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GROWTH ESTIMATES IN LINEAR ELASTICITY WITH A SUBLINEAR BODY FORCE WITHOUT DEFINITENESS CONDITIONS ON THE ELASTICITIES

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1. Introduction

In this paper, we study the boundary-initial value problem for a linear elastic body in a bounded domain, when the body force depends on the displacement vector **u** in a sublinear way.

Recently, much attention has been given to nonlinear body forces not only to study the fundamental solutions of the equations governing linear elastodynamics, see e.g. Kecs [**3**], but also to derive global non existence results in abstract problems with direct applications to nonlinear heat diffusion or to the nonlinear wave equation, see e.g. Ball [**1**], Levine and Payne [**10**].

In this context, Slemrod [**13**] has shown how such forced systems may be regarded as distributed control problems.

The boundary-initial value problem is therefore

$$\frac{\partial^2 u_i}{\partial t^2}-(a_{ijkh}(\mathbf{x})u_{k,h})_{,j}=f_i(\mathbf{u}),(\mathbf{x},t)\in\Omega\times(0,T)$$

$$u_i(\mathbf{x},t)=0 \qquad\qquad ,(\mathbf{x},t)\in\Gamma\times[0,T)$$

$$u_i(\mathbf{x},0)=u_0^i(\mathbf{x}),\frac{\partial u_i}{\partial t}(\mathbf{x},0)=v_0^i(\mathbf{x}),\mathbf{x}\in\Omega \tag{1.1}$$

In these equations, Ω is the three-dimensional elastic body bounded by a smooth surface Γ, $[0,T)$ is the maximal interval of existence, u_i are the cartesian components of the displacement vector about a reference configuration; $a_{ijkh}(\mathbf{x})$ are the elasticities which we suppose to satisfy the major symmetry condition namely $a_{ijkh}=a_{khij}$, f_i is the body force and u_0^i, v_0^i are prescribed functions; standard indicial notation is employed for spatial derivatives.

Let $G(\mathbf{u})$ be the potential associated with **f** such that ([**7**]):

$$\frac{d}{dt}G(\mathbf{u})=\int_\Omega\frac{\partial u_i}{\partial t}f_i(\mathbf{u})\,d\mathbf{x} \tag{1.2}$$

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It is convenient to let also $\|\cdot\|$, $\langle,\rangle$ denote the norm and the inner product on $(L^2(\Omega))^3$ and to let $A(\cdot,\cdot)$ denote the (symmetric) bilinear form generated by a_{ijkh}, i.e.

$$A(\mathbf{u},\mathbf{v})=\int_\Omega a_{ijkh}u_{i,j}v_{k,h}\,dx. \tag{1.3}$$

For a sufficiently regular solution (and we assume that such a solution exists, at least locally), we may easily establish the following energy equation, appropriate to (1.1):

$$E(t)=\tfrac{1}{2}\|\mathbf{u}'\|^2+\tfrac{1}{2}A(\mathbf{u},\mathbf{u})-G(\mathbf{u})=E(0) \tag{1.4}$$

where

$$'\equiv\frac{\partial}{\partial t}.$$

In [7], the writers showed that, for A positive definite, $E(0)\leqq 0$ and functions $\mathbf{f}, G$ satisfying the "sublinear" restriction

$$\langle\mathbf{u},\mathbf{f}\rangle+2(2\alpha-1)G(\mathbf{u})\leqq\mu\|\mathbf{u}\|^{2\gamma} \tag{1.5}$$

where α,γ,μ are real constants such that $\alpha>1$, $0<\gamma<1$, $\mu>0$, the solution to (1.1) decays to zero in a finite time.

Such behaviour is typical of sublinear equations, see e.g. Payne [**12**] and so we here turn our attention to the case of sublinear **f**, but without any requirement of sign definitess on the elasticities a_{ijkh}.

We find that, for a_{ijkh} indefinite, it is possible to exhibit specific conditions on the initial data which insure the solution will grow and lower bounds are determined for the appropriate growth results.

Since the linear problem is also unstable for indefinite a_{ijkh} (see [**5**]), it is perhaps not surprising that we again find unbounded behaviour. However, we are able to accurately assess the effect of the sublinear term **f** on the growth rate as compared with the linear results of Knops and Payne [**5**].

Finally, we observe that many writers have studied linear elastodynamics when the strain energy is not positive definite bounded below, that is when the elasticities do not satisfy the inequality $A(\mathbf{u},\mathbf{u})\geqq k\|\mathbf{u}\|^2, k>0$, much interest having been in determining the growth behaviour of solutions, see e.g. [**4, 6, 7, 9, 11**].

Galdi and Rionero [**2**] have also modified logarithmic convexity arguments to establish (Hölder) continuous data dependence in linear elastodynamics on exterior unbounded domains, again without any definitess assumption on the elasticities.

