CONSTRUCTION OF CERTAIN SEMI-SIMPLE GROUPS*

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Introduction. In **(3)**, Chevalley constructed, for every field K and every semi-simple Lie algebra $\mathfrak{g}$ over the complex number field, a group $G_K(\mathfrak{g})$ by using a system of root vectors X_r which satisfies a certain condition (more precisely, (1.2) in Section 1). The main point of Chevalley's construction lies in the fact that the above-mentioned root vectors furnish a basis of $\mathfrak{g}$ such that, for every root r, $\exp(t \operatorname{ad} X_r)$ is represented by a matrix $(A_{ij}(t))$ whose entries $A_{ij}(t)$ are polynomials in t with integral coefficients.

The purpose of this paper is to extend the above construction to an arbitrary faithful $\mathfrak{g}$-module V, including Chevalley's as the special case $V = \mathfrak{g}$, and to study the groups $G_K(V)$ thus obtained. Our construction is based on Theorem 1.6, which assures the existence of a necessary basis for V. The group $G_K(V)$ has all the properties of the Chevalley groups $G_K(\mathfrak{g})$ (see Section 3 for a list of these properties) except that $G_K(V)$ has, in general, a finite centre. There is a natural homomorphism $G_K(V) \to G_K(\mathfrak{g})$ whose kernel is the centre of $G_K(V)$. If Ω is a universal domain containing K, then $G_\Omega(V)$ is a semi-simple algebraic group of type $\mathfrak{g}$ and, conversely, any semi-simple algebraic group of type $\mathfrak{g}$ is isomorphic to a group of the form $G_\Omega(V)$. The group $G_K(V)$ turns out to be the set of all rational points in $G_\Omega(V)$ over K.

Throughout the paper, **Z**, **Q**, and **C** denote, respectively, the ring of integers, the field of rational numbers, and the field of complex numbers.

1. Regular basis of a $\mathfrak{g}$-module. Let $\mathfrak{g}$ be a semi-simple Lie algebra over the complex number field **C**, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Chevalley **(3)** proved that one can choose a system of root vectors (X_r) with respect to $\mathfrak{h}$ satisfying the following condition:

(1.1) *For each root r, set $H_r = [X_r, X_{-r}]$. Then*

$$[H_r, X_r] = 2X_r, \qquad [H_r, X_{-r}] = -2X_{-r}.$$

Moreover, whenever $r, s, r+s$ are roots, we have $[X_r, X_s] = \pm(p+1)X_{r+s}$, where p is the greatest integer $i \geqslant 0$ such that $s - ir$ is a root.

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*The referee has drawn the author's attention to the communication to the Seminaire Bourbaki by C. Chevalley (March 1961, No. 219), "Certains schémas de groupes semi-simples," of which the author was unaware at the time of writing the manuscript. It appears that there is a great deal of overlapping between the above paper of Chevalley and the present one. The author has consequently decided to remove all the proofs from Section 5 where the overlapping is the heaviest. The contents of the present paper, in their present form, will be the basis of a subsequent paper on integral points of the group G.

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A system of root vectors satisfying (1.1) will be called *regular*. We shall fix such a root system throughout this paper, and denote by $\mathfrak{g}_{\mathbf{Z}}$ $[\mathfrak{h}_{\mathbf{Z}}]$ the additive group generated by the X_r's and the H_r's [the H_r's]. Then (1.1) shows that $\mathfrak{g}_{\mathbf{Z}}$ is a Lie subring of $\mathfrak{g}$ and that the following holds.

(1.2) *For any root r and any integer $k > 0$, $(k!)^{-1}(\operatorname{ad} X_r)^k$ maps $\mathfrak{g}_{\mathbf{Z}}$ into itself (writing operators on the right,* $\operatorname{ad} X$ *is defined by* $Y(\operatorname{ad} X) = [Y, X]$).

A $\mathfrak{g}$-*module* is a finite-dimensional vector space V over $\mathbf{C}$ on which $\mathfrak{g}$ acts on the right as linear transformations such that

$$v[X, Y] = (vX)Y - (vY)X \qquad \text{for all } v \in V \text{ and } X, Y \in \mathfrak{g}.$$

A $\mathfrak{g}$-module V is said to be *faithful* if for any $X \neq 0$ in $\mathfrak{g}$ there is $v \in V$ such that $vX \neq 0$. A $\mathfrak{g}$-module V is *irreducible* if V and 0 are the only $\mathfrak{g}$-submodules of V. It is known that any $\mathfrak{g}$-module is completely reducible, i.e. a direct sum of irreducible modules. A linear function $\mathrm{M}(H)$ on $\mathfrak{h}$ is called a *weight* for $\mathfrak{h}$ in the $\mathfrak{g}$-module V if there is an element $v \neq 0$ in V such that $vH = \mathrm{M}(H)v$ for all $H \in \mathfrak{h}$. The element v is called a *weight vector* of weight M. We shall make use of the following theorem (**6**, p. 112).

(1.3) *For any root r, set $\mathfrak{g}^{(r)} = \mathfrak{h} \oplus \mathbf{C}X_r \oplus \mathbf{C}X_{-r}$. Then $\mathfrak{g}^{(r)}$ is a subalgebra of $\mathfrak{g}$, and any $\mathfrak{g}$-module V is completely reducible as a $\mathfrak{g}^{(r)}$-module. Any irreducible $\mathfrak{g}^{(r)}$-submodule of V has a basis $(v_0, v_1, \ldots, v_m)$ such that*

$$\begin{aligned} v_i H &= (\mathrm{M} - ir)(H)v_i \qquad (0 \leqslant i \leqslant m),\\ v_i X_r &= (i+1)v_{i+1}, \qquad v_m X_r = 0 \qquad (0 \leqslant i \leqslant m-1),\\ v_0 X_{-r} &= 0, \qquad v_i X_{-r} = (m-i+1)v_{i-1} \qquad (1 \leqslant i \leqslant m), \end{aligned}$$

where M is a weight and $m = \mathrm{M}(H_r)$. Moreover, if $v \in V$ is a weight vector of any weight such that $vX_{-r} = 0$, then v generates an irreducible $\mathfrak{g}^{(r)}$-submodule of V.

As a corollary to (1.3) we obtain the following.

(1.4) *For any $\mathfrak{g}$-module V and any root r, X_r acts on V as a nilpotent linear transformation.*

(1.5) DEFINITION. *Let V be a $\mathfrak{g}$-module. For $X \in \mathfrak{g}$ denote by $\rho(X)$ the linear transformation of V obtained by the action of X on V. A basis $(v_1, v_2, \ldots, v_n)$ will be called regular if every v_i is a weight vector, and if for any root r and any integer $k > 0$, $(k!)^{-1}\rho(X_r)^k$ maps $V_{\mathbf{Z}} = \mathbf{Z}v_1 \oplus \ldots \oplus \mathbf{Z}v_n$ into itself.*

The following theorem is basic in this work.

(1.6) THEOREM. *Every $\mathfrak{g}$-module has a regular basis.*

For the proof of (1.6) we shall first prove some lemmas.

(1.7) LEMMA. *If $(u_1, u_2, \ldots, u_m)$ and $(v_1, v_2, \ldots, v_n)$ are regular bases for* *the* $\mathfrak{g}$*-modules* U *and* V*, respectively, then* $(u_1 \otimes v_1, u_1 \otimes v_2, \ldots, u_m \otimes v_n)$ *is a regular basis for the* $\mathfrak{g}$*-module* $U \otimes V$*.*

Proof. Recall that the action of $\mathfrak{g}$ on $U \otimes V$ is defined by

$$(u \otimes v)X = (uX) \otimes v + u \otimes (vX).$$

From this it follows that $u_i \otimes v_j$ are all weight vectors in $U \otimes V$. By repeated application of the above formula we obtain

$$(k!)^{-1}(u_i \otimes v_j)X^k = \textstyle\sum_{\mu+\nu=k} ((\mu!)^{-1}u_i X^\mu) \otimes ((\nu!)^{-1}v_j X^\nu).$$

Setting $X = X_r$ we obtain the lemma.

