism ring of a module M will be denoted by $S = \mathrm{End}({}_RM)$ and we will use the convention of writing endomorphisms opposite scalars. We refer the reader to [**2**] and [**6**] for all the ring-theoretic notions used in the text.

In [**11**], a module M is called a QF-3 *module* if there exists a minimal M-cogenerator

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in $\sigma[M]$, i.e., a M-cogenerator in $\sigma[M]$ which is a direct summand of every M-cogenerator in $\sigma[M]$. These modules can be characterised in a similar way to left QF-3 rings.

PROPOSITION 1. *Let M be a finitely generated module. The following conditions are equivalent.*

(i) *M is a* QF-3 *module.*

(ii) *There exist (pairwise non-isomorphic) simple submodules $S_1, \dots, S_n$ of M such that $E\left(\bigoplus_1^n S_i\right)_M$ is a M-torsionless M-cogenerator.*

(iii) *There exists a finitely cogenerated M-injective submodule Q of M such that M is Q-torsionless.*

Proof. See [**3**, Theorem 1] or [**11**, Proposition 2.2].

Observe that the hypothesis of M being finitely generated cannot be deleted from Proposition 1. For instance, if M is an infinite direct sum of pairwise non-isomorphic simple modules, then M is clearly QF-3, but there is no finitely cogenerated M-cogenerator. However, conditions (ii) and (iii) imply (i) even if M is not assumed to be finitely generated.

Let M be a Σ-quasi-projective module and T the smallest torsion class of $\sigma[M]$ containing all the modules of the form X/X_M, with X in $\sigma[M]$, which in this case consists precisely of the modules N such that $\mathrm{Hom}_R(M, N) = 0$ [**5**]. A module N of $\sigma[M]$ is called M-faithful when it is T-torsion-free, that is, when $\mathrm{Hom}_R(M, X) \neq 0$ for every non-zero submodule X of N. If M is M-faithful we will say that M is self-faithful. Then we have the following result.

THEOREM 2. *Let M be a self-faithful Σ-quasi-projective module and $S = \mathrm{End}(_RM)$. Then S is a left* QF-3 *ring if and only if there exists a finitely cogenerated M-injective M-cogenerator submodule of M.*

Proof. By Proposition 1, R is left QF-3 if and only if it has a finitely cogenerated injective faithful left ideal I. Assuming that S is left QF-3, we claim that $MI = \sum_{f \in I} \mathrm{Im}\, f$ is a finitely cogenerated M-injective M-cogenerator submodule of M. To see this, we recall from [**1**, Corollary 4.10] that, since M is a quasi-projective module, $Y \to MY$ and $X \to \mathrm{Hom}_R(M, X)$ define order-preserving inverse bijections between the sets of finitely generated left ideals Y of S and finitely M-generated submodules X of M (i.e. submodules of M which are quotients of finite direct sums of copies of M). Therefore we have, since I is finitely generated, that $I = \mathrm{Hom}_R(M, MI)$. On the other hand MI, being a submodule of M, is M-faithful and thus it follows from [**5**, Theorem 2.1] that MI is a M-injective module. To show that MI cogenerates M, consider the R-homomorphism $q: M \to (MI)^I$ defined by $q(x) = (xu)_{u \in I}$ (where $xu = u(x)$). If $x \in \mathrm{Ker}\, q$, then $xu = 0$ for every u of I and hence $(Rx)u = 0$ for each $u \in I$. Now, since M is self-faithful, if $x \neq 0$ there exists

$0 \neq f: M \to Rx$ and hence, composing with the inclusion $Rx \to M$, we get $0 \neq g \in S$ such that $\operatorname{Im} g \subseteq Rx$. Thus we have that $gu = 0$ for each $u \in I$ and so I is not a faithful left ideal of S; contradiction. Therefore q is a monomorphism. Finally, we show that MI is a finitely cogenerated R-module. Let $I_1, \ldots, I_n$ be minimal left ideals of S such that $\operatorname{Soc}(I) = \bigoplus_1^n I_j$. Then, by the bijective correspondence above mentioned, each MI_j is minimal among the finitely M-generated non-zero submodules of M. Since M is self-faithful, each non-zero submodule of M contains a non-zero finitely M-generated submodule and this implies that each MI_j is actually a simple R-module. To see that MI is finitely cogenerated it remains to be shown that $\bigoplus_1^n MI_j$ is an essential submodule of MI. If Z is a non-zero submodule of MI, then, since M is self-faithful, we have that $0 \neq \operatorname{Hom}_R(M, Z) \subseteq \operatorname{Hom}_R(M, MI) = I$ and hence there exists a minimal left ideal K of S such that $K \subseteq \operatorname{Hom}_R(M, Z)$. As before, we see that MK is a simple R-module contained in $\bigoplus_1^n MI_j$ and $MK \subseteq M \operatorname{Hom}_R(M, Z) = Z_M \subseteq Z$.

