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UNIVERSAL PETTIS INTEGRABILITY

KEVIN T. ANDREWS

Since the invention of the Pettis integral over forty years ago [**11**], the problem of recognizing the Pettis integrability of a function against an individual measure has been much studied [**5**, **6**, **7**, **8**, **9**, **20**]. More recently, Riddle-Saab-Uhl [**14**] and Riddle-Saab [**13**] have considered the problem of when a function is integrable against every Radon measure on a fixed compact Hausdorff space. These papers give various sufficient conditions on a function that ensure this universal Pettis integrability. In this paper, we see how far these various conditions go toward characterizing universal Pettis integrability. We base our work on a w^*-analogue of the core of a vector-valued function [**8**].

We also give some sufficient conditions that ensure that a Banach space has the so-called universal Pettis integral property (UPIP) and consider some particular examples of spaces with this property. It is interesting that in these examples some of the special set theoretic axioms that play an important role in the study of the stronger Pettis integral property [**6**, **7**] make an appearance.

It is now time to fix some terminology and notation. Throughout this paper, μ will be a Radon probability measure on the σ-algebra Σ of Borel subsets of a compact Hausdorff space Ω. A subset of Ω is said to be *μ-measurable* if it is in the completion of the measure space (Ω, Σ, μ). The letters X, Y, Z will denote real Banach spaces with duals X^*, Y^*, Z^* respectively. A function $f:\Omega \to X^*$ is said to be *μ-w^*-measurable* (respectively, *μ-scalarly measurable*) if for each x in X (respectively each x^{**} in X^{**}) and each $\epsilon > 0$ there is a compact set $E \subset \Omega$ such that $\mu(\Omega E) < \epsilon$ and $f(\cdot)x|_E$ (respectively, $x^{**}f(\cdot)|_E$) is continuous. The function f is *universally scalarly measurable* if it is μ-scalarly measurable for each Radon probability measure μ on Ω.

A subset W of X^* is called a *weak Baire set* if it is a member of the σ-algebra generated by all sets of the form $g^{-1}(B)$ where $g:X^* \to R$ is any continuous (for the weak topology of X^*) function and B is a Borel subset of R. Edgar has shown [**5**] that a function $f:\Omega \to X^*$ is μ-scalarly measurable if and only if, for any weak Baire subset W of X^*, the set $f^{-1}(W)$ is μ-measurable.

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A function $f:\Omega \rightarrow X^*$ is said to be *universally Lusin measurable* if for every Radon probability measure μ on Ω, every $\epsilon > 0$, and every compact subset K_1 of Ω there is a compact subset K_2 of K_1 such that $\mu(K_1 \backslash K_2) < \epsilon$ and $f|_{K_2}:K_2 \rightarrow X^*$ is w^*-continuous. We note that if f is universally scalarly measurable and has relatively w^*-metrizable range, then f is universally Lusin measurable. Conversely, if f is universally Lusin measurable, then f is *universally Borel measurable* [**20**, p. 26] i.e., for each Radon probability measure μ on Ω and w^*-Borel subset B of X^*, the set $f^{-1}(B)$ is μ-measurable.

Suppose that $x^{**}f(\cdot) \in L_1(\mu)$ for all x^{**} in X^{**} and some fixed Radon probability measure μ. Then the *w^*-integral of f over a set E* in Σ is that element of X^* denoted by $w^* - \int_E f d\mu$ and defined by

$$\left(w^* - \int_E f d\mu\right)(x) = \int_E f(\cdot)x d\mu.$$

The *Dunford integral of f over E* is that element of X^{***} denoted by $D - \int_E f d\mu$ and defined by

$$\left(D - \int_E f d\mu\right)(x^{**}) = \int_E x^{**} f d\mu.$$

We note that

$$P\left(D - \int_E f d\mu\right) = w^* - \int_E f d\mu$$

where $P:X^{***} \rightarrow X^*$ is the canonical projection. If

$$J\left(w^* - \int_E f d\mu\right) = D - \int_E f d\mu$$

where $J:X^* \rightarrow X^{***}$ is the canonical injection, then f is said to be *μ-Pettis integrable* over E and we write $P - \int_E f d\mu$ for the common value of the integrals. The function f is said to be *μ-Pettis integrable* if it is μ-Pettis integrable over every set E in Σ. It is *universally Pettis integrable* if it is μ-Pettis integrable for each Radon probability measure μ. A Banach space is said to have the *universal Pettis integral property* (UPIP) if every bounded universally scalarly measurable function taking values in the space is universally Pettis integrable.

A subset K of a Banach space X is called a *weak Radon-Nikodym set* if for every probability measure space (Ω, Σ, μ) and every vector measure $G:\Sigma \rightarrow X$ such that $G(E) \in \mu(E)K$ for every E in Σ, there exists a Pettis integrable function $g:\Omega \rightarrow K$ such that

$$G(E) = P - \int_E g d\mu \quad \text{for every } E \text{ in } \Sigma.$$

A subset K of a Banach space X is called *weakly precompact* if every sequence in K has a weak Cauchy subsequence. Two functions $f, g:\Omega \rightarrow X^*$ are said to be *μ-w^*-equivalent* (respectively *μ-scalarly equivalent*) if $f(\cdot)x =$

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$g(\cdot)x$ μ a.e. for all x in X (respectively $x^{**}f(\cdot) = x^{**}g(\cdot)$ a.e. for all x^{**} in X^{**}). The *oscillation* of a real-valued function h over a subset A of its domain is defined by

$$\text{osc}(h, A) = \sup\{ |h(s) - h(t)|, s, t \in \text{A}\}.$$

Let (Ω, Σ, μ) be a measure space, let E be a measurable set and let $f:\Omega \to X^*$ be a bounded function. Following [**8**], we defined the *w^*-core of f over E*, denoted by $\text{cor}_f^*(E)$, to be that subset of X^* given by

$$\text{cor}_f^* (E) = \bigcap_{\mu A = 0} w^*\text{-clco}\, f(E \setminus A).$$

We summarize the basic (and essentially known) properties of these subsets of X^* in the following proposition.

