Can. J. Math., Vol. XXXIII, No. 1, 1981, pp. 168–180

TOPOLOGICALLY VERSAL DEFORMATIONS OF MATRICES; CODIMENSION AT MOST TWO

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1. Introduction. The reduction of a matrix to its Jordan normal form is an unstable operation in that both the normal form itself and the reducing mapping depend discontinuously on the elements of the original matrix. For example, the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ trivially reduces to itself in Jordan form, but there are arbitrarily small perturbations of this matrix that reduce to the form $\begin{bmatrix} 2+\epsilon & 0 \\ 0 & 2 \end{bmatrix}$ which is certainly not a small perturbation of the original matrix, and moreover the reducing mapping is not a small perturbation of the identity. In [**1**], Arnol'd derives the simplest possible normal forms to which families of matrices may be linearly reduced in a 'stable' manner. In this paper, we consider a 'topological' version of the problem, using the classification of matrices up to topological conjugacy given in [**8**] and the classification of linear dynamical systems up to orbital equivalence given in [**9**]. The classification given in [**8**] was not complete. It has since been completed by Cappell and Shaneson [**6**] for $GL(\mathbf{R}^n)$, $n \leqq 6$.

The work in this paper extends and modifies that by the author in [**5**]. For the sake of clarity we repeat the motivation and definitions given in [**5**]. In fact the definition of topological equivalence of deformations is modified. The definition was modified, partly to simplify the problem, and partly to relate the matrix deformation problem to the linear dynamical system deformation problem. The results in this paper can be viewed as a natural (though partial) solution to either problem.

2. Definitions, preliminaries and main result. A *smooth k-parameter family* of matrices A is a smooth map, $A: U \to GL(\mathbf{R}^n)$ where U is a neighbourhood of $0 \in \mathbf{R}^k$ and $GL(\mathbf{R}^n)$ is identified with $\mathbf{R}^{n^2}$. Write $A(0) = A_0$. A *k-parameter deformation of* A_0 is the germ at $0 \in \mathbf{R}^k$ of such a family. We denote the germ by $\bar{A}$ and a representative of it by A. The word *unfolding* is sometimes used in place of the word deformation.

We shall restrict our attention to the class of matrices $A_0 \in GL^+(\mathbf{R}^n)$ where $GL^+(\mathbf{R}^n)$ is the image of the set of all $n \times n$ matrices, $M(n, n)$ under the standard exponential map, exp: $M(n, n) \to GL(\mathbf{R}^n)$. Matrices

Received July 6, 1979.

in the class $GL^+(\mathbf{R}^n)$ are of course those associated with the solution curves in linear dynamical systems. It therefore seems a natural restriction to introduce; moreover various technical complications and 'special' cases are avoided.

If $A_0 \in GL^+(\mathbf{R}^n)$, the orbit system of A_0 in $\mathbf{R}^n$ is the set of A_0-invariant curves (called orbits) in $\mathbf{R}^n$. Explicitly an A_0-invariant curve through $y \in \mathbf{R}^n$ is given by $t \rightarrow A_0{}^t y$ $(t \in \mathbf{R})$ where $A_0{}^t = \exp(tQ_0)$ and $A_0 = \exp Q_0$. The orbit system is precisely the set of solution curves to the system $dy/dt = Q_0 y$.

Let $A: U \rightarrow GL^+(\mathbf{R}^n)$ be a k-parameter family of matrices; denote by $\Sigma^A(\omega)$ the subset of U such that if $x \in \Sigma^A(\omega)$, A_x has at least one pair of complex conjugate eigenvalues of modulus one. It will be clear later that if A is one of the 'stable' normal forms, $\Sigma^A(\omega)$ is nowhere dense in U.

Remark. The class of matrices with more than one pair of complex conjugate eigenvalues of modulus one were those that kept back Kuiper and Robbin [**8**] from a complete topological classification of linear maps. However, the associated problem for linear dynamical systems was completely solved by Kuiper [**9**] (the corresponding matrices are those having at least one pair of pure imaginary eigenvalues).