(1.8) LEMMA. *Let* V *be a* $\mathfrak{g}$*-module with a regular basis* $(v_1, v_2, \ldots, v_n)$*. If* $u \in V$ *is a weight vector and a linear combination of* $v_1, \ldots, v_n$ *with rational coefficients, then the* $\mathfrak{g}$*-submodule* U *generated by* u *has a regular basis.*

Proof. Clearly U is spanned by elements of the form

$$uX_{r_1} X_{r_2} \ldots X_{r_\nu} \qquad (\nu = 0, 1, 2, \ldots),$$

where the r_i are roots of $\mathfrak{g}$. Any of the above elements which is not zero can easily be seen to be a weight vector, and is a linear combination of $v_1, \ldots, v_n$ with rational coefficients. For any weight M, denote by U_{M} the subspace spanned by weight vectors of weight M which are contained in U. Let $V_{\mathbf{Z}} = \mathbf{Z}v_1 \oplus \ldots \oplus \mathbf{Z}v_n$. Then $U_{\mathrm{M}} \cap V_{\mathbf{Z}}$ is a finitely generated free abelian group, and contains a basis of U_{M}. For each weight M, choose a basis of $U_{\mathrm{M}} \cap V_{\mathbf{Z}}$, and let $(u_1, u_2, \ldots, u_m)$ be the union of these bases, where M runs over all the weights of V. Since U is the direct sum of the subspaces U_{M}, it follows that $(u_1, \ldots, u_m)$ is a basis for U. Clearly, each u_i is a weight vector. Consider $u' = (k!)^{-1}u_i X_r^k$. Since $u_i \in V_{\mathbf{Z}}$, the regularity of the basis $(v_1, \ldots, v_n)$ implies $u' \in V_{\mathbf{Z}}$. If $u' \neq 0$, then u' is clearly a weight vector. Hence $u' \in U_{\mathrm{M}}$ for some weight M. Thus, $u' \in U_{\mathrm{M}} \cap V_{\mathbf{Z}}$. Since $(u_1, \ldots, u_m)$ contains a basis of $U_{\mathrm{M}} \cap V_{\mathbf{Z}}$, it follows that u' is a linear combination of $u_1, u_2, \ldots, u_m$ with integral coefficients. This proves that the basis $(u_1, u_2, \ldots, u_m)$ is regular.

Throughout the rest of this section, we shall fix a usual lexicographic ordering of the additive group generated by weights, so that we can talk about the highest weights, fundamental weights, etc. Recall that the highest weight of an irreducible $\mathfrak{g}$-module is a dominant integral form on $\mathfrak{h}$, i.e. a linear function on $\mathfrak{h}$ which takes non-negative integral values on each H_r, and that for every dominant integral form on $\mathfrak{h}$ there exists an irreducible $\mathfrak{g}$-module which has the given integral form as the highest weight. Moreover, this correspondence between the dominant integral forms on $\mathfrak{h}$ and the irreducible $\mathfrak{g}$-modules is one-to-one. The irreducible $\mathfrak{g}$-module which has the given dominant integral form M is generated by a weight vector belonging to M.

(1.9) LEMMA. *Let* U*,* V *be irreducible* $\mathfrak{g}$*-modules with the highest weight* M, Λ *respectively. If both* U *and* V *have regular bases, then the irreducible* $\mathfrak{g}$*-module which has* $\mathrm{M} + \Lambda$ *as the highest weight has a regular basis.*

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Proof. Let $(u_1, u_2, \ldots , u_m)$ and $(v_1, v_2, \ldots , v_n)$ be regular bases for U and V, respectively. Let M_i and Λ_i be the weight of u_i and v_i respectively. Then $\mathrm{M} = \mathrm{M}_i$, $\Lambda = \Lambda_j$ for some i, j. Now the irreducible $\mathfrak{g}$-module W which has $\mathrm{M} + \Lambda$ as the highest weight is a submodule of $U \otimes V$ generated by $u_i \otimes v_j$. By (1.7), $(u_1 \otimes v_1, \ldots , u_m \otimes v_n)$ is a regular basis for $U \otimes V$. Then from (1.8) it follows that W has a regular basis.

We are now ready to prove (1.6). For brevity, we shall make the following definition. A $\mathfrak{g}$-module U will be said to be *regularly derived from the m-fold tensor product* of the $\mathfrak{g}$-module V if V has a regular basis $(v_1, v_2, \ldots , v_n)$ such that U is generated by a linear combination with rational coefficients of the elements

$$v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_m}.$$

Thus, U is a $\mathfrak{g}$-submodule of the m-fold tensor product of V, and has a regular basis by (1.7) and (1.8).

In view of the complete reducibility of $\mathfrak{g}$-modules, it suffices to prove (1.6) for V irreducible. Then, by (1.9), one can assume that the highest weight for V is a fundamental dominant integral form. Then, if $\mathfrak{g} = \sum \mathfrak{g}_i$ is the decomposition of $\mathfrak{g}$ as a direct sum of simple Lie algebras, $V\mathfrak{g}_i = 0$ for all but only one i. This implies that we can assume that $\mathfrak{g}$ is simple. Now let $\Pi_1, \Pi_2, \ldots , \Pi_l$ be the fundamental weights of $\mathfrak{g}$ in the notation of *E.* Cartan (**2**, p. 367). Denote by V_i the irreducible $\mathfrak{g}$-module whose highest weight is Π_i. We shall follow Cartan (**2**, pp. 386–398), considering each type of $\mathfrak{g}$ separately.

(i) $\mathfrak{g} = (A_l)$, $l \geqslant 1$. V_1 is the vector space which represents $\mathfrak{g}$ as the Lie algebra of all $(l + 1) \times (l + 1)$ matrices of trace 0. The basis used for this representation is easily seen to be regular with a suitable identification of the root vectors X_r (**7**). V_m, $2 \leqslant m \leqslant l$, is the space of all skew-symmetric tensors (or m-vectors) in the m-fold tensor product of V_1, and is easily seen to be regularly derived from V_1.

(ii) $\mathfrak{g} = (B_l)$, $l \geqslant 2$. V_2 is the $(2l + 1)$-dimensional vector space which represents $\mathfrak{g}$ as the Lie algebra of the orthogonal group corresponding to the quadratic form $\sum_0^l x_i x_{-i}$. The basis of V_2 used for this representation is easily seen to be regular with a suitable identification of the root vectors X_r (**7**). V_m, $3 \leqslant m \leqslant l$, is the space of $(m - 1)$-vectors of V_2, and is regularly derived from the $(m - 1)$-fold tensor product of V_2. V_1 is the space of spin representation. The basis given in (**2**, p. 388) is easily seen to be regular with a suitable identification of the X_r.

(iii) $\mathfrak{g} = (C_l)$, $l \geqslant 3$. V_1 is the $2l$-dimensional vector space which represents $\mathfrak{g}$ as the Lie algebra of the symplectic group; the basis used is regular (**7**). V_m, $2 \leqslant m \leqslant l$, is again the space of m-vectors of V_1, and is regularly derived from the m-fold tensor product of V_1.

(iv) $\mathfrak{g} = (D_l)$, $l \geqslant 4$. V_1 and V_2 are the spin representations of $\mathfrak{g}$; the formulas given in (**2**, p. 392) show clearly that these modules have regular bases. V_3 is the $2l$-dimensional vector space which represents $\mathfrak{g}$ as the Lie

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algebra of the orthogonal group corresponding to the quadratic form $\sum_1^l x_i x_{-i}$. V_m, $4 \leqslant m \leqslant l$, is the space of $(m-2)$-vectors of V_3, and is regularly derived from the $(m-2)$-fold tensor product of V_3.

(v) $\mathfrak{g} = (E_6)$. V_1 is 27-dimensional; the basis given in **(2**, p. 273**)** is easily seen to be regular. V_2 is $\mathfrak{g}$ itself, giving the adjoint representation, and has a regular basis by (1.2). V_3 is the dual $\mathfrak{g}$-module of V_1, i.e. the space of all linear functions on V_1. The action of $\mathfrak{g}$ on V_3 is defined by $(fX)(v) = -f(vX)$, where $v \in V_1, f \in V_3, X \in \mathfrak{g}$. It is immediate from the definition that the dual basis of a regular basis is regular. V_4 is regularly derived from the 2-fold tensor product of V_1, and V_5 is the dual of V_4. Finally, V_6 is derived from the 3-fold tensor product of V_1.

(vi) $\mathfrak{g} = (E_7)$. V_1 is $\mathfrak{g}$ itself, giving the adjoint representation, and has a regular basis by (1.2). V_2 is 56-dimensional, and the basis given by Cartan **(2**, p. 273**)** turns out to be regular, if one identifies the root vectors X_r suitably. V_3 and V_6 are regularly derived from the 3-fold tensor product of V_2. V_4 is regularly derived from the 2-fold tensor product of V_2. V_5 and V_7 are, respectively, regularly derived from the 2- and 3-fold tensor products of V_1.

(vii) $\mathfrak{g} = (E_8)$. V_1 is $\mathfrak{g}$ itself, giving the adjoint representation, and has a regular basis by (1.2). $V_2, \ldots, V_8$ are, respectively, regularly derived from the 2-, 2-, 4-, 3-, 4-, 4-, 5-fold tensor products of V_1; cf. **(2**, p. 396**)**.