We add that, although we restrict ourselves to strong solutions to (1.1), it is easy to generalize the ideas of this note to weak solutions or to certain abstract differential equations in Hilbert spaces as considered in [**5, 7**].

2. Growth results

We put:

$$F(t)=\|\mathbf{u}(t)\|^2 \tag{2.1}$$

and then consider the class of sublinear non linearities satisfying the restriction

$$2G(\mathbf{u})-\langle\mathbf{u},f\rangle\leqq\mu F^{\gamma} \tag{2.2}$$

for some real μ, γ, with $\mu>0$, $0<\gamma<1$.

Remark 1. Note that (2.2) is compatible with (1.5) in many situations; for example, when $\mathbf{f}=\mathbf{u}|\mathbf{u}|^{p-1}, 0<p<1$, it is possible to find an appropriate choice of the constants so that both (1.5) and (2.2) are satisfied. It is perhaps noteworthy to observe that, if (1.5) and (2.2) hold simultaneously and $A(\mathbf{u},\mathbf{u})$ is positive definite bounded below, then it is easy to show that $\|\mathbf{u}\|^2$ remains bounded. This result follows directly from the energy equation by using the mentioned conditions and by an application of Hölder's inequality.

We now establish the following theorem:

Theorem. *Let* $\mathbf{u}$ *be a classical solution to* (1.1) *with* $\mathbf{f}$ *and* G *satisfying the relation* (2.2). *Then, we have*:

(a) *If* $E(0)\leqq 0$ *and* $F'(0)>2[\mu/(1-\gamma)]^{1/2}F^{(1+\gamma)/2}(0)$, $F(t)$ *is bounded below by an increasing exponential function for all* $t>0$.
(b) *If* $E(0)>0$ *and* $F'(0)>B$, *where* $B=[8E(0)F(0)+4\mu F^{1+\gamma}(0)/(1-\gamma)]^{1/2}$, *then the conclusion of the preceding statement holds.*
(c) *If* $E(0)>0$ *and* $F'(0)=B$, $F(t)$ *is bounded below by a polynomial of order greater than* 2.

Proof. The proof employs logarithmic convexity arguments in the vein of Knops and Payne [5]. By computation:

$$F'=2\langle\mathbf{u},\mathbf{u}'\rangle \tag{2.3}$$

$$F''=2\langle\mathbf{u},\mathbf{u}''\rangle+2\|\mathbf{u}'\|^2=2\|\mathbf{u}'\|^2+2\langle\mathbf{u},\mathbf{f}\rangle-2A(\mathbf{u},\mathbf{u}) \tag{2.4}$$

as can be readily seen from (1.1). Substitute now in (2.4) for $A(\mathbf{u},\mathbf{u})$ using (1.4) to find that:

$$F''=4\|\mathbf{u}'\|^2+2\langle\mathbf{u},\mathbf{f}\rangle-4G(\mathbf{u})-4E(0) \tag{2.5}$$

Hence, from (2.3) and (2.5), we may form the expression:

$$\begin{aligned}FF''-(F')^2&=4\|\mathbf{u}\|^2\|\mathbf{u}'\|^2+2F\langle\mathbf{u},\mathbf{f}\rangle-4FG(\mathbf{u})-4FE(0)\\&\quad-4\langle\mathbf{u},\mathbf{u}'\rangle^2\geqq 2F[\langle\mathbf{u},\mathbf{f}\rangle-2G(\mathbf{u})]-4FE(0)\end{aligned} \tag{2.6}$$

where the term $4\|\mathbf{u}\|^2\|\mathbf{u}'\|^2-4\langle\mathbf{u},\mathbf{u}'\rangle^2$ has been discarded. Next, $F'(0)>0$ and so either

(i) $F'(t)>0\ \forall t>0$

or

(ii) $\exists$ (a smallest) $\eta<\infty$, such that $F'(\eta)=0$.

Suppose the second case. Then, on $[0,\eta)$, we employ (2.2) in (2.6) to derive:

$$(\log F)'' \geqq -4\frac{E(0)}{F} - 2\mu F^{\gamma-1}. \tag{2.7}$$

Let us now turn our attention to the initial conditions of part (a). Thus, we may discard the first term on the right of (2.7), multiply by F'/F and then integrate to find:

$$\left(\frac{F'}{F}\right)^2 \geqq \lambda^2, t\in[0,\eta), \tag{2.8}$$

where

$$\lambda^2 = \left(\frac{F'(0)}{F(0)}\right)^2 - [4\mu/(1-\gamma)]F^{\gamma-1}(0). \tag{2.9}$$

Taking into account the initial conditions of (a), $\lambda^2>0$ and so, by continuity, $F'(\eta)\neq 0$, which is a contradiction. Hence $F'(t)>0\,\forall t$ and, by the same reasoning as above, (2.8) holds $\forall t$ and so:

$$F' \geqq \lambda F \tag{2.10}$$

which leads to the conclusions of part (a) of the theorem.