Two k-parameter deformations of $A_0(=B_0)$, $\bar{A}$ and $\bar{B}$ are said to be *topologically equivalent* if for some neighbourhood U of 0 in $\mathbf{R}^k$, there exists a fibre-preserving homeomorphism $\Phi: U \times \mathbf{R}^n \rightarrow U \times \mathbf{R}^n$ such that for $x \in U - \Sigma(\omega)$,

$$B_x = \Phi_x \cdot A_x \cdot \Phi_x{}^{-1}$$

and for $x \in \Sigma(\omega)$ $(=\Sigma^A(\omega) = \Sigma^B(\omega))$, Φ_x takes orbits of A_x to orbits of B_x. On $\{0\} \times \mathbf{R}^n$, Φ_0 is required to be the identity map. In the linear equivalence of deformations of Arnol'd Φ_x is a similarity transformation for all $x \in U$. Let ϕ: $(V, 0) \rightarrow (U, 0)$ be a smooth map from a neighbourhood of 0 in $\mathbf{R}^n$ to a neighbourhood of 0 in $\mathbf{R}^k$. The m-parameter family induced from the family $A: U \rightarrow GL(\mathbf{R}^n)$ by ϕ, $\phi^*(A)$ is defined by the equation $\phi^*(A)(x) = A(\phi(x))$, $x \in V$. The family $\phi^*(A)$ is simply a reparameterisation of A. For example, if A is of the form

$$\begin{bmatrix} 2+x & 0 \\ 0 & 3+x^2 \end{bmatrix}, x \in \mathbf{R}$$

and $\phi: \mathbf{R}^2 \rightarrow \mathbf{R}$ is given by $\phi(x, y) = x + y$ then $\phi^*(A)$ is of the form

$$\begin{bmatrix} 2+(x+y) & 0 \\ 0 & 3+(x+y)^2 \end{bmatrix}.$$

The germ of $\phi^*(A)$ at 0 is called the *deformation induced from* $\bar{A}$ *by* ϕ. A deformation of a matrix A_0 is said to be a *topologically versal* (or just versal if there is no likelihood of ambiguity) *k-parameter deformation of* A_0, if any deformation of A_0 is topologically equivalent to a deformation induced from $\bar{A}$ by some smooth map ϕ. Thus a versal family is equivalent up to parameterisation, to any other deformation of A_0. It is said to be *miniversal* if k is minimal with respect to topological versality. Imprecisely it is the 'smallest' family containing at least one element from each topological equivalence class of matrices near A_0. The partition of the parameter space determined by these topological equivalence classes of matrices near A_0 is called the *bifurcation diagram* of A_0. The miniversal deformations provide us with the 'normal forms' to which matrices can be topologically reduced in a stable manner. That the reduction is stable follows from our main theorem.

$GL^+(\mathbf{R}^n)$ is partitioned into equivalence classes called *topological orbits* by the two non-intersecting equivalence relations of topological conjugacy and orbital equivalence. By abuse of language, a matrix is said to be *of codimension p* if its topological orbit is a submanifold of $GL(\mathbf{R}^n)$ of codimension p. In Section 3, we establish which classes of matrices are of codimension $\leqq 2$. It is conjectured that the partition of $GL^+(\mathbf{R}^n)$ into topological orbits defines a Whitney stratification of $GL^+(\mathbf{R}^n)$. For $n = 2$ or 3 it is easily verified and certainly holds for orbits of codimension $\leqq 2$.

We are now able to state our main theorem.

MAIN THEOREM. *If the topological orbit $\theta(A_0)$ of A_0 is a submanifold in $GL^+(\mathbf{R}^n)$ of codimension at most two, then a deformation of A_0 is topologically versal if and only if it is transversal to $\theta(A_0)$.*

The theorem establishes the stability of versal deformations. For, any versal deformation of A_0 is transversal to $\theta(A_0)$ and trivially remains transversal under small perturbations and, therefore, versal. Moreover, the versal deformation and its perturbation are, up to a smooth invertible change of parameters, topologically equivalent. Thus we have established the 'stability of the reduction' to versal (normal) form.

Remark. In our definition of topological equivalence of deformations of A_0, it would be more natural to require either that Φ_x be a topological conjugacy between B_x and A_x for all $x \in U$ or that Φ_x preserves orbits for all $x \in U$. The former is rejected for reasons indicated in the introduction. The latter leads to severe technical difficulties in the proof that 'versal implies transversal' in the main theorem. Nevertheless, I conjecture that the main theorem dealing with the classification of versal deformations of matrices relative to the notion of topological equivalence defined above, is still true relative to the more general notion of 'orbital equivalence'. The following informal remark is offered as intuitive justification for the conjecture. Any homeomorphism in a sequence of orbit-preserving homeomorphisms would appear to need a certain degree of 'regularity' or 'niceness' (possibly only relative to the other homeomorphisms in the sequence) in order for the limit of that sequence of homeomorphisms to be itself an orbit-preserving homeomorphism, especially if the orbit systems preserved by the limit homeomorphism are topologically distinct from those preserved by the other homeomorphisms in the sequence. Requiring that each homeomorphism in the sequence also conjugates the matrices giving rise to the orbit systems would provide (possibly) just that degree of 'regularity' or 'niceness' needed, though in general, of course, that might not be enough.