(viii) $\mathfrak{g} = (F_4)$. V_1 is 26-dimensional **(**cf. **2**, p. 275**)**. If one changes the basis elements y and z of Cartan by $u = 2z$, $v = y + z$, then one sees easily that $(u, v, x_i, x_{\alpha\beta\gamma\delta})$ is a regular basis under a suitable identification of the root vectors X_r. V_2 and V_3 are both regularly derived from the 2-fold tensor product of V_1, and V_4 from the 4-fold tensor product of V_1.

(ix) $\mathfrak{g} = (G_2)$. V_1 is 7-dimensional, and can be obtained from the representation of $\mathfrak{g}$ as the derivation algebra of the Cayley algebra. The basis given by Cartan **(2**, p. 276**)** becomes regular if a suitable identification of the root vectors is made. V_2 is $\mathfrak{g}$ itself, giving the adjoint representation. By (1.2), it has a regular basis.

This completes the proof of (1.6).

Remark. In the above proof we relied heavily on Cartan's results. The extent of this reliance would be much lessened if one could prove, by an argument common to all types, the following: if V is a $\mathfrak{g}$-module whose weights generate the additive group generated by all weights, then any irreducible $\mathfrak{g}$-module can be derived, for some integer $m > 0$, from the m-fold tensor product of V. At any rate, a direct proof of (1.6) is desirable.

2. Notations and remarks. We shall fix some notations which will be used in the rest of this paper. As in Section 1, $\mathfrak{g}$ will denote a semi-simple Lie algebra, and $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$. Hereafter, weights, roots, root vectors will all mean those taken with respect to $\mathfrak{h}$. Also, we fix a regular system (X_r) of root vectors of $\mathfrak{g}$. W will denote the Weyl group, and w_r the

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reflection attached to the root r. P will denote the additive group generated by all the weights, and $P(\mathfrak{g})$ the subgroup of P generated by the roots.

We shall denote by V a faithful $\mathfrak{g}$-module, and by $P(V)$ the subgroup of P generated by the weights in V. We shall fix a regular basis $(v_1, v_2, \ldots, v_n)$ of V as follows: if V is irreducible, let $(v_1, \ldots, v_n)$ be any regular basis of V; if V is not irreducible, take a regular basis for each irreducible constituent, and let $(v_1, \ldots, v_n)$ be the union of these regular bases. We shall denote by M_i the weight of v_i. By definition, $P(V)$ is generated by $\mathrm{M}_1, \mathrm{M}_2, \ldots, \mathrm{M}_n$. The representation of $\mathfrak{g}$ obtained from V by using the basis $(v_1, \ldots, v_n)$ will be denoted by ρ_V or ρ.

For a field K, K^* will denote the multiplicative group of K, $X(V, K)$ the multiplicative group of all homomorphisms χ: $P(V) \rightarrow K^*$, and $X'(V, K)$ the group of all $\chi \in X(V, K)$ which can be extended to a homomorphism $P \rightarrow K^*$. For any root r and $z \in K^*$, $\chi_{r,z,V}$ will denote the element in $X'(V, K)$ defined by $\chi_{r,z,V}(\mathrm{M}) = z^{M(H_r)}$, where $\mathrm{M} \in P(V)$. When there is no danger of confusion, we shall frequently write $\chi_{r,z}$ for $\chi_{r,z,V}$.

From the regularity of the basis $(v_1, \ldots, v_n)$ it is immediate that, for every root r, $\exp(t\rho(X_r))$ is a matrix $(A_{ij}(t))$ whose entries $A_{ij}(t)$ are all polynomials in t with integral coefficients. Hence, for any field K and $t \in K$, the matrix $(A_{ij}(t))$ with entries in K is well defined. We shall denote this matrix by $x_{r,K}(t; V)$, or simply by $x_r(t)$ when there is no danger of confusion. For $\chi \in X\ (V, K)$, the diagonal matrix with $\chi(\mathrm{M}_1), \chi(\mathrm{M}_2), \ldots, \chi(\mathrm{M}_n)$ on the diagonal will be denoted by $h(\chi, V)$ or $h(\chi)$. Now we shall define the following groups:

$\mathfrak{X}_{r,K}(V)$: the group generated by $x_{r,K}(t; V)$, where $t \in K$.
$\mathfrak{U}_K(V)$: the group generated by $\mathfrak{X}_{r,K}(V)$, where r runs over all the positive roots (when an ordering of the roots is given).
$\mathfrak{V}_K(V)$: the group generated by $\mathfrak{X}_{r,K}(V)$, where r runs over all the negative roots.
$G_K'(V)$: the group generated by $\mathfrak{U}_K(V)$ and $\mathfrak{V}_K(V)$.
$\mathfrak{H}_K(V)$: the group $\{h(\chi, V) | \chi \in X(V, K)\}$.
$\mathfrak{H}'_K(V)$: the group $\{h(\chi, V) | \chi \in X'(V, K)\}$.
$G_K(V)$: the group generated by $G_K'(V)$ and $\mathfrak{H}_K(V)$.
$\mathfrak{U}_{w,K}(V)$: the group generated by $\mathfrak{X}_{r,K}(V)$, where r runs over all positive roots such that $w(r) < 0$ $(w \in W)$.
$\mathfrak{U}'_{r,K}(V)$: the group generated by $\mathfrak{X}_{r,K}(V)$, where r runs over all positive roots such that $w(r) > 0$.

Also we shall define the following elements:

$\omega_{r,K}(V)$: the element $x_{r,K}(1; V)x_{-r,K}(-1; V)x_{r,K}(1; V)$.
$\omega(w, V)$: the element $\omega_{r_1,K}(V)\,\omega_{r_2,K}(V) \ldots \omega_{r_m,K}(V)$ with $w_{r_1}w_{r_2} \ldots w_{r_m} = w$. For given $w \in W$ such an element is not unique, but one will be fixed; $\omega(1, V) = 1$.

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When there is no danger of confusion, K or V or both will be dropped from the above symbols.

In the rest of this section, we shall collect some information which will be needed later on.

(2.1) $P(\mathfrak{g}) \subseteq P(V) \subseteq P$.

Proof. The second inclusion is clear. To prove the first, let r be any given root, let $\mathfrak{g}^{(r)}$ be as in (1.3), and regard V as a $\mathfrak{g}^{(r)}$-module. Since $VX_r \neq 0$, by (1.3) it can be seen easily that V contains a weight vector v such that $vX_r \neq 0$. If M is the weight of v, then vX_r is a weight vector of weight $\mathrm{M} - r$. Hence $r \in P(V)$. Since r is arbitrary, this proves the first inclusion.

(2.2) *If V is an irreducible $\mathfrak{g}$-module, then for any two weights* M_1, M_2 *in* V, $\mathrm{M}_1 - \mathrm{M}_2 \in P(\mathfrak{g})$.

Proof. Since V is irreducible, for any non-zero element $v \in V$, V is generated by the elements of the form

$$vX_{r_1}X_{r_2}\ldots X_{r_m}.$$

If v is a weight vector of weight M, the above element, when it is not zero, is a weight vector of weight $\mathrm{M} - (r_1 + \ldots + r_m)$. From this, (2.2) follows easily.

(2.3) *If* M *is a weight in the g-module V, then for any $w \in W$, $w(\mathrm{M})$ is also a weight in V.*

This follows easily from (1.3).

(2.4) *$X'(V, K)$ is generated by the elements* $\chi_{r,K,V}$.

Proof. Let $a_1, a_2, \ldots, a_l$ be a system of fundamental roots. Then there exists a basis $(\Pi_1, \Pi_2, \ldots, \Pi_l)$ of P such that $\Pi_i(H_{a_j}) = \delta_{ij}$ (Kronecker delta). Let $\chi \in X'(V, K)$, and $\chi(\Pi_i) = z_i$. Then it can easily be verified that $\chi = \prod \chi_{a_i,z_i,V}$.

3. Basic properties of $G_K(V)$.

We shall list some basic properties of the group $G_K(V)$. The proofs will be given in the next section.