PROPOSITION 1. *Let f, $g:\Omega \to X^*$ be w^*-measurable functions.*

(i) *If $\mu E > 0$, then $\text{cor}_f^*(E)$ is a nonempty w^*-compact convex subset of X^*.*

(ii) *For each E in Σ,*

$$\text{cor}_f^*(E) = w^*\text{-clco}\left\{\frac{1}{\mu B}\left(w^* - \int_B f d\mu\right): B \subset E, B \in \Sigma, \mu B > 0\right\}.$$

(iii) *Let $E \in \Sigma$. Then $\text{cor}_f^*(B) = \text{cor}_g^*(B)$ for all $B \subset E$, $B \in \Sigma \Leftrightarrow$ for each x in X, $f(\cdot)x = g(\cdot)x$ μ-a.e. on E.*

(iv) *Define an operator $T:X \to L_\infty(\mu)$ by $Tx(\cdot) = f(\cdot)x$. If $0 \in \text{cor}_f^*(\Omega)$, then T^* (positive part of $Bl(L_\infty(\mu)^*)$) $= \text{cor}_f^*(\Omega)$. Consequently, $\{f(\cdot)x:x \in X, \|x\| \leqq 1\}$ is weakly precompact in $L_\infty(\mu) \Leftrightarrow \text{cor}_f^*(\Omega)$ is a weak RN set.*

(v) *If μ is a Radon probability measure on a compact Hausdorff space Ω and the function $f:\Omega \to X^*$ is scalarly measurable, then $\text{cor}_f^*(\Omega)$ is a w^*-separable subset of X^*.*

(vi) *If $P:X \to X$ is a bounded linear operator such that $P^*|_{\text{cor}_f^*(\Omega)}$ is the identity, then f and P^*f are w^*-equivalent.*

Proof. (i) is obvious. (ii) and (iii) follow from the Hahn-Banach theorem, as in Theorems 2.2 and 2.6 of [**8**]. (iv) follows from (iii) and Theorem 3 of [**14**]. (v) is a consequence of (ii) and Stegall's observation [7] that, if μ is a Radon measure, the set

$$\left\{D - \int_E f d\mu : E \text{ in } \Sigma\right\}$$

is relatively norm compact in X^{***} and hence the set

$$\left\{w^* - \int_E f d\mu : E \text{ in } \Sigma\right\}$$

is relatively norm compact in X^*. (vi) follows from (iii).

In [**14**], Riddle, Saab, and Uhl show that if X is a separable Banach space and $f:\Omega \to X^*$ a bounded μ-w^*-measurable function such that $\{f(\cdot)x:\|x\| \leqq 1\}$ is almost weakly precompact in $L_\infty(\mu)$, then f is μ-Pettis integrable. In a later paper [**13**], Riddle and Saab offer a universal converse to this result under the additional condition that the function f is universally Lusin measurable. Here is another universal converse with an extra condition on the Banach space X rather than the function f.

LEMMA 2. *Suppose that every w^*-compact separable subset of X^* is w^*-metrizable. Let $f:\Omega \to X^*$ be a bounded universally scalarly measurable function. Then for each Radon measure μ and each $\epsilon > 0$, there exists a compact set E in Σ with $\mu(\Omega\backslash E) < \epsilon$ such that $\mathrm{cor}_f^*(E)$ is a w^*-metrizable weak RN set.*

Proof. Define an operator $T:X \to L_\infty(\mu)$ by $Tx = f(\cdot)x$. Since μ is a Radon measure, the set $\mathrm{cor}_f^*(\Omega)$ is w^*-separable and thus so is $\mathrm{cor}_f^*(\Omega) - \mathrm{cor}_f^*(\Omega)$; therefore, $\mathrm{cor}_f^*(\Omega) - \mathrm{cor}_f^*(\Omega)$ is w^*-metrizable by our hypothesis. Now without loss of generality we have that $0 \in \mathrm{cor}_f^*(\Omega)$. Hence the set $T^*(Bl(L_\infty(\mu)^*))$ is contained in $\mathrm{cor}_f^*(\Omega) - \mathrm{cor}_f^*(\Omega)$ as a consequence of Proposition 1 (iv). It follows that T has norm separable range. Choose x_n in X with $\|x_n\| \leqq 1$ such that Tx_n is norm dense in $T(BlX)$. Let $\epsilon > 0$. Then there exists a compact set E in Σ with $\mu(\Omega\backslash E) < \epsilon$ and such that the restriction $f(\cdot)x_n|_E$ is continuous for all n. Let

$$A = \{f(\cdot)x_n|_E:n \in \mathbf{N}\}.$$

Since f is a universally scalarly measurable function, the set A is relatively compact (for the topology of pointwise convergence on E) in $M_r(E)$. By a theorem of Bourgain, Fremlin, and Talagrand [**1**, p. 854], the set A is weakly precompact in $C(E)$ and hence in $L_\infty(\mu)$. Since A is norm dense in

$$\{f(\cdot)x|_E:\|x\| \leqq 1\},$$

this latter set is weakly precompact in $L_\infty(\mu)$. Hence $T^*(Bl(L_\infty(E, \mu)^*))$ is a weak *RN* set and thus so is

$$\mathrm{cor}_f^*(E) \subseteq T^*(BlL_\infty(E, \mu)^*) \quad [\mathbf{16}].$$

This completes the proof.

It is instructive at this point to consider Phillips' example [**12**] of a bounded universally scalar measurable function

$$f:[0, 1] \to X^* = l_\infty[0, 1]$$

that is not Lebesgue Pettis integrable. In this case, $\mathrm{cor}_f^*[0, 1] = 0$ and the functions $x^{**}f(\cdot)$ are each constant except on a countable set. The difficulty here is that the w^*-core of f is too small to capture the behavior of the function. On the other hand, if $f:\Omega \to X^*$ is universally Lusin measurable, then for each Radon measure μ on Ω there is a μ-null set N such that

$$w^*\text{-clco}\, f(\Omega \backslash N) = \text{cor}_f^*(\Omega)$$

[**13**, Lemma 9] and as a consequence, f is universally Pettis integrable. Thus the relative size of $\text{cor}_f^*(\Omega)$ seems to play a crucial role in the integrability of f, a suspicion that is confirmed by the following theorem.