3. Matrices of codimension at most two. The following lemma established in [**5**] is useful in determining the codimension of a matrix. Recall that if a deformation $\bar{A}$ is linearly versal, then trivially $\bar{A}$ is topologically versal.

LEMMA 1. *Let A be a linearly miniversal deformation of A_0 with representative $A: U \to GL(\mathbf{R}^n)$, $0 \in U \subset \mathbf{R}^k$. If $U_T \subset U$ is a submanifold of codimension p in U, where $U_T = \{x \in U | A(x) \text{ is of topological type } T\}$, then the germ at A_0 of the set of all matrices of type T is the germ of a submanifold of codimension p in $GL(\mathbf{R}^n)$.*

In other words to show that a certain topological orbit $\theta(A_0)$ is a submanifold of codimension k, it is sufficient to consider a representative A of the Arnol'd normal form for a deformation $\bar{A}$ of A_0 and establish that there is a smooth k-parameter 'subfamily' embedded in A corresponding to an open subset of $\theta(A_0)$.

A matrix $A_0 \in GL^+(\mathbf{R}^n)$ is said to be *hyperbolic* if all of its eigenvalues are of modulus different from one. A_0 is said to be of type $1^i \times \omega^j \times \text{hyp.}$ $(i, j = 0, 1, 2)$ if it has just one $i \times i$ Jordan block with associated eigenvalue one and just one $j \times j$ complex Jordan block with associated complex eigenvalue of modulus one; the remaining eigenvalues of A_0 are of modulus different from one. The real form for a 2×2 complex Jordan block with complex eigenvalue $\omega = x + iy$ is the 4×4 real matrix

$$\begin{bmatrix} x & 1 & -y & 0 \\ 0 & x & 0 & -y \\ y & 0 & x & 1 \\ 0 & y & 0 & x \end{bmatrix}$$

The following lemma further simplifies the problem of classifying the elements of $GL^+(\mathbf{R}^n)$ up to topological equivalence. It also offers a considerable simplification to the proof of the main theorem. Essentially, the lemma allows us to 'forget about' the hyperbolic part of A_0 and its deformations.

LEMMA 2. *Let $A_0 \in GL^+(\mathbf{R}^n)$ be a matrix of the form $\begin{bmatrix} B_0 & 0 \\ 0 & H_0 \end{bmatrix}$ where B_0*

is a $p \times p$ matrix with eigenvalues of modulus one and where H_0 is hyperbolic. Then any k-parameter deformation $\bar{A}$ of A_0 is topologically equivalent to a deformation of the form $\begin{bmatrix} \bar{B} & 0 \\ 0 & H_0 \end{bmatrix}$ *where $\bar{B}$ is a k-parameter deformation of $B_0 \in GL^+(\mathbf{R}^k)$.*

Proof. Arnol'd proves in [**1**] that any deformation of a matrix A_0 stably reduces to a deformation in block diagonal form. Each block corresponds to an eigenvalue of A_0. Rearranging these blocks, so that those corresponding to eigenvalues of A_0 of modulus one precede those blocks corresponding to eigenvalues of modulus different from one, we have that A is linearly equivalent (up to reparameterisation) to a family C of the form

$$\begin{bmatrix} B_x & 0 \\ 0 & H_x \end{bmatrix}, \quad x \in U \subset \mathbf{R}^k$$

where H_x is the block of hyperbolic blocks and B_x is the block of non-hyperbolic blocks. Now using the absolute structural stability of hyperbolic endomorphisms [**10**], we can topologically reduce the family C to the form claimed in the lemma.

COROLLARY 1. *$\bar{A}$ is a topologically versal deformation of A_0 if and only if $\bar{B}$ is a topologically versal deformation of B_0.*

COROLLARY 2. *$\bar{A}$ is transversal to $\theta(A_0)$ if and only if $\bar{B}$ is transversal to $\theta(B_0)$.*

Proof. From the lemma, it is clear that the codimension of $\theta(A_0)$ equals the codimension of $\theta(B_0)$. It is sufficient to show that A is transversal to $\theta(A_0)$ at 0 if and only if C is transversal to $\theta(A_0)$ at 0.