(3.1) *If r and s are linearly independent roots, then*

$$x_{r,K}(t)x_{s,K}(u)x_{r,K}(-t) = x_{s,K}(u)\prod_{ij} x_{ir+js,K}(C_{ij;r,s}t^i u^j)$$

(note that we have omitted the symbol V in the above). *Here the product is taken over all couples (i, j) of integers > 0 such that $ir + js$ is a root, the couples being arranged such that the roots $ir + js$ form an increasing sequence relative to an order of P for which r and s become positive. The coefficients $C_{ij;r,s}$ are integers independent of K, V, t, and u.*

(3.2) *For any root r, there exists a homomorphism ϕ_r: $SL(2; K) \rightarrow G_K(V)$ such that*

$$\phi_r\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix} = x_{-r,K}(t; V), \qquad \phi_r\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} = x_{r,K}(t; V) \qquad (t \in K).$$

(3.3) *For any root r and $z \in K^*$, we have*

$$\phi_r\begin{pmatrix}z & 0\\ 0 & z^{-1}\end{pmatrix} = h(\chi_{r,z}; V).$$

(3.4) *For any root r and $\chi \in X(V, K)$, we have*

$$h(\chi; V)x_{r,K}(t; V)h(\chi; V)^{-1} = x_{r,K}(\chi(r)t; V) \qquad (t \in K).$$

(3.5) *For any $w \in W$ and $\chi \in X(V, K)$, we have*

$$\omega(w, V)h(\chi; V)\omega(w, V)^{-1} = h(\chi'; V),$$

where χ' is defined by $\chi'(\mathrm{M}) = \chi(w^{-1}(\mathrm{M}))$, where $\mathrm{M} \in P(V)$.

(3.6) *For any roots r, s we have*

$$\omega_{r,K}(V)x_{s,K}(t; V)\omega_{r,K}(V)^{-1} = x_{w_r(s),K}(\eta_{r,s}t; V),$$

where $\eta_{r,s}$ are certain constants ± 1 which are independent of V, K, and t.

(3.7) *For any roots $r_1, r_2, \ldots, r_m$ such that*

$$w_{r1}\, w_{r2} \ldots w_{rm} = 1,$$

we have

$$\omega_{r1,K}(V)\omega_{r2,K}(V) \ldots \omega_{rm,K}(V) = h(\chi; V)$$

for some homomorphism $\chi: P \rightarrow \{\pm 1\}$ which is independent of V and K.

(3.8) *Every element x in $\mathfrak{U}_K(V)$ can be written in one and only one way in the form $\prod_r x_{r,K}(t_r; V)$, where the product is taken over all positive roots in increasing order, and where $t_r \in K$.*

(3.9) *Every element in $G_K(V)$ [in $G_K'(V)$] can be written in one and only one way in the form $xh\omega(w, v)x_1$, where $x \in \mathfrak{U}_K(V)$; $h \in \mathfrak{H}_K(V)$ [$h \in \mathfrak{H}_K'(V)$]; $w \in W$; $x_1 \in \mathfrak{U}_{w,K}(V)$.*

(3.10) *If V_1, V_2 are faithful $\mathfrak{g}$-modules such that $P(V_2) \subseteq P(V_1)$, then there exists a homomorphism $\phi: G_K(V_1) \rightarrow G_K(V_2)$ such that*

$$\phi(x_{r,K}(t; V_1)) = x_{r,K}(t; V_2), \qquad \phi(h(\chi; V_1)) = h(\chi|P(V_2); V_2)$$

for all roots r and all $\chi \in X(V, K)$, where $\chi|P(V_2)$ denotes the restriction of χ to $P(V_2)$.

The image of ϕ is the subgroup of $G_K(V_2)$ consisting of the elements $xh(\chi; V_2)\omega(w)x_1$, where x, w, and x_1 are as in (3.9) (replacing V by V_2), and where χ runs over all elements in $X(V_2, K)$ which can be extended to a homomorphism $P(V_1) \rightarrow K^*$. The kernel of ϕ consists of all $h(\chi; V_1) \in \mathfrak{H}_K(V_1)$ such *that $\chi|P(V_2) = 1$, and is contained in the centre of $G_K(V_1)$. The restriction of ϕ to $G_K'(V_1)$ is an epimorphism $\phi':G_K'(V_1) \rightarrow G_K'(V_2)$. The kernel of ϕ' consists of all $h(\chi; V_1)$ in $\mathfrak{H}_K'(V_1)$ such that $\chi|P(V_2) = 1$.*

(3.11) *The groups $G_K(V)$ and $G_K'(V)$ are determined uniquely (up to isomorphisms) by K and the $\mathfrak{g}$-module V, and are independent of the regular basis of V used to define them.*

(3.12) *The centre of $G_K(V)$ [of $G_K'(V)$] consists of all $h(\chi; V)$ in $\mathfrak{H}_K(V)$ [in $\mathfrak{H}_K'(V)$] such that $\chi(r) = 1$ for all roots r, and is isomorphic to*

$$\mathrm{Hom}(P(V)/P(g), K^*).$$

If V is irreducible, then the centre of $G_K(V)$ [of $G_K'(V)$] consists of all matrices in $G_K(V)$ [in $G_K'(V)$] of the form zI, where I is the identity matrix and $z \in K^*$.

4. Proofs of (3.1)–(3.12).

(4.1) *Proof of* (3.1). If $K = \mathbf{C}$, then the identity given in (3.1) is clearly equivalent to a set of polynomial relations (with polynomials of integral coefficients) among the coefficients of the polynomials $A_{ij}(t)$ appearing as entries of $x_{r,\mathbf{C}}(t; V)$. Hence, if (3.1) is true for $K = \mathbf{C}$, then it is true for any field K. Now we shall prove (3.1) assuming $K = \mathbf{C}$.

Let $X \rightarrow \rho(X)$ be the representation of $\mathfrak{g}$ obtained from V. Then it can be seen easily that

$$\rho(Xx_{r,\mathbf{C}}(t; \mathfrak{g})) = x_{r,\mathbf{C}}(t; V)\rho(X)x_{r,\mathbf{C}}(t; V)^{-1} \tag{4.1.1}$$

holds true for all $X \in \mathfrak{g}$, $t \in \mathbf{C}$, and all roots r. It follows from this that for any element $x \in G_{\mathbf{C}}'(\mathfrak{g})$ there exists an element $x' \in G_{\mathbf{C}}'(V)$ such that $\rho(Xx') = x\rho(X)x^{-1}$ for all $X \in \mathfrak{g}$. Moreover, the faithfulness of the $\mathfrak{g}$-module V implies that x' is uniquely determined by x. Also, it can be seen easily that the map $x \rightarrow x'$ gives an epimorphism $\phi:G_{\mathbf{C}}'(V) \rightarrow G_{\mathbf{C}}'(\mathfrak{g})$. It is known (**3**, p. 19) that for any two roots r, s there is a structure of ordered group on P such that r and s are positive. We shall fix such a structure on P, and prove that the restriction of ϕ to $\mathfrak{U}_{\mathbf{C}}(V)$ is an isomorphism. Suppose that $x \in \mathfrak{U}_{\mathbf{C}}(V)$, $\phi(x) = 1$. Then x commutes with all $\rho(X)$, $X \in \mathfrak{g}$. Then the complete reducibility implies that x is a diagonal matrix. Now, number the basis elements $v_1, v_2, \ldots, v_n$ of V such that $\mathrm{M}_1 \geqslant \mathrm{M}_2 \geqslant \ldots \geqslant \mathrm{M}_n$. Then it is immediate from the definition that, for any root $r > 0$, $x_{r,\mathbf{C}}(t; V)$ is a superdiagonal matrix with 1's on the diagonal. Hence the same holds true for x. Then $x = 1$, since x is diagonal. Thus, the restriction of ϕ to $\mathfrak{U}_{\mathbf{C}}(V)$ is an isomorphism, and the proof of (3.1) is reduced to the case $V = \mathfrak{g}$. But (3.1) is known (**3**, p. 36) for $V = \mathfrak{g}$. Thus (3.1) is proved.

(4.2) *Proof of* (3.2) *and* (3.3). Let $\mathfrak{g}^{(r)}$ be as in (1.3), and decompose V as a direct sum of irreducible $\mathfrak{g}^{(r)}$-modules V_λ. Then, by (1.3), each V_λ has a basis $(u_0, u_1, \ldots, u_m)$ such that

$$(4.2.1)\qquad \begin{aligned} u_kH &= (\mathrm{M} - kr)(H)u_k && (0 \leqslant k \leqslant m; H \in \mathfrak{h}),\\ u_k X_r &= (k+1)u_{k+1}, \quad u_m X_r = 0 && (0 \leqslant k \leqslant m-1),\\ u_0 X_{-r} &= 0, \quad u_k X_{-r} = (m-k+1)u_{k-1} && (1 \leqslant k \leqslant m),\end{aligned}$$

where M is a weight in V, and $m = \mathrm{M}(H_r)$. From this it follows that

$$(4.2.2)\qquad \begin{aligned} u_k x_{r,\mathbf{C}}(t; V) &= \sum_{i=k}^{m} C_{i,k}\, t^{i-k}\, u_i,\\ u_k x_{-r,\mathbf{C}}(t; V) &= \sum_{i=0}^{k} C_{m-i,m-k}\, t^{k-i}\, u_i,\end{aligned}$$

for $0 \leqslant k \leqslant m$, where $C_{i,k}$ denotes the binomial coefficient $\binom{i}{k}$. Now consider the space $V_\lambda' = \mathbf{C}_m[S, T]$ of all homogeneous polynomials of degree m in the indeterminates S, T with coefficients in $\mathbf{C}$, and let V' be the direct sum of the V_λ'. Let the group $GL(2; \mathbf{C})$ act on V' by the rule: for $F(S, T) \in V_\lambda'$ and

$$(4.2.3)\qquad \zeta = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in GL(2; \mathbf{C})$$

set $(F\zeta)(S, T) = F(aS + bT, cS + dT) \in V_\lambda'$.

