ONE MORE METHOD TO CONSTRUCT
THE IRREDUCIBLE UNITARY REPRESENTATIONS
OF NILPOTENT LIE GROUPS

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Abstract

In this paper one more canonical method to construct the irreducible unitary representations of a connected, simply connected nilpotent Lie group is introduced. Although we use Kirillov's analysis to deduce this procedure, the method obtained differs from that of Kirillov's, in that one does not need to consider the coadjoint representation of the group in the dual of its Lie algebra (in fact, neither does one need to consider the Lie algebra of the group, provided one knows certain connected subgroups and their characters). The method also differs from that of Mackey's as one only needs to induce characters to obtain all irreducible representations of the group.

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1. Introduction

Throughout this paper G will denote a connected, simply connected nilpotent Lie group and $\mathscr{G}$ its Lie algebra. It is well known that the irreducible unitary representations of G can be obtained by Kirillov's orbital method (cf. [2]). One can also apply Mackey's theory to constrsuct the irreducible representations of G (cf. [3]); since G always contains a commutative connected (and hence closed) normal subgroup containing the nontrivial centre of G, and we know (cf. [1, Theorem 1]) that such a normal subgroup is always regularly embedded in G in the sense of Mackey in [3]. The purpose of this paper is to introduce a third canonical way of constructing the irreducible representations of G. The method is deduced from the following three theorems.

2. The statements of the theorems

The first theorem was proved by Dixmier in 1958 (cf. [1, Theorem 2]), using the earlier results of Mackey. His theorem is true in the more general case. We only need the following theorem.

THEOREM 1. *Let H be a commutative connected normal subgroup of G. Then every irreducible representation U of G is induced by a character χ of a connected subgroup K of G which contains H.*

The second theorem we consider is the so called Shoda's criterion for irreducibility of monomial representations of G which was proved by Pukanszky using Kirillov's results and was first published in his book (cf. [4, Theorem 2, Section 3, Chapter III, Part II]). The analogue of this theorem for finite groups was first stated and proved by Shoda around 1933.

THEOREM 2. *Let K be a connected subgroup of G, and χ a character of K. For $x \in G$ let $K_x = K \cap x^{-1}Kx$ and denote by χ_x the character of $x^{-1}Kx$ which is defined by*

$$(1) \qquad \chi_x(k') = \chi(xk'x^{-1}), \qquad k' \in x^{-1}Kx.$$

Then the representation $\operatorname{ind}_K^G \chi$ is irreducible if and only if for each $x \notin K$ the characters χ and χ_x have distinct restrictions to K_x.

We next extend another criterion of Shoda to connected, simply connected nilpotent Lie groups by using Kirillov's results.

THEOREM 3. *Let K_i, $i = 1, 2$, be connected subgroups of the same dimension of G, and let χ^i be a character of K_i such that $\operatorname{ind}_{K_i}^G \chi^i$ is irreducible. Then*

$$(2) \qquad \operatorname{ind}_{K_1}^G \chi^1 \cong \operatorname{ind}_{K_2}^G \chi^2$$

if and only if there exists $x \in G$ such that the respective characters χ^1 and χ_x^2 of K_1 and $x^{-1}K_2x$ have the same restriction to $K_1 \cap x^{-1}K_2x$, where χ_x^2 is defined as (1) in Theorem 2.

PROOF. For $i = 1, 2$, let f_i be the functional on $\mathscr{G}$ from which χ^i is defined by (see [4, Section 3, Chapter III, Part II])

$$(3) \qquad \chi^i(\exp k) = e^{if_i(k)}, \qquad k \in \mathscr{K}_i,$$

where $\mathscr{K}_i$ is the Lie algebra of K_i. We also know (cf. [4, Theorem 1, Section 3, Chapter III, Part II]) that (2) holds if and only if there exists $x \in G$ such that

(4) $$f_1 = \mathrm{Coad}(x)f_2$$

where Coad denotes the coadjoint representation of G in the vector space dual of $\mathscr{G}$. On the other hand (4) holds if and only if

(5) $$f_1(k) = f_2\big(\mathrm{Ad}(x^{-1})k\big)$$

for all $k \in \mathscr{K}_1 \cap \mathrm{Ad}(x)\mathscr{K}_2$. Since the Lie algebra of $K_1 \cap xK_2x^{-1}$ is identified with $\mathscr{K}_1 \cap \mathrm{Ad}(x)\mathscr{K}_2$ (see [4, D, Section 1, Chapter I, Part II]), having also (3) in mind, we have that (5) holds if and only if

(6) $$\chi^1(\exp k) = \chi^2(x^{-1}\exp(k)x)$$

for all $\exp(k) \in K_1 \cap xK_2x^{-1}$. Since (6) means that χ^1 and χ_x^2 have the same restrictions to $K_1 \cap xK_2x^{-1}$ the proof of the theorem is complete.