In some neighbourhood U of 0 in $\mathbf{R}^k$

$$C_x = P_x A_x P_x^{-1}$$

where $x \in U$ and $P_x \in GL(\mathbf{R}^n)$. Taking the associated tangent bundle maps of both sides at $x = 0$, we have

$$C_0^* = A_0^* + [P_0^*, A_0].$$

Since $[P_0^*, A_0]$ maps vectors in the tangent space at $0 \in U$, to vectors in the tangent space to $\theta(A_0)$ at A_0, it follows immediately that C is transversal to $\theta(A_0)$ at 0 if and only if A is transversal to $\theta(A_0)$ at 0.

LEMMA 3. *The following table gives the complete list of matrices of codimension at most two:*

Codimension	0	1	2
Topological Type	*hyp.*	$1 \times$ *hyp.* $\omega \times$ *hyp.*	$1 \times \omega \times$ *hyp.* $1^2 \times$ *hyp.*

Proof. From Lemma 1, it is sufficient to give a linearly miniversal deformation of the appropriate matrices. In view of Lemma 2 we can ignore the hyperbolic parts of the matrices considered. The hyp. and $1 \times$ hyp. cases are then trivial.

Consider a matrix A_0 of type ω. A representative of a linearly miniversal deformation of A_0 is of the form:

$$\begin{bmatrix} a + x_1 & b + x_2 \\ -(b + x_2) & a + x_1 \end{bmatrix}$$

where $\omega = a \pm ib$ and $(x_1, x_2) \in U \subset \mathbf{R}^2$. The subset of U associated with matrices of the same topological type as A_0 is given by the equation

$$(a + x_1)^2 + (b + x_2)^2 = 1$$

which clearly defines a submanifold of codimension one in U.

Consider A_0 of type $\omega \times 1$. A representative of a linearly miniversal 3-parameter deformation of A_0 is of the form:

$$\begin{bmatrix} a + x_1 & b + x_2 & 0 \\ -(b + x_2) & a + x_1 & 0 \\ 0 & 0 & 1 + x_3 \end{bmatrix}$$

where $\omega = a \pm ib$, and $(x_1, x_2, x_3) \in U \subset \mathbf{R}^3$. The subset of U corresponding to matrices of type $\omega \times 1$ is given by the set

$$\{(x_1, x_2, x_3) | (a + x_1)^2 + (b + x_2)^2 = 1 \text{ and } x_3 = 0\}$$

which clearly defines a submanifold of codimension two in $\mathbf{R}^3$.

Consider A_0 of type 1^2. A representative of a linearly miniversal deformation of A_0 is of the form:

$$\begin{bmatrix} 1 & 1 \\ x_1 & 1 + x_2 \end{bmatrix}$$

where $(x_1, x_2) \in U \subset \mathbf{R}^2$. The subset of U corresponding to matrices of the same topological type of A_0 is trivially the set $\{(x_1, x_2) | x_1 = x_2 = 0\}$. Thus 1^2 is of codimension two.

That there are no other matrices of codimension two may easily be checked. We take as an example one possible candidate and show it is of codimension three. Let A_0 be of type $\omega_1 \times \omega_2$ where $\omega_1 \neq \omega_2$ or $\bar{\omega}_2$; i.e., A_0 has two distinct complex conjugate pairs of eigenvalues of modulus one. For simplicity consider A_0 in $GL(\mathbf{C}^2)$; then a representative of a linearly miniversal deformation of A_0 is of the form:

$$\begin{bmatrix} \omega_1 + z_1 & 0 \\ 0 & \omega_2 + z_2 \end{bmatrix}$$

where $(z_1, z_2) \in U \subset \mathbf{C}^2$. The subset of U corresponding to matrices with two distinct eigenvalues of modulus one is given by the torus

$$\{(z_1, z_2) \in U| \quad |\omega_1 + z_1| = 1 \quad \text{and} \quad |\omega_2 + z_2| = 1\}.$$

The subset of the torus corresponding to matrices of the same 'orbital type' as A_0 is a curve given by the equation

$$\operatorname{Arg}(\omega_1 + z_1) = k \operatorname{Arg}(\omega_2 + z_2)$$

(see [**9**]) for some real constant $k \neq 0$. Hence A_0 is of codimension 3.

The main result of the paper proved in the next section relies heavily on the following two lemmas together with various technical extensions and modifications of them. The first is proved, in a slightly different form in [**3**] and the second is proved using similar arguments.