Let $\phi\colon V \to V'$ be the linear map defined as follows: on each V_λ, set $u_k\,\phi = \mathbf{C}_{m,k}\, S^{m-k}\, T^k \in V_\lambda'$. Then one can verify from (4.2.2) that

$$(v\phi)\begin{pmatrix} 1 & 0\\ t & 1\end{pmatrix} = (vx_{-r,C}(t; V))\phi, \qquad (v\phi)\begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix} = (vx_{r,C}(t; V))\phi$$

holds for all $v \in V$ and $t \in \mathbf{C}$. Hence, for any $\zeta \in SL(2; \mathbf{C})$ there exists $x \in G_{\mathbf{C}}(V)$ such that

$$(4.2.4)\qquad (v\phi)\zeta = (vx)\phi$$

for all $v \in V$. Since ϕ is an isomorphism, it follows that, for any given $\zeta \in SL(2; \mathbf{C})$, the element $x \in G_{\mathbf{C}}(V)$ satisfying (4.2.4) is unique. It can be seen from (4.2.4) that the map $\zeta \to x$ is a homomorphism $\phi_r\colon SL(2; \mathbf{C}) \to G_{\mathbf{C}}(V)$ such that

$$(4.2.5)\qquad \phi_r\begin{pmatrix} 1 & 0\\ t & 1\end{pmatrix} = x_{-r,\mathbf{C}}(t; V), \qquad \phi_r\begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix} = x_{r,\mathbf{C}}(t; V).$$

Now we shall show that for any $z \in \mathbf{C}^*$ we have

$$(4.2.6)\qquad \phi_r(\operatorname{diag}(z, z^{-1})) = h(\chi_{r,z}; V).$$

For the proof of this it suffices to show that

$$(4.2.7)\qquad v\phi_r(\operatorname{diag}(z, z^{-1})) = z^{\Lambda(H_r)}v$$

holds for any weight vector v of weight Λ. Since V is the direct sum of the V_λ, it suffices to prove (4.2.7) for $v = u_k \in V_\lambda$. Since $\zeta = \operatorname{diag}(z, z^{-1})$ induces the transformation $S \to zS$, $T \to z^{-1}T$ in $V_\lambda' = \mathbf{C}_m[S, T]$, we have

$$(u_k\,\phi)\zeta = C_{m,k}(zS)^{m-k}(z^{-1}\,T)^k = z^{m-2k}\, C_{m,k}\, S^{m-k}\, T^k = z^{m-2k}(u_k\,\phi).$$

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By (4.2.1), the weight of u_k is $\mathrm{M} - kr$, and $(\mathrm{M} - kr)(H_r) = m - 2k$. Hence $(v\phi)\zeta = (z^{\Lambda(H_r)}v)\phi$. By (4.2.4) we have $(v\phi)\zeta = (v\phi_r(\zeta))\phi$. Then, since ϕ is an isomorphism, we have $v\phi_r(\zeta) = z^{\Lambda(H_r)}v$. Thus, (4.2.6) is proved.

It is clear that the element ζ given in (4.2.3) is represented in each V_λ', and hence in V' also, by a matrix whose entries are polynomials in a, b, c, and d with coefficients in $\mathbf{C}$. From this and (4.2.4) it follows that there exist n^2 polynomials $F_{ij}(X, Y, Z, U)$ in $C[X, Y, Z, U]$ where $n = \dim V$, such that the matrix

$$f\begin{pmatrix} X & Y \\ Z & U \end{pmatrix} = (F_{ij}(X, Y, Z, U))$$

satisfies the following two conditions:

$$\text{(4.2.8)} \qquad f\begin{pmatrix} X & Y \\ Z & U \end{pmatrix} f\begin{pmatrix} X' & Y' \\ Z' & U' \end{pmatrix} = f\left(\begin{pmatrix} X & Y \\ Z & U \end{pmatrix}\begin{pmatrix} X' & Y' \\ Z' & U' \end{pmatrix}\right),$$

$$\text{(4.2.9)} \qquad f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \phi_r\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

whenever a, b, c, and d are complex numbers satisfying $ad - bc = 1$.

Now we shall show that there exist $2n^2$ polynomials $P_{ij}(X, Y, Z)$, $Q_{ij}(Z, U)$ in $\mathbf{Z}[X, Y, Z, U]$ such that

$$\text{(4.2.10)} \qquad F_{ij}(X, Y, Z, X^{-1}(YZ + 1)) = X^{-n} P_{ij}(X, Y, Z),$$

$$\text{(4.2.11)} \qquad F_{ij}(0, -Z^{-1}, Z, U) = Z^{-n} Q_{ij}(Z, U).$$

To prove this, let t, s, and $z \neq 0$ be arbitrary complex numbers. Applying the homomorphism ϕ_r to

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z & zs \\ tz & tzs + z^{-1} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -z & 0 \\ 0 & -z^{-1} \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -z^{-1} \\ z & zs \end{pmatrix},$$

and using (4.2.5), (4.2.6), and (4.2.9), one sees easily that $P_{ij}(z, zs, tz)$ and $Q_{ij}(z, zs)$ are in $\mathbf{Z}[z, s, t]$. From this it follows that $P_{ij}(X, Y, Z)$ and $Q_{ij}(Z, U)$ are in $\mathbf{Z}[X, Y, Z, U]$.

Now let K be the given field. For a, b, c, d in K such that $ad - bc = 1$, define $F_{ij}(a, b, c, d) \in K$ as follows: if $a \neq 0$, set $F_{ij}(a, b, c, d) = a^{-n} P_{ij}(a, b, c)$; if $a = 0$, set $F_{ij}(0, -c^{-1}, c, d) = c^{-n} Q_{ij}(c, d)$. We shall prove that the mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow (F_{ij}(a, b, c, d))$$

is a homomorphism $\phi_{r,K}: SL(2; K) \longrightarrow G_K(V)$. Let

$$\zeta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \zeta' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \qquad \zeta\zeta' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$$

be in $SL(2;K)$. We shall prove that

$$\phi_{r,K}(\zeta)\phi_{r,K}(\zeta') = \phi_{r,K}(\zeta\zeta'). \tag{4.2.12}$$

First consider the case where none of a, a', a'' is 0. From (4.2.8) we have the identity

$$\sum_{k=1}^{n} X^{-n} P_{ik}(X, Y, Z)X'^{-n} P_{kj}(X', Y', Z') = X''^{-n} P_{ij}(X'', Y'', Z''),$$

where

$$\begin{pmatrix} X & Y \\ Z & U \end{pmatrix}\begin{pmatrix} X' & Y' \\ Z' & U' \end{pmatrix} = \begin{pmatrix} X'' & Y'' \\ Z'' & U'' \end{pmatrix},$$

$$U = X^{-1}(YZ + 1), \qquad U' = X'^{-1}(Y'Z' + 1).$$

Substituting $X = a$, $Y = b, \ldots, Z' = c'$ in the above, we obtain (4.2.12). The cases where some of a, a', a'' are 0 can be treated similarly. From the definition, (4.2.5), and (4.2.9), we have

$$\phi_{r,K}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = x_{-r,K}(t;\, V), \qquad \phi_{r,K}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x_{r,K}(t;\, V).$$

Hence $\phi_{r,K}(\zeta) \in G_K(V)$ for all $\zeta \in SL(2;K)$. Also, from the definition and (4.2.6), we obtain

$$\phi_{r,K}(\operatorname{diag}(z, z^{-1})) = h(\chi_{r,z};\, V)$$

for any $z \in K^*$. Thus (3.2) and (3.3) are proved.