THEOREM 3. *Let μ be a Radon probability measure and let $f:\Omega \to X^*$ be a bounded scalarly measurable function. Suppose that $\text{cor}_f^*(\Omega)$ is a w^*-metrizable weak RN set. Then the following are equivalent:*

(i) *f is μ-Pettis integrable.*

(ii) *f is μ-scalarly equivalent to a function $g:\Omega \to X^*$ such that the set w^*-clco $g(\Omega)$ is w^*-metrizable.*

(iii) *For each x^{**} in X^{**}, there exists a sequence x_n in X and a μ-null set N with $\|x_n\| \leqq \|x^{**}\|$ for all n and*

$$\lim_n x^*(x_n) = x^{**}(x^*)$$

for all $x^* \in f(\Omega \backslash N) \cup \text{cor}_f^*(\Omega)$.

(iv) *For each x^{**} in X^{**}, each $\epsilon > 0$, and each μ-measurable set E of positive measure, there exists a μ-measurable set $B \subset E$ of positive measure such that*

$$\text{osc}(x^{**}, f(B) \cup \text{cor}_f^*(B)) < \epsilon.$$

(v) *For all μ-measurable sets E of positive measure*

$$\frac{1}{\mu E}\left(D - \int_E f d\mu\right) \in X^{**}\text{-closure of } \text{cor}_f^*(E).$$

(vi) *For each x^{**} in X^{**} and each μ-measurable set E of positive measure,*

$$x^{**}f(\cdot) \leqq \sup_{x^* \in \text{cor}_f^*(E)} x^{**}(x^*)\ \mu \text{ a.e. } on\ E.$$

(vii) *If E is a μ-measurable set and W is any weak Baire set containing $\text{cor}_f^*(E)$, then*

$$\mu(E \cap f^{-1}(W)) = \mu E.$$

Proof. (i) $\Rightarrow$ (ii). Define a measure $G:\Sigma \to X^*$ by

$$G(E) = w^* - \int_E f d\mu \quad \text{for all } E \text{ in } \Sigma.$$

Then

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$$\left\{\frac{G(E)}{\mu E}:E \text{ in } \Sigma,\ \mu E > 0\right\} \subset \text{cor}_f^*(\Omega).$$

Since this latter set is a weak RN set, the measure G has a Pettis integrable derivative g whose values lie in $\text{cor}_f^*(\Omega)$. Hence w^*-clco $g(\Omega)$ is a w^*-metrizable set and

$$P - \int_E g d\mu = G(E) = w^* - \int_E f d\mu = P - \int_E f d\mu$$

since f is Pettis integrable. It follows that

$$x^{**}f(\cdot) = x^{**}g(\cdot)\ \mu\text{-a.e. } x^{**}$$

so f and g are scalarly equivalent.

(ii) $\Rightarrow$ (iii). Suppose that f is scalarly equivalent to a function $g:\Omega \to X^*$ such that the set w^*-clco $g(\Omega)$ is w^*-metrizable. Then, except on a set of measure zero, g must take its values in $\text{cor}_f^*(\Omega)$. For if Z is a norm separable subspace of X such that the Z and X topologies agree on w^*-clco $g(\Omega)$, then there exists a null set M and partitions π_n of Ω into a finite number of measurable sets such that

$$g(\omega)z = \lim_n g_n(\omega)z$$

for all ω in $\Omega \backslash M$ and z in Z. Here

$$g_n = \sum_{E \in \pi_n} \frac{w^* - \int_E g d\mu}{\mu E} \chi_E.$$

It follows that

$$g(\omega) = w^* - \lim_n g_n(\omega)$$

and hence

$$g(\omega) \in \text{cor}_g^*(\Omega) = \text{cor}_f^*(\Omega) \quad \text{for all } \omega \text{ in } \Omega \backslash M.$$

Now define an operator $T:X \to L_\infty(\mu)$ by

$$Tx = g(\cdot)x.$$

Without loss of generality, we may suppose that

$$0 \in \text{cor}_g^*(\Omega).$$

Since $\text{cor}_g^*(\Omega)$ is a weak RN set [**16**], so is

$$T^*(BlL_\infty(\mu)^*)) \subseteq \text{cor}_g^*(\Omega) - \text{cor}_g^*(\Omega).$$

Consequently, T factors through a space $Y \not\supset l_1$ [**14**]. Since T has norm separable range, we may assume that Y is separable. Let $S:X \to Y$ and $J:Y \to L_\infty(\mu)$ be the factorization operators i.e., $T = JS$. We may take

$||J|| \leqq 1$. Then since

$$T^*(BlL_\infty(\mu)^*) \supset \text{cor}_g^*(\Omega)$$

we have that

$$S^*(BlY^*) \supset \text{cor}_g^*(\Omega).$$

Now fix x^{**} in X^{**}. We may assume $||x^{**}|| \leqq 1$. Since $S(BlX)$ is w^*-sequentially dense in $S^{**}(BlX^{**})$ [**15**], there is a sequence x_n in BlX such that

$$y^*Sx_n \to S^{**}x^{**}y^* \quad \text{for all } y^* \text{ in } Y^*.$$

It follows that

$$x^*(x_n) \to x^{**}(x^*) \quad \text{for all } x^* \in \text{cor}_g^*(\Omega) = \text{cor}_g^*(\Omega) \cup g(\Omega \backslash M).$$

Since $g(\cdot)x_n = f(\cdot)x_n$ a.e. and $x^{**}f(\cdot) = x^{**}g(\cdot)$ a.e., there is a null set N such that

$$x^*(x_n) \to x^{**}(x^*) \quad \text{for all } x^* \text{ in } f(\Omega \backslash N) \cup \text{cor}_f^*(\Omega)$$

as desired.