3. The deduction of the procedure

The above three theorems suggest the following procedure to obtain the irreducible unitary representations of G. Find a connected commutative normal subgroup of G, preferably of the largest dimension possible (as we mentioned in the introduction such an H always exists). For each $n = \dim H, (\dim H) + 1, \ldots, \dim G$, let the set $\mathscr{S}_n$ denote the collection of all connected subgroups K of G which contain H and have dimension n. Let also $\mathscr{C}_n$ denote the family of all characters of the subgroups belonging to $\mathscr{S}_n$ which satisfy the condition of Theorem 2. Theorems 1 and 2 ensure that if we induce all elements of all $\mathscr{C}_n$'s to G, then (up to equivalence) we get all irreducible representations of G. To ensure that in the process of inducing these characters we do not get undue equivalent irreducible representations we first note (see, [4, the proof of Theorem 1, Section 3, Chapter III, Part II]) that this may only happen between the characters belonging to the same $\mathscr{C}_n$. On the other hand the criterion of Theorem 3 clearly defines an equivalence relation in each $\mathscr{C}_n$. Let $\{[C_n]\}$ denote the set of all equivalence classes in $\mathscr{C}_n$ under this relation. Then Theorem 3 ensures that if we induce only one character from each member of $[C_n]$ we do not get undue equivalent representations.

4. An example

Let G denote the nilpotent Lie group consisting of all real matrices

$$\begin{pmatrix} 1 & x & z & w \\ 0 & 1 & y & v \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us write $[x, y, z, u, v, w]$ for the above element of G. Choose the subgroup of all elements of the form $[0, y, z, 0, v, w]$ as the subgroup H of Theorem 1. Then $\mathscr{S}_4$ consists only of H itself. Let $\chi^{\nu,\lambda,\beta,\gamma}$ denote a typical character of H which is defined by

$$\chi^{\nu,\lambda,\beta,\gamma}([0, y, z, 0, v, w]) = e^{i(\nu y + \lambda z + \beta v + \gamma w)}.$$

Applying the criterion of Theorem 2 it is easy to see that

$$\mathscr{C}_4 = \{\chi^{\nu,\lambda,\beta,\gamma} : \gamma \neq 0\}.$$

On the other hand, for arbitrary reals λ, β and $\gamma \neq 0$, $\chi^{\nu,\lambda,\beta,\gamma}$ and $\chi^{\nu,0,0,\gamma}$ belong to the same member of $[C_4]$; indeed $\chi^{\nu,0,0,\gamma}_{[\beta,0,0,\lambda,0,0]}$ and $\chi^{\nu,\lambda,\beta,\gamma}$ coincide on H. Hence $\{\chi^{\nu,0,0,\gamma} : \gamma \neq 0\}$ consists of one and only one character from each member of $[C_4]$. We then induce the elements of this set to get the following representations of G in $L^2(\mathbb{R}^2)$,

(i)

$$(\Pi^{\nu,\gamma}(g)f)(t, s) = e^{i\{\nu y + \gamma[w + tv - (z + ty)(s + u)]\}}f(t + x, s + u), \qquad \nu \in \mathbb{R}, \gamma \neq 0,$$

where $g = [x, y, z, u, v, w]$. Next we observe that $\mathscr{S}_5$ consists of the subgroups K_1, K_2 and also of all K_a, $a \neq 0$, with the respective typical elements $[0, y, z, u, v, w]$, $[x, y, z, 0, v, w]$ and $[x, y, z, ax, v, w]$. Denote by $\chi^{\nu,\lambda,\alpha}, \phi^{\mu,\nu,\beta}$ and $\psi^{\mu,\nu,\beta}$ the typical characters of K_1, K_2 and K_a, respectively, which are defined by

$$\chi^{\nu,\lambda,\alpha}([0, y, z, u, v, w]) = e^{i(\nu y + \lambda z + \alpha u)},$$

$$\phi^{\mu,\nu,\beta}([x, y, z, 0, v, w]) = e^{i(\mu x + \nu y + \beta v)},$$

and

$$\psi_a^{\mu,\nu,\beta}([x, y, z, ax, v, w]) = e^{i[\mu x + \nu y + a\beta(z - xy) + \beta v]}.$$

Applying Theorem 2 we have that

$$\mathscr{C}_5 = \{\chi^{\nu,\lambda,\alpha}, \phi^{\mu,\nu,\beta}, \psi_a^{\mu,\nu,\beta} : a\lambda\beta \neq 0\}.$$

It is also easy to check that for arbitrary ν and $a, \lambda, \beta \neq 0, \chi^{0,\lambda,\alpha}_{[\nu,0,0,0,0,0]} = \chi^{\nu,\lambda,\alpha}$, $\phi^{\mu,0,\beta}_{[0,0,0,-\nu,0,0]} = \phi^{\mu,\nu,\beta}$ and $(\psi_a^{\mu,0,\beta})_{[0,0,0,-\nu/\beta,0,0]} = \psi_a^{\mu,\nu,\beta}$, so that it is enough to find the equivalence classes of the following subset of $\mathscr{C}_5$:

$$\mathscr{C}_5^* = \{\chi^{0,\lambda,\alpha}, \phi^{\mu,0,\beta}, \psi_a^{\mu,0,\beta} : a\lambda\beta \neq 0\}.$$