LEMMA 4. *Let $\bar{A}$ and $\bar{B}$ be one-parameter deformations of $A_0 = \mathrm{Id}_{\mathbf{R}}$ with representatives $A, B: U \rightarrow GL(\mathbf{R})$, $0 \in U \subset \mathbf{R}$. If A_x is topologically conjugate to B_x for all $x \in U$, then $\bar{A}$ and $\bar{B}$ are topologically equivalent deformations of A_0 if and only if for some neighbourhood $V \subset U$ of 0, the function $c: V - \{0\} \rightarrow \mathbf{R}^+$ extends over V where*

$$c(x) = \frac{\log A(x)}{\log B(x)}.$$

LEMMA 5. *Let $\bar{A}$ and $\bar{B}$ be one parameter deformations of $A_0 = \omega \in GL(\mathbf{C})$ where $\omega = e^{i\beta}$, $\beta \neq 0$ or π, with representatives $A, B: U \rightarrow GL(\mathbf{C})$, $0 \in U \subset \mathbf{R}$. If A_x is topologically conjugate to B_x for all $x \in U - \{0\}$, then $\bar{A}$ and $\bar{B}$ are topologically equivalent deformations of A_0 if and only if for some neighbourhood $V \subset U$ of 0, the function $c_1: V - \{0\} \rightarrow \mathbf{R}^+$ extends over V where*

$$c_1(x) = \frac{\log \|A(x)\|}{\log \|B(x)\|}.$$

The significance of the function c (or c_1) will be appreciated once it has been noted that a 'typical' homeomorphism conjugating two linear maps on $\mathbf{R}$, for example, $f(y) = \lambda_1 y$ and $g(y) = \lambda_2 y$ ($\lambda_1, \lambda_2 > 1$) is of the form $y \rightarrow y^c$ where $c = \log \lambda_1/\log \lambda_2$.

4. Proof of the main theorem. The proof that 'transversal implies versal' given in [**5**] is false. The proof contains the appealing but false statement that $\alpha \cdot A \cdot \phi = \alpha \cdot B$ implies $A \cdot \phi = B$. The correct proof actually follows quickly after the construction of the map ϕ mentioned above (though the proof does not generalise to arbitrary codimension as suggested in [**5**]).

Proof of Theorem 1 (i). *Transversal Deformations are Versal.* Let $\bar{A}$ be a deformation of A_0 transversal to $\theta(A_0)$ with representative $A: U \rightarrow$

$GL^+(\mathbf{R}^n)$, $0 \in U \subset \mathbf{R}^k$. To show $\bar{A}$ is topologically versal, it is sufficient to show any linearly miniversal deformation of A_0 is topologically equivalent to $\bar{A}$ (possibly after reparameterisation). In fact we shall show that there is a reparameterisation of A that allows us to assume $\bar{A}$ is a p-parameter deformation of A_0 where p is the codimension of $\theta(A_0)$ ($p = 0, 1, 2$). More precisely we construct a map ϕ: $V \rightarrow U$, $0 \in V \subset \mathbf{R}^p$ such that $\overline{\phi^*(A)}$ is a p-parameter deformation of A_0 that is transversal to $\theta(A_0)$ at 0. The versality of $\overline{\phi^*(A)}$ implies the versality of $\bar{A}$.

Since $\theta(A_0)$ is a submanifold of codimension p ($p = 0, 1, 2$), there exists a smooth map

$$\alpha: (GL^+(\mathbf{R}^n), A_0) \rightarrow (\mathbf{R}^p, 0)$$

such that α is a submersion at 0 and such that the germ of $\alpha^{-1}(0)$ at A_0 is the germ of $\theta(A_0)$ at A_0. Since A is transversal to $\theta(A_0)$, $\alpha \cdot A$: $U \rightarrow \mathbf{R}^p$ is a submersion at 0 and we may choose local coordinates for $\mathbf{R}^k$ such that

$$\alpha \cdot A(x_1, \ldots, x_k) = (x_1, \ldots, x_p)$$

where $(x_1, \ldots, x_k) \in U$. Let $U' \subset U$ be the set

$$\{(x_1, \ldots, x_k) \in U | x_{p+1} = \ldots = x_k = 0\}.$$

Then $\alpha \cdot A|U'$ is a local diffeomorphism at 0. Let

$$(\alpha \cdot A)|U' = V \subset \mathbf{R}^p \quad \text{and} \quad \phi = (\alpha \cdot A|U')^{-1}.$$

Then $\overline{\phi^*(A)}$ is a deformation of A_0 transversal to $\theta(A_0)$.

From Corollary 1 to Lemma 2, it is sufficient to consider deformations of the non-hyperbolic part of A_0. We therefore assume (i) $A_0 \in GL(\mathbf{R})$ if A_0 is $1 \times$ hyp (ii) $A_0 \in GL(\mathbf{R}^2)$ if A_0 is $1^2 \times$ hyp or $\omega \times$ hyp (iii) $A_0 \in GL(\mathbf{R}^3)$ if A_0 is $\omega \times 1 \times$ hyp. The proof now becomes a straightforward construction of explicit topological equivalences. In fact in the case of $1 \times$ hyp and $1^2 \times$ hyp, the equivalence is the identity. For the remaining cases the reader is referred to the constructions in [**4**].