As a corollary to (3.3), we obtain

$$\omega_{r,K}(V) = \phi_r\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.4}$$

(4.5) *Proof of* (3.4). Let $(v_1, \ldots, v_n)$ be the regular basis used to define $x_{r,K}(t;\, V)$, and let M_i be the weight of v_i. Let $x_{r,K}(t;\, V) = (A_{ij}(t))$. If $r > 0$, then from the definition of $x_{r,K}(t)$ we have, for $r > 0$,

(4.5.1) $A_{ij}(t) = 0$ if $\mathrm{M}_i < \mathrm{M}_j$, or $\mathrm{M}_i = \mathrm{M}_j$, $i \neq j$, or if $\mathrm{M}_i > \mathrm{M}_j$ and $\mathrm{M}_i - \mathrm{M}_j$ is not an integral multiple of r; $A_{ij}(t) = a_{ij}t_{\nu}$, where a_{ij} is an integer, if $\mathrm{M}_i - \mathrm{M}_j = \nu r$ with an integer ν.

From this it follows that for any $\chi \in X(V, K)$ and any i, j, we have $\chi(\mathrm{M}_i)A_{ij}(t)\chi(\mathrm{M}_j)^{-1} = A_{ij}(\chi(r)t)$. From this, (3.4) follows immediately.

(4.6) Lemma. *For any weight vector $v \in V$ of weight Λ and for any root r, $v\omega_{r,\mathbf{C}}(V)$ is a weight vector of weight $w_r(\Lambda)$.*

Proof. We shall use the notation introduced at the beginning of (4.2). Clearly, it suffices to prove the case $v = u_k \in V_\lambda$. Set

$$\zeta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2;\mathbf{C}).$$

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Then, by (4.4), we have $\omega_{r,\mathbf{C}}(V) = \phi_r(\zeta)$. Then ζ induces the transformation $S \to T$, $T \to -S$ in $V_\lambda' = \mathbf{C}_m[S, T]$. Hence

$$(v\phi)\zeta = (u_k\,\phi)\zeta = C_{m,k}\,T^{m-k}(-S)^k = (-1)^k\,u_{m-k}\,\phi.$$

By (4.2.4) we have $(v\phi)\zeta = (v\phi_r(\zeta))\phi$. Hence $v\phi_r(\zeta) = (-1)^k\,u_{m-k}$. But, by (4.2.1), the weight of u_k is $\mathrm{M} - kr = \Lambda$. We have

$$\begin{aligned} w_r(\Lambda) &= w_r(\mathrm{M} - kr) = \mathrm{M} - kr - (\mathrm{M} - kr)(H_r)r \\ &= \mathrm{M} - kr - (m - 2k)r = \mathrm{M} - (m - k)r. \end{aligned}$$

Hence, the weight of $v\omega_{r,K}(V)$ is $w_r(\Lambda)$.

(4.7) *Proof of* (3.5). It suffices to prove the case $K = \mathbf{C}$, since $\omega(w, V)$ is a matrix with integral entries. Let $(v_1, \ldots, v_n)$ be the regular basis of V, and M_i the weight of v_i. By repeated application of (4.6) it follows that $v_i\,\omega(w, V)$ is a weight vector of weight $w^{-1}(\mathrm{M}_i)$. Hence,

$$(v_i\,\omega(w, V))h(\chi; V) = \chi(w^{-1}(\mathrm{M}_i))v_i\,\omega(w, V),$$

from which one obtains (3.5).

(4.8) *Proof of* (3.6). It suffices to discuss the case $K = \mathbf{C}$. Let ρ be the representation of $\mathfrak{g}$ obtained from V. Then, by (4.1.1), we have

$$\omega_{r,\mathbf{C}}(V)\rho(tX_s)\omega_{r,\mathbf{C}}(V)^{-1} = \rho(tX_s\,\omega_{r,\mathbf{C}}(\mathfrak{g})) = \rho(\eta_{r,s}\,tX_{w_r(s)}),$$

where $\eta_{r,s} = \pm 1$; see (**3**, p. 31) or (**8**, p. 439). Taking the exponential of both sides, we get (3.6).

The formulae

$$\begin{aligned} G_K(V) &= \mathfrak{B}_K(V)\mathfrak{H}_K(V)\mathfrak{U}_K(V)\mathfrak{B}_K(V), \\ G_K'(V) &= \mathfrak{B}_K(V)\mathfrak{H}_K'(V)\mathfrak{U}_K(V)\mathfrak{B}_K(V) \end{aligned} \tag{4.9}$$

can be proved in exactly the same way as Lemma 3 of (**3**, p. 48). We omit the proof.

(4.10) *Proof of* (3.7). Since $\omega_{r,K}(V)$ is a matrix with integral entries, it suffices to prove (3.7) for the case $K = \mathbf{C}$. Let $\phi: G_{\mathbf{C}}'(V) \to G_{\mathbf{C}}'(g)$ be the homomorphism considered in (4.1). Then clearly we have, for

$$\begin{aligned} \omega &= \omega_{r_1,\mathbf{C}}(V)\omega_{r_2,\mathbf{C}}(V)\ldots\omega_{r_m,\mathbf{C}}(V), \\ \phi(\omega) &= \omega_{r_1,\mathbf{C}}(\mathfrak{g})\omega_{r_2,\mathbf{C}}(\mathfrak{g})\ldots\omega_{r_m,\mathbf{C}}(\mathfrak{g}). \end{aligned} \tag{4.10.1}$$

The right-hand side of (4.10.1) is equal to $h(\chi; \mathfrak{g})$, where $\chi \in X(\mathfrak{g}, \mathbf{C})$ such that $\chi(r) = \pm 1$ for all roots (**3**, p. 37). Since $\omega_{r,\mathbf{C}}(\mathfrak{g}) \in G_{\mathbf{Q}}'(\mathfrak{g})$, where $\mathbf{Q}$ denotes the field of rational numbers, for all roots r, it follows that $h(\chi; \mathfrak{g}) \in G_{\mathbf{Q}}'(\mathfrak{g})$, and hence $h(\chi; \mathfrak{g}) \in \mathfrak{H}_{\mathbf{Q}}'(\mathfrak{g})$ (**3**, p. 49). In other words, χ can be extended to a homomorphism $P \to \mathbf{Q}^*$. Hence, $h(\chi; V) \in \mathfrak{H}_{\mathbf{Q}}'(V)$ can be defined. We shall show that $\phi(h(\chi; V)) = h(\chi; \mathfrak{g})$.

By (2.4), $h(\chi; V)$ can be written as a product of elements of the form

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$h(\chi_{r,z};\, V)$, where $z \in \mathbf{Q}^*$. Applying the homomorphism $\phi_r{:}SL(2;\mathbf{C}) \to G_{\mathbf{C}}'(V)$ to

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^{-1}-1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z-1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -z^{-1} \\ 0 & 1 \end{pmatrix},$$

and using (3.3), we obtain

$$h(\chi_{r,z};\, V) = x_{-r,\mathbf{C}}(z^{-1}-1;\, V)x_{r,\mathbf{C}}(1;\, V)x_{-r,\mathbf{C}}(z-1;\, V)x_{r,\mathbf{C}}(-z^{-1};\, V)$$

Applying ϕ to the above, we obtain $\phi(h(\chi_{r,z};\, V)) = h(\chi_{r,z};\mathfrak{g})$. Hence, $\phi(h(\chi;\, V)) = h(\chi;\mathfrak{g})$, as desired. Now, we have $\phi(h(\chi;\, V)\omega^{-1}) = 1$. In view of (4.1.1.) this implies that $h(\chi;\, V)\omega^{-1}$ commutes with all $\rho(X)$, $X \in \mathfrak{g}$. Hence, in each irreducible constituent V_α of V, $h(\chi;\, V)\omega^{-1}$ is a scalar multiple zI of the identity matrix I. Since $zI \in G_{\mathbf{C}}'(V_\alpha)$, we have $\det(zI) = 1$. Thus, z is a root of unity. On the other hand, since the regular basis $(v_1, \ldots, v_n)$ is assumed to be the union of regular bases of V_α's, both $h(\chi;\, V)$ and ω are represented on V_α by matrices with rational entries. Hence z is rational. Since z is a root of unity, $z = \pm 1$. Thus we have shown that $h(\chi;\, V)\omega^{-1}$ is a diagonal matrix with ± 1 on the diagonal. We already know that $\chi(r) = \pm 1$ for all roots r. Hence for any $\mathrm{M} \in P$ we have $\chi(\mathrm{M})^m = \pm 1$, where $m = [P{:}\, P(\mathfrak{g})]$. Hence $\chi(\mathrm{M}) = \pm 1$, since $\chi(\mathrm{M})$ is rational. Then, from the above it follows that ω is a diagonal matrix with ± 1 on the diagonal. By (4.9) we can write $\omega = yh(\chi';\, V)xy'$, where $y,\ y' \in \mathfrak{V}_{\mathbf{Q}}(V)$, $x \in \mathfrak{U}_{\mathbf{Q}}(V)$, $h(\chi';\, V) \in \mathfrak{H}_{\mathbf{Q}}'(V)$. Then

$$y^{-1}\omega y' = h(\chi';\, V)x. \tag{4.10.2}$$

Now number the basis elements $v_1, \ldots, v_n$ such that $\mathrm{M}_1 \geqslant \ldots \geqslant \mathrm{M}_n$. Then (4.5.1) shows that y^{-1}, y' are subdiagonal matrices with 1 on the diagonal, and that x is a superdiagonal matrix with 1 on the diagonal. Since both ω and $h(\chi';\, V)$ are diagonal, (4.10.2) implies that $\omega = h(\chi';\, V)$. This shows, since ω is a diagonal matrix with ± 1 on the diagonal, that $\chi'(\mathrm{M}) = \pm 1$ for all $\mathrm{M} \in P(V)$. Then $\chi'(\mathrm{M}) = \pm 1$ for all $\mathrm{M} \in P$, since $\chi'(\mathrm{M})$ is rational and $\chi'(\mathrm{M})^m = \pm 1$.