To establish part (b), we retain all of (2.7). By the same contradiction argument as above, we argue $F'(t)>0$ on $[0,\eta)$ and then multiply (2.7) by F'/F to obtain after integration:

$$\left(\frac{F'}{F}\right)^2 \geqq \left(\frac{F'(0)}{F(0)}\right)^2 - \frac{8E(0)}{F(0)} - \frac{4\mu}{(1-\gamma)}F^{\gamma-1}(0) + \frac{4\mu}{(1-\gamma)}F^{\gamma-1}. \tag{2.11}$$

Thanks to the assumptions of part (b), the right hand side of (2.11) is positive, which implies $F'(t)>0, \forall t$, hence we again deduce an inequality equivalent to (2.10).

Finally to prove also part (c), we begin by observing that (2.11) reduces to the form:

$$\left(\frac{F'}{F}\right)^2 \geqq \frac{4\mu}{1-\gamma}F^{\gamma-1}. \tag{2.12}$$

Employing once more reductio ad absurdum, since $F'(t)>0$ on $[0,\eta)$, certainly $F(\eta)>0$ and so, from (2.12) $F'(\eta)\neq 0$. Therefore (2.12) holds for each $t>0$ and this inequality may be manipulated to yield:

$$F(t) \geqq \{F^{(1-\gamma)/2}(0) + [\mu/(1-\gamma)]^{1/2}t\}^{2/(1-\gamma)} \tag{2.13}$$

which completes the proof of the theorem.

Remark 2. It is worth making some observations on the conclusions of Remark 1 and the results of the Theorem.

Firstly, note that the proof of Remark 1 was carried out for a class of strong solutions $\mathbf{u}$ satisfying more restrictive hypotheses than the ones necessary to establish the Theorem; any strong solution of this class can be expected to be smoother than the ones we require to exist to prove our Theorem, hence the Theorem should hold also for the solutions of the first class, which yields a contradiction.

However no contradiction exists as we can easily show that the solutions of the first class are not compatible with the particular choice of the initial conditions under which the Theorem is true, as follows.

Appendix. Let us begin by examining initial data such that $E(0)\leqq 0$. When the conditions (1.5) and (2.2) hold simultaneously, we restrict attention to functions $G(\mathbf{u})$ satisfying the inequality:

$$2\alpha G(\mathbf{u})\leqq \mu(F(t))^{\gamma}. \tag{A.1}$$

By virtue of the positive definitess of the elasticities and by (A.1), (1.4) gives:

$$\|\mathbf{u}'(t)\|^2\leqq \frac{\mu}{\alpha}(F(t))^{\gamma}, \forall t\geqq 0. \tag{A.2}$$

Consider next $F(0)=\|\mathbf{u}(0)\|^2$. An application of Schwarz's inequality leads to:

$$F'(0)\leqq 2\|\mathbf{u}(0)\|\,\|\mathbf{u}'(0)\|. \tag{A.3}$$

Suppose now that the solutions $\mathbf{u}$ satisfying (A.1) and (A.2) satisfy also the initial conditions stated to prove part (a) of the Theorem.

We have:

$$\frac{4\mu}{(1-\gamma)}\|\mathbf{u}(0)\|^2\|\mathbf{u}(0)\|^{2\gamma}<(F'(0))^2\leqq 4\|\mathbf{u}(0)\|^2\|\mathbf{u}'(0)\|^2 \tag{A.4}$$

hence, we arrive at the inequality:

$$\|\mathbf{u}'(0)\|^2>\frac{\mu}{(1-\gamma)}\|\mathbf{u}(0)\|^{2\gamma}. \tag{A.5}$$

This inequality is compatible with (A.2) only if $\alpha<1-\gamma$, which is not possible as we have considered $\alpha>1$.

Consider next $E(0)>0$ and assume the initial conditions of part (b). From (1.4), for the solutions verifying (A.1) at $t=0$, we obtain:

$$\|\mathbf{u}'(0)\|^2\leqq 2E(0)+\frac{\mu}{\alpha}(F(0))^{\gamma}. \tag{A.6}$$

By the same reasoning as above, the second choice of initial data yields:

$$2E(0)+\frac{\mu}{(1-\gamma)}F(0)^{\gamma}<2E(0)+\frac{\mu}{\alpha}F(0)^{\gamma} \tag{A.7}$$

which leads to the same contradiction argument as in the first case. Analogously, it is possible to prove that the classical solutions considered in Remark 1, violate also the initial conditions of part (c).

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