The proof that 'versal implies transversal' is given in [**5**] for the hyp and $1 \times$ hyp cases. We omit the proof of the '$\omega \times$ hyp' case because of its similarity to the '$1 \times$ hyp' case. The remaining two cases are dealt with below.

Proof of Theorem 1(ii). *Versal deformations are Transversal.*

(a) $\omega \times 1 \times$ hyp case. In view of Lemma 2, it is sufficient to consider $A_0 \in GL(\mathbf{R}^3)$ of type $\omega \times 1$. Let A: $U \rightarrow GL(\mathbf{R}^3)$, $0 \in U \subset \mathbf{R}^k$ be a representative of a versal deformation $\bar{A}$ of A_0. Let $\bar{B}$ be a 2-parameter deformation of A_0, transversal to $\theta(A_0)$ with representative B: $V \rightarrow GL(\mathbf{R}^3)$, $0 \in V \subset \mathbf{R}^2$. By an argument similar to that used in Lemma 2, Corollary 2, it may be shown that it is sufficient to consider A in a linearly

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reduced form. Precisely we assume A is of the form

$$\begin{bmatrix} a_1(x) & (1 + a_2(x)) & 0 \\ -(1 + a_2(x)) & a_1(x) & 0 \\ 0 & 0 & 1 + a_3(x) \end{bmatrix}, x \in U \subset \mathbf{R}^k.$$

We take B in a particularly simple form. Namely B is of the form:

$$\begin{bmatrix} x_1 & 1 + x_1 & 0 \\ -(1 + x_1) & x_1 & 0 \\ 0 & 0 & 1 + x_2 \end{bmatrix} (x_1, x_2) \in V \subset \mathbf{R}^2.$$

The versality of A now implies there exists a smooth map $\phi: V \to U$ such that $\phi^*(A)$ is topologically equivalent to B. Notice that this immediately implies $\phi^*(A)$ and B have the same bifurcation diagrams. Thus

$$\Sigma^{\phi^*(A)}(1) = \Sigma^B(1)(=\Sigma(1)) \quad \text{and} \quad \Sigma^{\phi^*(A)}(\omega) = \Sigma^B(\omega)(=\Sigma(\omega)).$$

It follows from Lemmas 4 and 5 that this topological equivalence between $\phi^*(A)$ and B exists only if the functions $c: V - \Sigma(1) \to \mathbf{R}^+$ and $c_1: V - \Sigma(\omega) \to \mathbf{R}^+$ extend over V where

$$c(x) = \log(1 + a_3(\phi(x_1, x_2)))/\log(1 + x_2)$$

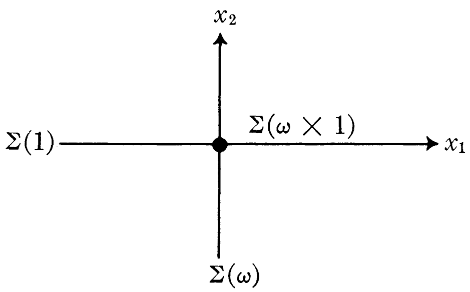
and

$$c_1(x) = \log[a_1^2(\phi(x_1, x_2)) + (1 + a_2(\phi(x_1, x_2)))^2]/ \log(x_1^2 + (1 + x_1)^2).$$

Both c and c_1 are ratios of logarithms of moduli of eigenvalues. It is then shown that these conditions on c and c_1 imply the transversality of A. We choose axes in $GL(\mathbf{R}^3)$ at A_0 such that $\phi^*(A)$ and B can be regarded as the maps $V \to \mathbf{R}^3$ given by:

$$\phi^*(A)(x_1, x_2) = (a_1(\phi(x_1, x_2)), a_2(\phi(x_1, x_2)), a_3(\phi(x_1, x_2)))$$

and $B(x_1, x_2) = (x_1, x_1, x_2)$.