(iii) $\Rightarrow$ (iv). It suffices to prove the result for compact sets E so let x^{**} in X^{**}, E a compact subset of Ω and $\epsilon > 0$ be given and let g be as in the implication (i) $\Rightarrow$ (ii). For each w^*-Borel subset B of $\text{cor}_f^*(E) = \text{cor}_g^*(E)$ we define

$$\lambda(B) = \mu(g^{-1}(B) \cap E).$$

Since μ is a Radon probability measure and w^*-clco $g(\Omega)$ is w^*-metrizable, λ is a finite nonzero Radon measure [**21**, p. 31]. Let A be the support of λ. Since $\text{cor}_g^*(E)$ is a weak RN set, we have by Theorem 1 of [**16**] and Proposition 7 of [**17**] that there is a w^*-open slice S of $\text{cor}_g^*(E)$ such that $A \cap S \neq \emptyset$ and

$$\text{osc}(x^{**}, w^*\text{-clco}(A \cap S)) < \epsilon/6.$$

Since A is the support of λ, it follows that

$$\lambda(A \cap S) = \lambda(S) > 0.$$

Since each x^{**} is λ-measurable [**16**], we can now find a w^*-compact set $K \subset \text{cor}_g^*(E)$ with the following properties:

1) $x^{**}|_K$ is continuous for the w^*-topology of K
2) $\text{osc}(x^{**}, w^*\text{-clco } K) < \epsilon/6$
3) $\mu(E \cap g^{-1}(K)) > 0$.

If we now put $G = E \cap g^{-1}(K)$, then since

$$w^*\text{-clco } g(G) \subset w^*\text{-clco } K$$

we have that

$$\text{osc}(x^{**}, \text{cor}_g^*(G)) < \epsilon/6.$$

Now, by hypothesis, choose $x_n \in X$ with $\|x_n\| \leqq \|x^{**}\|$ and a μ-null set N such that

$$\lim_n x^*(x_n) = x^{**}(x^*) \quad \text{for all } x^* \in f(\Omega \backslash N) \cup \text{cor}_f^*(\Omega).$$

Using Egorov's theorem, we may assume that

$$\lim_n f(\omega)x_n = x^{**}f(\omega)$$

uniformly for ω in G and

$$\lim_n x^*(x_n) = x^{**}(x^*)$$

uniformly for x^* in $\text{cor}_f^*(G)$. Since $x^{**}f(\cdot)$ is μ-measurable, there is a measurable set $B \subset G$ with $\mu B > 0$ such that

$$\text{osc}(x^{**}f, B) < \epsilon/6.$$

Choose an integer N such that

$$\text{osc}(x^{**}f - fx_N, B) < \epsilon/6 \quad \text{and}$$

$$\text{osc}(x^{**} - x_N, \text{cor}_f^*(B)) < \epsilon/6.$$

Then

$$\text{osc}(f(\cdot)x_N, B) < \epsilon/2.$$

It follows that

$$\text{osc}(x_N, w^*\text{-clco}\, f(B)) \leqq \epsilon/2.$$

We now have that

$$\text{osc}(x^{**}, f(B)) < \epsilon/6 \quad \text{and} \quad \text{osc}(x^*, \text{cor}_f^*(B)) < \epsilon/6.$$

If now we take $x_1^* \in f(B)$ and $x_2^* \in \text{cor}^*(B)$, then

$$\begin{aligned} |x^{**}(x_1^*) - x^{**}(x_2^*)| &\leqq |(x^{**} - x_N)x_1^*| \\ &\quad + |(x_1^* - x_2^*)(x_N)| + |(x_N - x^{**})(x_2^*)| \\ &< \epsilon/6 + \epsilon/2 + \epsilon/6 < \epsilon. \end{aligned}$$

Thus

$$\text{osc}(x^{**}, f(B) \cup \text{cor}_f^*(B)) < \epsilon,$$

as desired.

(iv) $\Rightarrow$ (v). Suppose not. Then there is a set E in Σ with $\mu E > 0$ and an x^{**} in X^{**}, such that

$$\sup_{x^* \in \operatorname{cor}_f^*(E)} x^{**}(x^*) = a < b = \frac{1}{\mu E} \int_E x^{**} f d\mu.$$

Let $\epsilon = (b - a)/2$. An application of Zorn's lemma now produces a maximal collection $\{B_\alpha\}_{\alpha \in A}$ of mutually disjoint subsets of E with positive measure such that

$$\operatorname{osc}(x^{**}, f(B) \cup \operatorname{cor}_f^*(B)) < \epsilon/2.$$

Since μ is a finite measure, the collection $\{B_\alpha\}_{\alpha \in A}$ must necessarily be countable. Since the collection is maximal, the set $E \setminus \bigcup_{\alpha \in A} B_\alpha$ has measure zero. Choose a finite number of sets $B_{\alpha_1}, \ldots, B_{\alpha_n}$ such that

$$\mu(E \setminus \bigcup_{i=1}^n B_{\alpha_i}) < \frac{\epsilon \mu E}{2\|x^{**}f\|_\infty}.$$

For each i, choose any x_i^* in $\operatorname{cor}_f^*(B_{\alpha_i})$. Since

$$\operatorname{osc}(x^{**}, f(B_{\alpha_i}) \cup \operatorname{cor}_f^*(B_{\alpha_i})) < \epsilon/2,$$

we have that

$$\left| \left(\frac{1}{\mu B_{\alpha_i}} \int_{B_{\alpha_i}} x^{**} f d\mu \right) - x^{**}(x^*) \right| < \epsilon/2.$$

Thus

$$\left| \int_{B_{\alpha_i}} x^{**} f d\mu - \mu B_{\alpha_i} x^{**}(x_i^*) \right| < \frac{\epsilon \mu B_{\alpha_i}}{2}$$

and so

$$\left| \int_{\bigcup_{i=1}^n B_{\alpha_i}} x^{**} f d\mu - \sum_{i=1}^n \mu B_{\alpha_i} x^{**}(x_i^*) \right| < \frac{\epsilon}{2} \sum_{i=1}^n \mu B_{\alpha_i} \leqq \frac{\epsilon}{2} \mu E.$$