Conversely, assume that Q is a finitely cogenerated M-injective M-cogenerator submodule of M. Since M is Σ-quasi-projective and Q is M-faithful, it follows from [**5**, Corollary 2.2] that $\operatorname{Hom}_R(M, Q)$ is an injective left ideal of S. Moreover, since M is cogenerated by Q, it is clear that $\operatorname{Hom}_R(M, Q)$ is a faithful left ideal. Thus to complete the proof of the theorem it suffices to show that $\operatorname{Hom}_R(M, Q)$ is a finitely cogenerated S-module. Using as before the bijective correspondence between finitely M-generated submodules of M and finitely generated left ideals of S, we see that if $\operatorname{Soc}(Q) = \bigoplus_1^n Q_i$, with each Q_i simple, then $\operatorname{Soc}(\operatorname{Hom}_R(M, Q)) = \bigoplus_1^n \operatorname{Hom}_R(M, Q_i)$, where each $\operatorname{Hom}_R(M, Q_i)$ is a minimal left ideal of S. Also, if J is a finitely generated left ideal of S contained in $\operatorname{Hom}_R(M, Q)$, then MJ contains at least one of the Q_i, say Q_k, and hence J contains the minimal left ideal $\operatorname{Hom}_R(M, Q_k)$, so that $\operatorname{Soc}(\operatorname{Hom}_R(M, Q))$ is essential in $\operatorname{Hom}_R(M, Q)$ and the proof is complete.

Note that the necessity in Theorem 2 is true under the more general assumption that M is quasi-projective instead of Σ-quasi-projective. From the proof it also follows that, in the hypotheses of Theorem 2, S has an injective faithful left ideal if and only if it has an M-injective M-cogenerator submodule. A similar result holds if we assume that M is a generator of $\sigma[M]$ (in this case the Σ-quasi-projectivity of ${}_RM$ is not needed because M_S is flat and so $\operatorname{Hom}_R(M, Q)$ is injective). A weaker result is [**10**, Proposition 2.5] where M is supposed to be a generator of R-mod and only the sufficiency is given.

Corollary 3. *Let* $M = \bigoplus_I S_i$ *be a semisimple module, with each* S_i *simple. Then* $\operatorname{End}({}_RM)$ *is left* QF-3 *if and only if there is only a finite number of pairwise non-isomorphic modules among the* S_i.

Corollary 3 shows that a self-faithful Σ-quasi-projective module M may be a QF-3 module without $\mathrm{End}({}_RM)$ being a left QF-3 ring. Nevertheless, we have the following result.

COROLLARY 4. *Let M be a self-faithful Σ-quasi-projective module and $S = \mathrm{End}({}_RM)$. If S is left* QF-3, *then M is a* QF-3 *module. If M is, furthermore, finitely generated, then the converse holds.*

Proof. It is a straightforward consequence of Proposition 1 and Theorem 2.

We also remark that if M is not self-faithful, then S may be left QF-3 without M being a QF-3 module, even in the case when M is also assumed to be projective and finitely generated. For instance, if $R = \begin{pmatrix} \mathbf{Z} & \mathbf{Q} \\ \mathbf{O} & \mathbf{Q} \end{pmatrix}$ and $e = e_{22}$, then $\mathrm{End}({}_RRe)$ is QF-3 but Re is not a QF-3 module. Nevertheless, if M is Σ-quasi-projective and $t(M)$ denotes the torsion submodule of M with respect to the torsion class T of $\sigma[M]$, then $\bar{M} = M/t(M)$ is easily seen to be a self-faithful Σ-quasi-projective module and $\mathrm{End}({}_R\bar{M}) \simeq S$, so that Corollary 4 gives an answer to the problem of characterising the finitely generated quasi-projective modules with left QF-3 endomorphism ring (and Theorem 2 does the same for Σ-quasi-projective modules in general).

We recall that a ring R is said to be *left-Kasch* (or a *left S-ring*) when each simple left R-module is isomorphic to a minimal left ideal. In [**5**, Theorem 3.1] it is shown that if M is a self-faithful Σ-quasi-projective module, then S is left Kasch if and only if M is a finitely generated RZ-module (RZ-module means that M cogenerates each of its simple quotients). From this we get the following result.

COROLLARY 5. *Let M be a self-faithful quasi-projective module with endomorphism ring S. Then S is left* PF *if and only if M is a finitely generated* QF-3 *RZ-module.*

Proof. If S is left PF, then M is finitely generated by [**5**, Proposition 3.4]. Thus the result follows from the above remark, bearing in mind Corollary 4 and the fact that S is left PF if and only if it is left QF-3 and left Kasch [**9**, Theorem 2].

Let P be a projective module and T its trace ideal on R. We recall that P is *distinguished* if, for $x \in P$, $Tx = 0$ implies $x = 0$ [**7**]. This is equivalent to P being self-faithful [**5**]. Thus we get the following partial improvement of [**7**, Proposition 6] and [**7**, Corollary 7].