Bifurcation diagram for $\phi^*(A)$ and B

On $\Sigma(1)$, $a_3(\phi(x_1, 0)) = 0$, which implies

$$\left(\frac{\partial a_3 \cdot \phi}{\partial x_1}\right) = 0.$$

Since the functions c and c_1 extend over V, $\lim_{x_2 \to 0} c(x)$ and $\lim_{x_1 \to 0} c_1(x)$ exist and are non-zero. By an elementary application of L'Hopital's rule it follows

$$\left(\frac{\partial a_1 \cdot \phi}{\partial x_1}\right)_{(0,0)} \neq 0 \quad \text{or} \quad \left(\frac{\partial a_2 \cdot \phi}{\partial x_1}\right)_{(0,0)} \neq 0$$

and that

$$\left(\frac{\partial a_3 \cdot \phi}{\partial x_3}\right)_{(0,0)} \neq 0.$$

It is clear that the map $A: V \to GL(\mathbf{R}^3)$ is an embedding at 0. Hence A is transversal to $\theta(A_0)$ at 0.

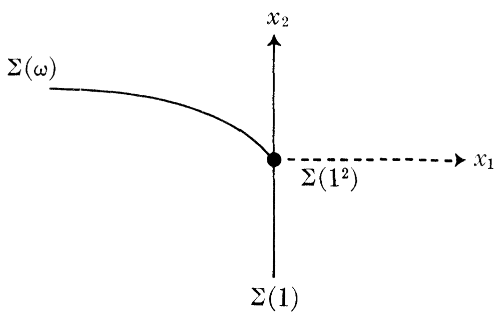
(b) $1^2 \times$ hyp case. From Lemma 2, it is sufficient to consider $A_0 \in GL(\mathbf{R}^2)$ of type 1^2. Let $A: U \to GL(\mathbf{R}^2)$, $0 \in U \subset \mathbf{R}^k$ be a representative of a versal deformation $\bar{A}$ of A_0. Let $\bar{B}$ be a 2-parameter deformation of A_0 transversal to $\theta(A_0)$ with representative $B: V \to GL(\mathbf{R}^2)$, $0 \in V \subset \mathbf{R}^2$. We may assume that A is in a linearly reduced form. We take A and B to be in the following respective forms:

$$\begin{bmatrix} 1 & 1 \\ a_1(x) & 1 + a_2(x) \end{bmatrix}, \quad x \in U \subset \mathbf{R}^k,$$

$$\begin{bmatrix} 1 & 1 \\ x_1 & 1 + x_2 \end{bmatrix}, \quad (x_1, x_2) \in V \subset \mathbf{R}^2.$$

Since $\bar{A}$ is versal, there exists a smooth map ϕ: $V \to U$ such that $\phi^*(A)$ and B are topologically equivalent. We have immediately that

$$\Sigma^{\phi^*(A)}(1) = \Sigma^B(1)(= \Sigma(1)) \quad \text{and} \quad \Sigma^{\phi^*(A)}(\omega) = \Sigma^B(\omega)(= \Sigma(\omega)).$$



Bifurcation diagram for $\phi^*(A)$ and B.

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In order to prove $\bar{A}$ is transversal to $\theta(A_0)$, it is sufficient to regard $\phi^*(A)$ and B as maps from V into $\mathbf{R}^2$ and show that

$$\left(\frac{\partial a_1 \cdot \phi}{\partial x_1}\right)_{(0,0)} \neq 0 \quad \text{and} \quad \left(\frac{\partial a_2 \cdot \phi}{\partial x_2}\right)_{(0,0)} \neq 0.$$

Notice that $a_1 \cdot \phi$ is zero on $\Sigma(1)$ and hence

$$\left(\frac{\partial a_1 \cdot \phi}{\partial x_2}\right)_{(0,0)} = 0.$$

The general form for a conjugacy Φ between $\phi^*(A)$ and B on $V - (\Sigma(1) \cup \Sigma(\omega))$ is very complicated. However, certain subspaces of $\mathbf{R}^2$ may be assumed 'invariant' for each parameter value, and the general form for Φ on these subspaces considered. For $x = (x_1, x_2) \in V$ such that $x_1 \geqq 0$, both $\phi^*(A)_x$ and B_x have one eigenvalue greater than one and one eigenvalue less than one. Since the eigenspaces for $\phi^*(A)_x$ and B_x merge as $x \rightarrow 0$, we make the reasonable assumption that the eigenspaces for $\phi^*(A)_x$ and B_x coincide for $x \in V^+ = \{x \in V | x_1 \geqq 0\}$. Then by considering Φ_x on the eigenspaces in $\mathbf{R}^2$, it may be shown, using Lemma 4, that $\phi^*(A)$ and B are topologically equivalent only if the functions

$$c_1 \colon V^+ - \Sigma(1) \rightarrow \mathbf{R}^+ \quad \text{and} \quad c_2 \colon V^+ - \Sigma(1) \rightarrow \mathbf{R}^+$$

extend over V^+ where

$$c_1(x) = \frac{[\log\,(1 + a_2(\phi(x)) - (a_2(\phi(x))^2 + a_1(\phi(x)))^{1/2}]}{\log\,(1 + x_2 - (x_2^{\,2} + x_1)^{1/2})}$$

$$c_2(x) = \frac{[\log\,(1 + a_2(\phi(x)) + (a_2(\phi(x))^2 + a_1(\phi(x)))^{1/2}]}{\log\,(1 + x_2 + (x_2^{\,2} + x_1)^{1/2})}.$$