Since

$$\mu\left(E \setminus \bigcup_{i=1}^n B_{\alpha_i} \right) < \frac{\epsilon \mu E}{2\|x^{**}f\|_\infty},$$

we have that

$$\left| \int_{\bigcup_{i=1}^n B_{a_i}} x^{**} f d\mu - \int_E x^{**} f d\mu \right| < \frac{\epsilon}{2} \mu E$$

and hence

$$\left| \int_E x^{**} f d\mu - x^{**}\left(\sum_{i=1}^n \mu B_{\alpha_i} x_i^* \right) \right| < \epsilon \mu E.$$

But then

$$\left|\frac{1}{\mu E}\int_E x^{**}f d\mu - x^{**}\left(\sum_{i=1}^{n}\frac{\mu B_{\alpha i}}{\mu E_i}x_i^*\right)\right| < \epsilon$$

and

$$\sum_{i=1}^{n}\frac{\mu B_{\alpha_i}}{\mu E}x_i^* \in \operatorname{cor}_f^*(E),$$

a contradiction.

(v) $\Rightarrow$ (vi). Suppose not. Then there exists a set E of positive measure and an x^{**} in X^{**} such that

$$x^{**}f(\omega) > \sup_{x^* \in \operatorname{cor}_f^*(E)} x^{**}(x^*) \quad \text{for all } \omega \text{ in } E.$$

Hence

$$\frac{1}{\mu E}\int_E x^{**}f d\mu > \sup_{x^* \in \operatorname{cor}_f^*(E)} x^{**}(x^*).$$

But this implies that

$$\frac{1}{\mu E}\left(D - \int_E f d\mu\right) \notin X^{**}\text{-closure of } \operatorname{cor}_f^*(E),$$

a contradiction.

(vi) $\Rightarrow$ (vii). The hypothesis implies that (v) holds since otherwise there is a set E in Σ and an x^{**} and X^{**} such that

$$\frac{1}{\mu E}\int_E x^{**}f d\mu > \sup_{x^* \in \operatorname{cor}_f^*(E)} x^{**}(x^*).$$

But this is impossible since then

$$\begin{aligned}\sup_{x^* \in \operatorname{cor}_f^*(E)} x^{**}(x^*) &< \frac{1}{\mu E}\int_E x^{**}f d\mu \\ &\leqq \frac{1}{\mu E}\int_E \sup_{x^* \in \operatorname{cor}_f^*(E)} x^{**}(x^*) d\mu \\ &\leqq \sup_{x^* \in \operatorname{cor}_f^*(E)} x^{**}(x^*),\end{aligned}$$

a contradiction. Hence (v) holds. We now show (v) $\Rightarrow$ (i).

Let x^{**} in X^{**} and E in Σ be fixed and let λ and g be as in the implication (iii) $\Rightarrow$ (iv). Let $\epsilon > 0$. Since $\operatorname{cor}_f^*(E)$ is a w^*-metrizable weak RN set, there exists a w^*-compact metrizable subset K of $\operatorname{cor}_g^*(E) = \operatorname{cor}_f^*(E)$ such that

$$\lambda(\operatorname{cor}_f^*(E)\backslash K) < \epsilon$$

and $x^{**}|_K$ is continuous [**16**]. Let $\delta > 0$. Since $x^{**}|_K$ is uniformly continuous, there exists an $\eta > 0$ such that if $G \subset K$ and w_* diam $G < \eta$ then

$$\operatorname{diam} x^{**}(G) < \delta.$$

Now let $F = g^{-1}(K) \cap E$ and note that $\mu(E\backslash F) < \epsilon$. Since $g\chi_F$ has relatively w^*-compact metrizable range, there exist disjoint measurable sets of positive measure $E_1, \ldots, E_n$ such that

$$F = \bigcup_{i=1}^{n} E_i \quad \text{and} \quad w^*\text{-diam } g(E_i) < \eta \quad \text{for all } i.$$

Consequently,

$$w^*\text{-diam } \operatorname{cor}_g^*(E_i) = w^*\text{-diam } \operatorname{cor}_f^*(E_i) < \eta$$

and so

$$\operatorname{diam} x^{**}(\operatorname{cor}_f^*(E_i)) < \delta \quad \text{for all } i.$$

It follows that

$$\left|x^{**}\left(w^* - \int_{E_i} f d\mu\right) - x^{**}(x_i^*)\mu E_i\right| < \delta\mu E_i$$

for each i and any x_i^* in $\operatorname{cor}_f^*(E_i)$. But, by hypothesis,

$$\frac{1}{\mu E_i}\left(D - \int_{E_i} f d\mu\right) \in X^{**}\text{-closure of } \operatorname{cor}_f^*(E_i), \quad \text{for all } i.$$

Hence $\left|x^{**}\left(w^* - \int_{E_i} f d\mu\right) - x^{**}\left(D - \int_{E_i} f d\mu\right)\right| < \delta\mu E_i$

and so

$$\left|x^{**}\left(w^* - \int_F f d\mu\right) - x^{**}\left(D - \int_F f d\mu\right)\right| < \delta\mu F.$$

Since $\delta > 0$ was arbitrary, we have that

$$x^{**}\left(w^* - \int_F f d\mu\right) = x^{**}\left(D - \int_F f d\mu\right)$$

and since $\epsilon > 0$ was arbitrary, we have that

$$x^{**}\left(w^* - \int_E f d\mu\right) = x^{**}\left(D - \int_E f d\mu\right).$$

Because this holds for all x^{**} in X^{**}, we have that

$$w^* - \int_E f d\mu = D - \int_E f d\mu$$

so f is Pettis integrable. Since f and g have the same w^*-core and both are

Pettis integrable, it follows that they are scalarly equivalent. Hence if W is any weak Baire set, we have that

$$\chi_W \circ f = \chi_W \circ g \ \mu\text{-a.e. } [\mathbf{5}].$$

Now suppose that $W \supseteq \mathrm{cor}_f^*(E)$. Since the essential range of $g|_E$ is contained in $\mathrm{cor}_f^*(E)$, we have that

$$\mu(E \cap f^{-1}(W)) = \int_E \chi_W \circ f d\mu = \int_E \chi_W \circ g d\mu = \int_E d\mu = \mu E,$$

as desired.