COROLLARY 6. *Let R be a left* QF-3 *ring with minimal faithful left ideal Re (e an idempotent). If Re is distinguished, then eRe is a left* PF *ring.*

Proof. Clearly, Re is a QF-3 module. Moreover, Re decomposes as a direct sum $\bigoplus_1^n Re_i$, where $Re_i \simeq E(S_i)$, with the $S_1, \ldots, S_n$, pairwise non-isomorphic simple modules. Since the Re_i are indecomposable injective modules, they have local endomorphism rings e_iRe_i and, as is well known, this implies that, if J denotes the radical of R, Re_i/Je_i is a simple module [**6**, 11.4.1.]. Furthermore, it follows from [**6**, 12.5.1] that all the Re_i/Je_i,

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$i = 1, \ldots, n$, are pairwise nonisomorphic and these are all the simple quotients of Re. On the other hand, since Re is distinguished, each S_i is isomorphic to one of the Re_i/Je_i and hence Re is a RZ-module. Thus the result follows from Corollary 5.

COROLLARY 7. *Let R be a ring with a distinguished injective faithful left ideal Re. Then R is left* QF-3 *if and only if eRe is left* QF-3.

Proof. Assume that R is left QF-3. Then there exists a minimal faithful left R-module X which is a direct summand of Re. X is finitely cogenerated and hence Re is a QF-3 module by Proposition 1. Then eRe is a QF-3 ring by Corollary 4. Conversely, if eRe is a left QF-3 ring, then we get from Corollary 4 that Re is a QF-3 module and hence it follows from Proposition 1 that Re has a finitely cogenerated Re-injective submodule Y which cogenerates Re. Since Re is injective, so is Y, and since Re is faithful by hypothesis, Y is also faithful. Thus R is left QF-3 by Proposition 1.

Note that, unlike what happens in Corollary 6, from the hypotheses of Corollary 7 we cannot conclude that eRe is a left *PF* ring. For instance, if V is an infinite dimensional vector space over a field k, then $R = \mathrm{End}({}_kV)$ is left QF-3 by Corollary 3 and R is left self-injective but not left PF.

It is an immediate consequence of Theorem 2 that the endomorphism ring of a distinguished faithful projective module P over a left QF-3 ring is left QF-3. However, if P is not assumed to be faithful this is no longer true. For instance, if A is a (non-trivial) simple left noetherian domain with an injective simple A-module S and $C = \mathrm{End}({}_AS)$, then $R = \begin{pmatrix} A & S \\ 0 & C \end{pmatrix}$ is left QF-3 [**8**, p. 60] but Re_{11} is a distinguished finitely generated projective R-module whose endomorphism ring is isomorphic to A and hence is not left QF-3. The following corollary shows that this cannot happen if R is a *QF* ring (i.e. a (left) self-injective (left) artinian ring) and provides more examples of left QF-3 endomorphism rings.

COROLLARY 8. *Let R be a left artinian ring and M an injective self-faithful Σ-quasi-projective module. Then* $\mathrm{End}({}_RM)$ *is a left* QF-3 *ring. In particular, if R is a* QF *ring and P a distinguished projective R-module, then* $\mathrm{End}({}_RP)$ *is a left* QF-3 *ring.*

Proof. By a well known result of Faith and Walker [**4**, Corollary 1.5], M is isomorphic to a direct sum $\bigoplus_I E_i$ of injective envelopes of simple R-modules. Since the number of pairwise non-isomorphic simple R-modules over a left artinian ring is finite, there is a finite set $E_1, \ldots, E_n$ of representatives of the isomorphism classes of the E_i. Let $X = \bigoplus_1^n E_i$. Then X is a finitely cogenerated injective submodule of M which obviously cogenerates M. Thus $\mathrm{End}({}_RM)$ is left QF-3 by Theorem 2.

The last assertion follows from the first, bearing in mind that over a QF ring every projective module is injective.

REFERENCES

1. T. Albu and C. Năstăsescu, *Relative finiteness in module theory* (Marcel Dekker, 1984).

2. F. W. Anderson and K. R. Fuller, *Rings and categories of modules* (Springer-Verlag, 1974).

3. R. R. Colby and E. A. Rutter, Jr., QF-3 rings with zero singular ideal, *Pacific J. Math.* **28** (1969), 303–308.

4. C. Faith and E. A. Walker, Direct sum representations of injective modules, *J. Algebra* **5** (1967), 203–221.

5. J. L. García Hernández and J. L. Gómez Pardo, Self-injective and PF endomorphism rings, *Israel J. Math.*, **58** (1987), 324–350.

6. F. Kasch, *Modules and rings* (Academic Press, 1982).

7. S. S. Page, FPF endomorphism rings with applications to QF-3 rings, *Comm. Algebra* **14** (1986), 423–435.

8. C. M. Ringel and H. Tachikawa, QF-3 rings, *J. Reine Angew. Math.* **272** (1975), 49–72.

9. E. A. Rutter, Jr., PF and QF-3 rings, *Arch. Math.* **19** (1968), 608–611.

10. H. Tachikawa, On left QF-3 rings, *Pacific J. Math.* **32** (1970), 255–268.

11. R. Wisbauer, Decomposition properties in module categories *Acta Univ. Carolinae Math. Phys.* **26** (1985), 57–68.

Departamento de Algebra
Facultad de Ciencias Químicas y Matemáticas
Universidad de Murcia
30001 Murcia
Spain