In order to show that the homomorphism $\chi{:}P \to \{\pm 1\}$ can be taken independently of V, write $\omega(V)$ for the element ω considered above. Now fix a faithful $\mathfrak{g}$-module V_0 and a homomorphism $\chi_0{:}P \to \{\pm 1\}$ such that $P(V_0) = P$ and $\omega(V_0) = h(\chi_0;\, V_0)$. Let $V' = V \oplus V_0$. Then, from what we have proved above there exists $\chi'{:}P \to \{\pm 1\}$ such that $\omega(V') = h(\chi';\, V')$. Clearly, the matrix $\omega(V')$ is a direct sum of the matrices $\omega(V)$ and $\omega(V_0)$, and $h(\chi';\, V')$ is a direct sum of the matrices $h(\chi'|P(V);\, V)$ and $h(\chi'|P(V_0);\, V_0)$. Hence, we have $h(\chi'|P(V);\, V) = h(\chi;\, V)$ and $h(\chi'|P(V_0);\, V_0) = h(\chi_0;\, V_0)$. Then $\chi = \chi'|P(V)$ and $\chi_0 = \chi'|P(V_0) = \chi'$. Hence, $\chi = \chi_0|P(V)$ with χ_0 independent of V. Thus (3.7) is proved.

(4.11) *Proof of* (3.8). The part that any element in $\mathfrak{U}_K(V)$ can be written

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in the form stated in (3.8) can be proved in the same way as Lemma 6 of **(3**, p. 39**)**. We shall prove only the uniqueness part.

(4.11.1) LEMMA. *Let V be a faithful $\mathfrak{g}$-module, and $(v_1, \ldots, v_n)$ a basis of V consisting of weight vectors. Then, for any root r, there exists an i such that $v_i X_{-r} = 0, v_i X_r \neq 0$.*

In order to prove this, let $\mathfrak{g}^{(r)}$ be as in (1.3), and regard V as a $\mathfrak{g}^{(r)}$-module. Then V is a direct sum of irreducible $\mathfrak{g}^{(r)}$-modules V_λ. Assume $r > 0$. Let Λ_λ be the highest weight of V_λ, and Λ the highest among Λ_λ for which $V_\lambda X_r \neq 0$. We shall show that for any weight vector v of weight Λ, we have $vX_{-r} = 0$. To prove this, one can clearly assume that $v \in V_\lambda$ for some λ. Then $\Lambda \leqslant \Lambda_\lambda$. If $\Lambda = \Lambda_\lambda$, then $vX_{-r} = 0$, since otherwise vX_{-r} would be a weight vector of weight $\Lambda_\lambda + r$ contained in V_λ. Assume $\Lambda < \Lambda_r$. Then (1.3) shows that there exists $v' \in V_\lambda$ such that $v'X_r = v$. In particular, $V_\lambda X_r \neq 0$. This contradicts the assumption on Λ.

Now let $\Lambda = \Lambda_\lambda$ and $v \in V_\lambda$, a weight vector of weight Λ. Then $V_\lambda X_r \neq 0$, and (1.3) shows that $vX_r \neq 0$. Since $(v_1, \ldots, v_n)$ is a basis of V consisting of weight vectors, v is a linear combination $\sum a_i v_i$ of v_i's of weight Λ. Since $vX_r = \sum a_i(v_i X_r) \neq 0$, it follows that $v_i X_r \neq 0$ for some v_i of weight Λ. But, as shown above, $v_i X_{-r} = 0$. This proves (4.11.1) for the case $r > 0$. The case $r < 0$ can be treated similarly.

(4.11.2) LEMMA. *Let m be a positive integer, and p a prime. Then $m!$ is not divisible by p^m.*

It is well known that if p^e is the highest power of p which divides m, then

$$e = [m/p] + [m/p^2] + \ldots,$$

where $[\alpha]$ denotes the greatest integer $\leqslant \alpha$. Hence

$$e < \sum_{i>0} mp^{-i} = m/(p-1) \leqslant m.$$

(4.11.3) LEMMA. *Let $x_{r,\mathbf{C}}(t; V) = (A_{ij}(t))$. Then the greatest common divisor d of the integers which can appear as the coefficient of t in some $A_{ij}(t)$, $i \neq j$, is* 1.

Suppose that (4.11.3) is false. Then there is a prime p which divides d. Set $v_i X_{-r} = \sum_j c_{ij} v_j$, $1 \leqslant i \leqslant n$. Then, if $c_{ij} \neq 0$, c_{ij} appears as the coefficient of $A_{ij}(t) = c_{ij} t$. Hence, p divides all c_{ij}. This implies that if $V_\mathbf{Z} = \mathbf{Z}v_1 + \ldots + \mathbf{Z}v_n$, then

$$(*) \qquad (V_\mathbf{Z})X_r \subseteq pV_\mathbf{Z}.$$

By (4.11.1) there exists a v_i such that

$$v_i X_{-r} = 0, \qquad v_i X_r \neq 0.$$

Set

$$u_0 = v_i, \qquad u_k = (k!)^{-1} u_0 X_r^k \qquad (k = 1, 2, \ldots).$$

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Since X_r acts on V as a nilpotent linear transformation by (1.4), there exists an integer $m > 0$ such that $u_m \neq 0$, $u_{m+1} = 0$. Since $u_1 = u_0 X_r \neq 0$, we have $m \geqslant 1$, and

$$u_0 x_{r,\mathbf{C}}(1; V) = u_0 + u_1 + \ldots + u_m.$$

By the regularity of the basis $(v_1, \ldots, v_n)$ it follows that $u_k \in V_{\mathbf{Z}}$ for $0 \leqslant k \leqslant m$. Also, from $(*)$ and the definition of u_k we have $(k!)p^{-k} u_k = u_k' \in V_{\mathbf{Z}}$. Set $u_m = \sum c_i v_i$, $u_m' = \sum c_i' v_i$, where $c_i, c_i' \in \mathbf{Z}$. Then $(m!)p^{-m} c_j = c_j'$ for $1 \leqslant j \leqslant n$. By (4.11.2) it follows that $c_i \equiv 0 \pmod{p}$ for $1 \leqslant j \leqslant n$. In other words,

$$(**) \qquad u_m \in pV_{\mathbf{Z}}.$$

On account of $u_0 X_{-r} = 0$, it can be seen easily that

$$u_j X_{-r} = (m - j + 1)u_{j-1} \qquad \text{for } 1 \leqslant j \leqslant m.$$

Hence,

$$u_m x_{-r,\mathbf{C}}(1; V) = u_0 + u_1 + \ldots + u_m.$$

Since $x_{-r,\mathbf{C}}(1; V)$ is a matrix with integral entries, it maps $V_{\mathbf{Z}}$ into itself. Then $(**)$ and the above implies that

$$(***) \qquad u_0 + u_1 + \ldots + u_m \in pV_{\mathbf{Z}}.$$

Now, one sees from the definition that u_k is a weight vector of weight $\mathrm{M}_i - kr$. Hence, u_k is a linear combination of v_j's of weight $\mathrm{M}_i - kr$. Therefore, $(***)$ implies that $u_k \in pV_{\mathbf{Z}}$ for $0 \leqslant k \leqslant m$. In particular, $v_i = u_0 \in pV_{\mathbf{Z}}$. This is a contradiction. Thus, (4.11.3) is proved.

(4.11.4) LEMMA. *$x_{r,K}(t; V) = 1$ implies $t = 0$.*

Suppose that $x_{r,K}(t; V) = 1$ for some $t \neq 0$. In view of (4.5.1) this implies that if $A_{ij}(T)$, $i \neq j$, is of degree 1, then its coefficient of T is divisible by p, where p is the characteristic of K. But this is impossible by (4.11.3).