(vii) $\Rightarrow$ (i). By the proof of the above implication, it suffices to show that (vi) holds. Suppose (vi) doesn't hold i.e., there is a subset E of positive measure and an x^{**} in X^{**} such that

$$x^{**}f(\omega) > \sup_{x^* \in \mathrm{cor}_f^*(E)} x^{**}(x^*) \quad \text{for all } \omega \text{ in } E.$$

Let

$$W = \{ y^* \in X^* | x^{**}(y^*) \leqq \sup_{x^* \in \mathrm{cor}_f^*(E)} x^{**}(x^*) \}.$$

Then W is a weak Baire set containing $\mathrm{cor}_f^*(E)$ but

$$\mu(E \cap f^{-1}(W)) = 0,$$

a contradiction. This completes the proof.

We note that condition (iv) of the above theorem owes its origin to Saab and Saab [**17**] and comes close to saying that w^*-clco $f(\Omega)$ is a weak *RN* set. It is also worth mentioning that conditions (vi) and (vii) of the above theorem have their antecedents in the work of Uhl [**24**] and Edgar [**5**] on the question of when scalarly measurable functions are scalarly equivalent to strongly measurable functions. Condition (iii) is related to the Bourgain property discussed by Riddle and Saab [**13**] which also ensures that a function is Pettis integrable.

Recently [**2**], various one to one operators weaker than embeddings have been used to study the Radon-Nikodym property. In light of condition (vii) of Theorem 3, it seems natural to introduce here the concept of a weak Baire embedding. A one to one adjoint operator $T:X^* \to Y^*$ is called a *weak Baire embedding* if for every weak Baire subset W of X^* there is a weak Baire subset S of Y^* such that

$$TW = TX^* \cap S.$$

Obviously any dual embedding is also a weak Baire embedding; hence the dual of any separable space weak Baire embeds into l_∞. But the concept is not limited to dual embeddings; for example, any separable dual space weak Baire embeds into l_2 [**2**, Proposition 1.2]. We now have the following stability result.

COROLLARY 4. *Let $T:X^* \to Y^*$ be a weak Baire embedding and suppose that the w^*-compact separable subsets of X^* and Y^* are w^*-metrizable. Then a function $f:\Omega \to X^*$ is universally Pettis integrable if and only if the function $Tf:\Omega \to Y^*$ is universally Pettis integrable. Consequently, X^* has the* UPIP *if Y^* does.*

Proof. One direction is clear. For the other, suppose that Tf is universally Pettis integrable and let μ be a Radon probability measure on Ω. If W is a weak Baire subset of X^*, then $TW = TX^* \cap S$ for some weak Baire subset S of Y^*. Hence

$$f^{-1}(W) = (Tf)^{-1}(TW) = (Tf)^{-1}(S)$$

since T is one to one and so f is universally scalarly measurable. Let $\epsilon > 0$. By Lemma 2 there is a compact subset E of Ω such that $\mu(\Omega \backslash E) < \epsilon$ and $\text{cor}_f^*(E)$ is a w^*-metrizable weak RN set. Hence without loss of generality we may assume that $\text{cor}_f^*(\Omega)$ is a w^*-metrizable weak RN set. Let E be a μ-measurable subset of Ω and suppose that W is a weak Baire subset of X^* such that W contains $\text{cor}_f^*(E)$. Then

$$S \supset TW \supset T(\text{cor}_f^*(E)) = \text{cor}_{Tf}^*(E)$$

and so by Theorem 3,

$$\mu(E \cap (Tf)^{-1}(S)) = \mu E.$$

But

$$(Tf)^{-1}(S) = f^{-1}(W)$$

so another application of Theorem 3 shows that f is μ-Pettis integrable. Since μ is an arbitrary Radon probability measure, we are done.

We turn next to the problem of what conditions imply that a Banach space has the universal Pettis integral property. Riddle, Saab, and Uhl [**14**] have shown that the dual of a separable space has this property and ask if the dual of a weakly compactly generated space also does. In light of Theorem 4, we may view the fact the dual of a separable space has the UPIP as a consequence of the fact that l_∞ has the UPIP.

We now give two general conditions that ensure that a Banach space has the UPIP. The first of these requires a property (which we call property (*)) that is enjoyed by separable or reflexive Banach spaces X. It is akin to requiring that (X^*, weak) be Lindelof or that X^* have property (C) of R. Pol, conditions that ensure that X^* has the stronger Pettis integral property [**6**].

(*) Let $\{A_\alpha\}$ be a collection of nonempty w^*-compact convex subsets of X^*. Let $x^{***} \in X^{***}$ be a w^*-sequentially continuous functional such that for any countable subcollection $\{A_{\alpha_i}\}_{i=1}^{+\infty}$ of $\{A_\alpha\}$ we have that $x^{***} \in X^{**}$-closure of $\bigcap_{i=1}^{+\infty} A_{\alpha_i}$. Then $x^{***} \in X^{**}$-closure of $\bigcap_\alpha A_\alpha$.

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COROLLARY 5. *If every w^*-compact separable subset of X^* is w^*-metrizable and X has property* (*), *then X^* has the* UPIP.