As usual, $c_1(x)$ and $c_2(x)$ are the ratios of the logarithms of the eigenvalues of $\phi^*(A)_x$ and B_x. We show that $\lim_{x\rightarrow 0^+} c_1(x)$ and $\lim_{x\rightarrow 0^+} c_2(x)$ exist and are non-zero only if

$$\left(\frac{\partial a_1 \cdot \phi}{\partial x_1}\right)_{(0,0)} \neq 0 \quad \text{and} \quad \left(\frac{\partial a_2 \cdot \phi}{\partial x_2}\right)_{(0,0)} \neq 0,$$

which then means that A is transversal to $\theta(A_0)$. The problem here is complicated by the fact that the eigenvalues are not smooth functions of the parameters.

Claim. If $\lim_{x\rightarrow 0^+} c_1(x)$ exists and is non-zero then

$$\left(\frac{\partial a_1 \cdot \phi}{\partial x_1}\right)_{(0,0)} \neq 0$$

Proof of Claim. Write

$$a_1(\phi(x_1, 0)) = a(x) \quad \text{and} \quad a_2(\phi(x_1, 0)) = b(x).$$

Then

$$\lim_{x\to 0^+} c_1(x) = \lim_{x\to 0^+} \frac{\log\,(1 + b(x) + (b^2(x) + a(x))^{1/2})}{\log\,(1 + x^{1/2})}$$

which, using L'Hopital's rule, equals

$$\lim_{x\to 0^+} \frac{2b'(x) - (b^2(x) + a(x))^{-1/2}(2b(x)b'(x) + a'(x))}{x^{-1/2}}$$

$$= \lim_{x\to 0^+} x^{1/2}b'(x) - \frac{b'(x)x^{1/2}}{\left(1 + \dfrac{a(x)}{b^2(x)}\right)^{1/2}} - \frac{a'(x)x^{1/2}}{a^{1/2}(x)\left(1 + \dfrac{b^2(x)}{a(x)}\right)^{1/2}}.$$

The first two terms in the expression vanish as $x \to 0$ (note that $a(x) > 0$ for $x > 0$). The third term also vanishes unless

$$\lim_{x\to 0^+}(a'(x))^2 x/a(x) \quad \text{exists and is non-zero.}$$

Since the limit is non-zero, there is a neighbourhood of 0 in which $a'(x) \neq 0$ for all $x \neq 0$, $a(x) > 0$ for $x > 0$, hence we may apply L'Hopital's rule and conclude the limit is non-zero only if $a'(0) \neq 0$. Thus

$$\left(\frac{\partial a_1 \cdot \phi}{\partial x_1}\right)_{(0,0)} \neq 0.$$

The proof that if $\lim_{x\to 0^+} c_2(x) \neq 0$ then

$$\left(\frac{\partial a_2 \cdot \phi}{\partial x_2}\right)_{(0,0)} \neq 0,$$

is much simpler. For

$$\lim_{x\to 0^+} c_2(x) = \lim_{x_2\to 0^+} \log(1 + 2a_2(\phi(0, x_2)))/\log(1 + 2x_2)$$

and this limit exists and is non-zero only if

$$\left(\frac{\partial a_2 \cdot \phi}{\partial x_2}\right)_{(0,0)} \neq 0.$$

This completes the proof of the main theorem.

COROLLARY 1. *The miniversal forms for deformations of matrices A_0 of*

https://doi.org/10.4153/CJM-1981-015-2 Published online by Cambridge University Press

type $1 \times$ hyp, $\omega \times$ hyp, $\omega \times 1 \times$ hyp, $1^2 \times$ hyp *are respectively:*

$$\begin{bmatrix} 1+x & 0 \\ 0 & H_0 \end{bmatrix}, \quad \begin{bmatrix} x & 1+x & \\ -(1+x) & x & 0 \\ & 0 & H_0 \end{bmatrix},$$

$$\begin{bmatrix} x & 1+x & 0 & \\ -(1+x) & x & 0 & 0 \\ 0 & 0 & 1+y & \\ & 0 & & H_0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ x & 1+y & \\ 0 & & H_0 \end{bmatrix},$$

where H_0 is the hyperbolic part of A_0.

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