Since K_1, K_2 and also K_a are normal subgroups and since

$$K_1 \cap K_2 = K_1 \cap K_a = K_2 \cap K_a = H,$$

one immediately checks, by applying the criterion of Theorem 3, that $\mathscr{C}_5^*$ indeed contains one and only one character from each member of $\{[C_5]\}$. We therefore induce the characters $\chi^{0,\lambda,\alpha}$, $\phi^{\mu,0,\beta}$ and $\psi_a^{\mu,0,\beta}$ of K_1, K_2 and K_a (in respective

order) to obtain the following irreducible representations of G in $L^2(\mathbb{R})$: for $g = [x, y, z, u, v, w]$

(ii) $\left(U^{\lambda,\alpha}(g)f\right)(t) = e^{i\lambda[(z+ty)+\alpha u]}f(t+x),$ $\lambda \neq 0, \alpha \in \mathbb{R},$

(iii) $\left(V^{\mu,\beta}(g)f\right)(t) = e^{i[\mu x+\beta(v-yu-ty)]}f(t+u),$ $\beta \neq 0, \mu \in \mathbb{R},$

(iv)

  $\left(W^{\mu,a\beta,\beta}(g)f\right)(t) = e^{i[\mu x+a\beta z+\beta(v-yu-ty)]}f(t+u-ax),$ $a\beta \neq 0, \mu \in \mathbb{R}.$

Finally $\mathscr{S}_6$ consists of G only, and the characters of G are defined by

(v) $\chi^{\mu,\nu,\alpha}([x, y, z, u, v, w]) = e^{i(\mu x+\nu y+\alpha u)},$ $\mu, \nu, \alpha \in \mathbb{R}.$

We end this section by mentioning that the representations (ii), (iii) and (iv) can be put into one formula; indeed for a real number $r \neq 0$, let T^r be the unitary operator in $L^2(\mathbb{R})$ defined by

$$(T^rf)(t) = (|r|/2\pi)^{1/2}\int_\mathbb{R} e^{irt\tau}f(\tau)\,d\tau.$$

Then it is easy to see that the respective equivalent representations $U_1^{\lambda,\alpha} = T^\lambda U^{\lambda,\alpha}(T^\lambda)^{-1}$, $V_1^{\mu,\beta} = (T^\beta)^{-1}V^{\mu,\beta}T^\beta$ and $W_1^{\mu,a\beta,\beta} = (T^\beta)^{-1}W^{\mu,a\beta,\beta}T^\beta$ to $U^{\lambda,\alpha}$, $V^{\mu,\beta}$ and $W^{\mu,a\beta,\beta}$ are defined by

(ii)* $\left(U_1^{\lambda,\alpha}(g)f\right)(t) = e^{i[\lambda(z-xy-tx)+a\mu]}f(t+y),$ $\lambda \neq 0, \alpha \in \mathbb{R},$

(iii)* $\left(V_1^{\mu,\beta}(g)f\right)(t) = e^{i[\mu x+\beta(v+tu)]}f(t+y),$ $\beta \neq 0, \mu \in \mathbb{R},$

(iv)*

  $\left(W_1^{\mu,a\beta,\beta}(g)f\right)(t) = e^{i[\mu x+a\beta(z-xy-tx)+\beta(v+tu)]}f(t+y),$ $a\beta \neq 0, \mu \in \mathbb{R}.$

Now if we put $\lambda = a\beta$ in (iv)* then (ii)*–(iv)* can be considered as

(ii)** $\left(\rho^{\mu,\lambda,\alpha,\beta}(g)f\right)(t) = e^{i[\mu x+\lambda(z-xy-tx)+\beta(v+tu)+\alpha u]}f(t+y).$

with the conditions $\lambda^2 + \beta^2 \neq 0$, $\alpha\beta = 0$, and that $\mu = 0$ whenever $\lambda \neq 0$ but $\beta = 0$.

  We also mention that the representations $\Pi^{\nu,\gamma}$, $\gamma \neq 0$, $\nu \in \mathbb{R}$, of G given by (i) are equivalent to $\sigma^{\nu,\gamma}$ defined by

(i)* $\left(\sigma^{\nu,\gamma}(g)f\right)(t, s) = e^{i[\nu y+\gamma(w+tv+su)]}f(t+x, s+z+ty),$

since $\sigma^{\nu,\gamma} = (T^\gamma)^{-1}\Pi^{\nu,\gamma}T^\gamma$ where

$$(T^\gamma f)(t, s) = (|\gamma|/2\pi)^{1/2}\int_\mathbb{R} e^{i\gamma s\tau}f(t, \tau)\,d\tau.$$

Acknowledgement

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