Proof. Let $f:\Omega \to X^*$ be a universally scalarly measurable function and let $\epsilon > 0$. If μ is a Radon probability measure on Ω, then by Lemma 2 there exists a compact set F with $\mu(\Omega \backslash F) < \epsilon$ such that $\mathrm{cor}_f^*(F)$ is a w^*-metrizable weak RN set. Let E be any measurable subset of F having positive measure and let $\{N_\alpha\}_{\alpha \in A}$ be the collection of all null subsets of E. If $\{N_i\}_{i=1}^{+\infty}$ is any countable subcollection, then

$$\bigcap_{i=1}^{+\infty} w^*\text{- clco} f(F \backslash N_{\alpha_i}) = w^*\text{-clco} f(E \backslash N)$$

where

$$N = \bigcup_{i=1}^{+\infty} N_{\alpha_i}$$

is null. Hence the w^*-sequentially continuous functional $\frac{1}{\mu E}(D - \int f d\mu)$ is in the X^{**}-closure of

$$\bigcap_{i=1}^{+\infty} w^*\text{-clco} f(E \backslash N_{\alpha_i})$$

for any countable subcollection of the N_α's. Since X has property (*), this implies that

$$\frac{1}{\mu E}\left(D - \int_E f d\mu\right) \in X^{**}\text{-closure of} \bigcap_{\mu N = 0} w^*\text{-clco} f(E \backslash N).$$

An appeal to Theorem 3 now establishes the Pettis integrability of $f|_F$. Since $\epsilon > 0$ was arbitrary and μ is any Radon measure, the function f is universally Pettis integrable.

The hypothesis for the next result may be viewed as a natural weakening of Mazur's condition on X^*: every w^*-sequentially continuous functional on X^{**} is w^*-continuous i.e., in X^*. Mazur's condition is known to imply that X^* has the Pettis integral property [**6**].

COROLLARY 6. *If every w^*-compact separable subset of X^* is w^*-metrizable and every w^*-sequentially continuous functional on X^{**} is the X^{**}-limit of a w^*-separable subset of X^*, then X^* has the* UPIP.

Proof. Let $f:\Omega \to X^*$ be a universally scalarly measurable function and let $\epsilon > 0$. If μ is a Radon probability measure on Ω, then, by Lemma 2, there exists a measurable set F with $\mu(\Omega \backslash F) < \epsilon$ such that $\mathrm{cor}_f^*(F)$ is a w^*-metrizable weak RN set. Let E be any measurable subset of F having positive measure and choose a w^*-separable subset S of X^* such that

$$D - \int_E f d\mu \in X^{**}\text{-closure of } S.$$

Let Y be the w^*-closed linear span of $S \cup \text{cor}_f^*(F)$ in X^*. Then the unit ball of Y is a w^*-metrizable subset of X^*. Choose a norm separable subspace Z of X such that the restriction map $R:X^* \to Z^*$ is an isometry from Y to $R(Y)$. By Theorem 3, the function $Rf:\Omega \to Z^*$ is μ-Pettis integrable. Now let x^{**} be in X^{**} and choose a net $\{x_\alpha\}_{\alpha \in A}$ in X such that

$$x^{**} = w^* - \lim_\alpha x_\alpha \quad \text{and} \quad \|x_\alpha\| \leqq \|x^{**}\| \quad \text{for all } \alpha \text{ in } A.$$

Define, for each α in A, linear functionals l_α on $R(Y)$ by

$$l_\alpha(Ry) = X(y_\alpha).$$

Each of these functionals is continuous for the bounded Z topology on $R(Y)$ and hence [**4**, p. 428] for the Z topology on $R(Y)$. Now apply the Hanh-Banach theorem to extend l_α to a z_α in Z with $\|z_\alpha\| \leqq \|x_\alpha\|$. Let z^{**} be a w^*-cluster point of the z_α's. Then there is a subnet $\alpha(\beta)$ of α such that

$$z^{**} = w^* - \lim_\beta z_{\alpha(\beta)}.$$

Consequently, for all y in Y

$$x^{**}(y) = \lim_\beta y(x_{\alpha(\beta)}) = \lim_\beta Ry(z_{\alpha(\beta)}) = z^{**}(Ry) = R^*z^{**}(y).$$

Since $D - \int_E f d\mu \in X^{**}$-closure of Y, we have that

$$\begin{aligned} x^{**}\left(D - \int_E f d\mu\right) &= R^*z^{**}\left(D - \int_E f d\mu\right) \\ &= \int_E z^{**} Rf d\mu \\ &= z^{**}\left(P - \int_E Rf d\mu\right) \\ &= z^{**}\left(w^* - \int_E Rf d\mu\right) \\ &= z^{**}R\left(w^* - \int_E f d\mu\right) \\ &= R^*z^{**}\left(w^* - \int_E f d\mu\right) \\ &= x^{**}\left(w^* - \int_E f d\mu\right) \end{aligned}$$

and hence $f\chi_F$ and thus f are μ-Pettis integrable. Since μ was an arbitrary Radon probability measure, this completes the proof.

We turn next to some particular examples of spaces with the UPIP for which some special axioms are required. We consider only dual spaces whose preduals are weakly K-analytic; for such spaces it is known that every w^*-compact separable set is w^*-metrizable [**23**, **25**] so Lemma 2 and hence Theorem 3 are applicable. Recall that a cardinal α is said to be a real valued measurable cardinal if there exists a set Γ having cardinality α and a finite nonzero measure defined on all subsets of Γ that vanishes on singleton sets. It is consistent with the usual axioms of set theory that no such cardinals exist; for further details we refer the reader to [**22**]. The least real valued measurable cardinal is denoted by m_r; the cardinality of the continuum by c. The following result is an easy modification of a result of Edgar [**6**, Theorem 5.9].

THEOREM 7. *Assume* $m_r \geqq c$. *Then, for any set* Γ, $l_1(\Gamma)$ *has the* UPIP.

Proof. Let $f:\Omega \to l_1(\Gamma)$ be a universally scalarly measurable function and let μ be a Radon measure on Ω. Let (Ω, Σ^*, μ) denote the completion of (Ω, Σ, μ). Then for each x^{**} in $l_\infty(\Gamma)$ and each Borel subset B of the real line we have that $(x^{**}f)^{-1}(B)$ is in Σ^* [**21**, p. 26]. Since μ is a Radon measure, the sets

$$\left\{D- \int_E fd\mu:E \text{ in } \Sigma^*\right\} \quad \text{and}$$
$$\{x^{**}f(\cdot):\|x^{**}\| \leqq 1\}$$

are norm compact in $l_1(\Gamma)^{**}$ and $L_1(\mu)$ respectively [**7**]. But then, by [**4**, p. 168], there is a sub σ-algebra Σ_1^* of Σ^* which is the completion of a countably generated σ-algebra Σ_1 such that

$$\left\{D - \int_E fd\mu:E \text{ in } \Sigma_1^*\right\}$$

is norm dense in $\{D - \int_E fd\mu:E \text{ in } \Sigma^*\}$ and $(x^{**}f)^{-1}(B)$ is in Σ_1^* for each x^{**} in $l_\infty(\Gamma)$ and each Borel subset B of the real line. We note that the cardinality of Σ_1 is c and that for each E in Σ_1^*,

$$\mu E = \sup\{\mu B:B \text{ in } \Sigma_1, B \subset E\}.$$

Consequently, the proof of Theorem 5.9 of [**6**] shows that the function f is Pettis integrable over each set E in Σ_1^*, since the proof of that result holds for any finite complete measure space that is countably generated up to null sets. It follows that

$$\left\{D - \int_E fd\mu:E \text{ in } \Sigma^*\right\} \subset l_1(\Gamma)$$

so f is universally Pettis integrable.

For our next result we recall that the density character of a space is the least cardinal such that there is a dense subset of the space with that cardinality.

THEOREM 8. *Let $C(K)$ be a weakly K-analytic space and suppose that the density character of K is not a real valued measurable cardinal. Then the space $M(K)$ has the* UPIP.

Proof. Let $f:\Omega \to M(K)$ be universally scalarly measurable and let μ be a Radon probability measure on Ω. For each Borel subset C of K, let $P_C:M(K) \to M(K)$ be the natural projection given by

$$P_C(\lambda) = \lambda(\cdot \cap C).$$

Since $\mathrm{cor}_f^*(\Omega)$ is a w^*-metrizable compact set, there is a compact metrizable $K_0 \subset K$ such that the natural projection P_{K_0} is the identity on $\mathrm{cor}_f^*(\Omega)$. Suppose that f is not μ-Pettis integrable. Then by Theorem 3 (v) there is a measurable set E of positive measure and a functional $b \in M(K)^*$ such that

$$\frac{1}{\mu E} \int_E < b, f(\cdot) > d\mu > \sup_{x^* \in \mathrm{cor}_f^*(E)} b(x^*).$$

Now define a (signed) Borel measure on K by

$$\lambda(C) = \frac{1}{\mu E} \int_E < b, P_{K_0 \cup C} f(\cdot) > d\mu$$

for each Borel subset C of K. Since the density character of K is not a real-valued measurable cardinal, the measure λ is a (signed) Radon measure [**23**]. But if C is any metrizable compact subset of K, then the function $P_{K_0 \cup C} f$ is Pettis integrable by Theorem 3. Since $P_{K_0} f$ and $P_{K_0 \cup C} f$ are w^*-equivalent by (vi) of Proposition 1 and both are Pettis integrable, these functions are scalarly equivalent. It follows that

$$\begin{aligned}\lambda(C) &= \frac{1}{\mu E} \int_E < b, P_{K_0 \cup C} f(\cdot) > d\mu \\ &= \frac{1}{\mu E} < b, P - \int_E P_{K_0 \cup C} f(\cdot) d\mu > \\ &= \frac{1}{\mu E} < b, P - \int_E P_{K_0 \cup C} f(\cdot) d\mu > \\ &= \frac{1}{\mu E} < b, w^* - \int_E f d\mu > \\ &\leqq \sup_{x^* \in \mathrm{cor}_f^*(E)} b(x^*)\end{aligned}$$

$$< \frac{1}{\mu E} \int_E < b, f(\cdot) > d\mu = \lambda(K).$$

Consequently, the measure λ does not have metrizable support. This contradiction of Théorème 6.7 of [**23**] completes the proof.

We note that the proof of the above result can easily be adapted to show that if K is a compact Hausdorff space such that every strongly additive measure of finite variation defined on the Borel subsets of K has metrizable support, then $L_\infty(v, K, B)$ has the UPIP (here v is any finite measure on the Borel subsets B of K).

Finally, we see that if we impose an additional condition on the function (namely, that $f(\cdot)\chi_B:\Omega \rightarrow X^*$ is universally scalarly measurable for each w^*-Borel subset B of X^*), we can adapt the result of Theorem 8 to any weakly K-analytic space. This condition holds, for example, if f is universally Borel measurable. Thus when X is weakly K-analytic, we may replace Lusin measurability with the weaker Borel measurability and still ensure Pettis integrability.

THEOREM 9. *Let X be a weakly K-analytic space and suppose that the norm density character of X is not a real valued measurable cardinal. Let $f:\Omega \rightarrow X^*$ be a universally scalarly measurable function. If, for each w^*-Borel subset B of X^*, the function $f(\cdot)\chi_B:\Omega \rightarrow X^*$ is universally scalarly measurable, then f is universally Pettis integrable.*

Proof. Suppose not. Then there exists a Radon measure μ defined on the Borel subsets of Ω and a measurable set E of positive measure such that

$$D - \int_E f d\mu \notin X^{**}\text{-linear span of } \operatorname{cor}_f^*(E)$$

since otherwise the argument of Corollary 6 would show that f is μ-Pettis integrable. Hence there is a functional x^{**} in X^{**} such that

$$\int_E x^{**} f d\mu > 0 \quad \text{and} \quad < x^{**}, \operatorname{cor}_f^*(E) > \, = 0.$$

We may assume that $x^{**}f(\cdot) \geqq 0$ μ-a.e. on E. Now define a nonzero positive measure λ on the w^* Borel subsets of w^*-clco $f(\Omega)$ by

$$\lambda(C) = \int_E < x^{**}, f(\cdot)\chi_C > d\mu.$$

By hypothesis and Théorème 6.11 of [**23**], λ is a Radon measure and therefore has metrizable support, but it is easily seen that $\lambda(C) = 0$ for every w^*-compact metrizable subset C. This contradiction completes the proof.

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Oakland University,
Rochester, Michigan