(4.11.5) LEMMA. *If $0 < r_1 < r_2 < \ldots < r_m$ is an increasing sequence of positive roots and if*

$$x_{r_1,K}(t_1; V)x_{r_2,K}(t_2; V) \ldots x_{r_m,K}(t_m; V) = 1,$$

then $t_1 = t_2 = \ldots = t_m = 0$.

Clearly it suffices to prove (4.11.5) under the assumption that K is algebraically closed. We shall proceed by induction on the smallest root r_1. If r_1 is the greatest positive root, then $m = 1$ and (4.11.4) can be applied. Let r be a positive root and assume that (4.11.5) is proved for $r_1 > r$. Suppose that $r_1 = r$. Take any χ in $X(V, K)$ such that $\chi(r_1) = 1$. Then by (3.4) we have

$$(*) \qquad \Big(\prod_{i \geqslant 2} x_{r_i,K}(t_i; V)\Big)\Big(\prod_{i \geqslant 2} x_{r_i,K}(\chi(r_i)t_i; V)\Big)^{-1} = 1.$$

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By (3.1) it can be seen easily that the left-hand side of the above equation can be written in the form

$$x_{r_2,K}(t_2 - \chi(r_2)t_2;\ V) \prod_{s>r_2} x_{s,K}(u_s;\ V).$$

Hence by the assumption of the induction we have $t_2 - \chi(r_2)t_2 = 0$. Similarly, from $(*)$ we obtain $t_i - \chi(r_i)t_i = 0$ for all $i > 1$. Suppose that $t_i \neq 0$ for some i. Then for any $\chi \in X(V, K)$ such that $\chi(r_1) = 1$ we have $\chi(r_i) = 1$. It can be seen easily that this cannot happen for two positive roots r_1 and r_i unless $r_1 = r_i$ when K is algebraically closed. Hence, $t_i = 0$ for all $i > 1$. Then we are in the case $m = 1$, and (4.11.4) can be applied. This proves (4.11.5).

Now we shall prove the uniqueness part of (3.8). Suppose

$$\prod x_{r_i,K}(t_i;\ V) = \prod x_{r_i,K}(t_i';\ V),$$

where the products on both sides run over positive roots $r_1 < r_2 < \ldots < r_m$ in this order. Then, by (3.1),

$$x_{r_1,K}(t_1 - t_1';\ V) \prod_{s>r_1} x_{s,K}(u_s; K) = 1.$$

Hence, by (4.11.5), we have $t_1 = t_1'$. Now, proceeding by induction, we get $t_i = t_i'$ for all i. Thus, the uniqueness part of (3.8) is proved.

(4.12) *Proof of* (3.9). A certain procedure of putting a given element in $G_K(V)$ into the form specified in (3.9) can be found in (**3**, pp. 38–40). As was pointed out in (**8**, p. 437), the properties (3.1)–(3.8) are sufficient to carry out this procedure. The uniqueness part of (3.9) can be proved in exactly the same way as the corresponding theorem in (**3**, p. 42, Theorem 2). We omit the details.

(4.13) *Proofs of* (3.10) and (3.11). Clearly $V = V_1 \oplus V_2$ is a faithful $\mathfrak{g}$-module. Take the union of regular bases of V_1 and V_2 as the regular basis of V. For $z \in G_K(V)$, define $\phi_i(z)$ to be the restriction of z to V_i. Then it is clear that

$$\phi_i(x_{r,K}(t;\ V)) = x_{r,K}(t;\ V_i), \qquad \phi_i(h(\chi;\ V)) = h(\chi|P(V_i);\ V_i)$$

and that ϕ_i is a homomorphism $G_K(V) \to G_K(V_i)$. Then we have $\phi_i(\omega(w,\ V)) = \omega(w,\ V_i)$. Since $P(V) = P(V_1) + P(V_2)$ and since $P(V_2) \subseteq P(V_1)$ by our assumption, $P(V) = P(V_1)$. Hence ϕ_1 is onto. Let z be in the kernel of ϕ_1. By (3.9), z can be written as $z = xh(\chi;\ V)\omega(w,\ V)x'$, where $x \in \mathfrak{U}_K(V)$, $x' \in \mathfrak{U}_{w,K}(V)$. Then

$$1 = \phi_1(z) = \phi_1(x)h(\chi|P(V_1);\ V_1)\omega(w,\ V_1)\phi_1(x').$$

Hence, by the uniqueness part of (3.9), we have

$$\phi_1(x) = 1, \qquad h(\chi|P(V_1);\ V_1) = 1, \qquad \omega(w,\ V_1) = 1, \qquad \phi_1(x') = 1,$$

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and consequently $x = x' = 1, w = 1, \chi|P(V_1) = 1$. Then $\chi = 1$, since $P(V_1) = P(V)$, as shown above. Hence, $z = 1$, and ϕ_1 is an isomorphism. It can be seen easily that $\phi = \phi_2 \circ \phi_1^{-1}$ is the desired homomorphism $G_K(V_1) \to G_K(V_2)$. The statement about the image and the kernel of ϕ is also clear from the above. Note that if $\chi|P(V_2) = 1$, then, by (2.1), $\chi(r) = 1$ for all roots r, and hence by (3.4) $h(\chi; V_1)$ must be in the centre of $G_K(V_1)$. The statement about the restriction ϕ' of ϕ to $G_K'(V_1)$ follows easily from the above.

By setting $V_1 = V_2$ in (3.10), we obtain (3.11).

(4.14) *Proof of* (3.12). Let z be the centre of $G_K(V)$. By (3.9), $z = xh\omega(w, V)x'$, where $x \in \mathfrak{U}_K(V), x' \in \mathfrak{U}_{w,K}(V), h \in \mathfrak{H}_K(V)$. Then $z = x'xh\omega(w, V)$. Hence, by the uniqueness part of (3.9), we have $x' = 1$. Suppose that $w \neq 1$. Then there exists a root $r > 0$ such that $w(r) < 0$. We have

$$x_{r,K}(1; V)xh\omega(w, V) = xh\omega(w, V)x_{r,K}(1; V).$$

This contradicts the uniqueness part of (3.9). Hence, $w = 1$, and $z = xh$. Suppose that $x \neq 1$, and write

$$x = x_{r_1,K}(t_1; V) \ldots x_{r_m,K}(t_m; V),$$

where $0 < r_1 < \ldots < r_m$, and $t_i \neq 0$ for all i.

Suppose that no r_i is a fundamental root. Then for any fundamental root a we have $0 < w_a(r_i) \leqslant r_i$ for all i. Choose a such that $w_a(r_1) < r_1$. Then by (3.6) and (3.1) we have

$$z' = \omega_{a,K}(V)z\omega_{a,K}(V)^{-1} = x_{s_1,K}(u_1; V) \ldots x_{s_k,K}(u_k; V)h',$$

where $h' \in \mathfrak{H}_K(V)$, $0 < s_1 < s_2 < \ldots < s_k, s_1 < r_1$, and no u_i is 0. Then $z = z'$ contradicts the uniqueness part of (3.9). Hence, some r_i is a fundamental root.

We may assume that $r_1 = a$ is fundamental. Then $w_a(r_i) > 0$ for $2 \leqslant i \leqslant m$. Hence $z' = \omega_{a,K}(V)z\omega_{a,K}(V)^{-1}$ is of the form $x_{-a,K}(\pm t_1; V)z_1$, where $z_1 \in \mathfrak{U}_K(V)\mathfrak{H}_K(V)$. Since $z' = z$ and $z \in \mathfrak{U}_K(V)\mathfrak{H}_K(V)$, it follows that $x_{-a,K}(\pm t_1)$ is in $\mathfrak{U}_K(V)\mathfrak{H}_K(V)$. If the basis elements $v_1, \ldots, v_n$ are numbered such that $\mathrm{M}_1 \geqslant \ldots \geqslant \mathrm{M}_n$, then $x_{-a,K}(\pm t_1)$ is a subdiagonal matrix with 1 on the diagonal, while $\mathfrak{U}_K(V)\mathfrak{H}_K(V)$ consists of superdiagonal matrices. Hence, we must have $x_{-a,K}(\pm t_1) = 1, t_1 = 0$. This is a contradiction. Hence, $x = 1$, and $z = h(\chi; V)$. Since z commutes with $x_{r,K}(1; V)$, we have, by (3.4), $\chi(r) = 1$ for all roots r. Conversely, if $\chi(r) = 1$ for all roots r, then clearly $h(\chi; V)$ belongs to the centre of $G_K(V)$. Since $\chi(r) = 1$ for all roots r if and only if $\chi|P(V) = 1$, it follows that the centre of $G_K(V)$ is isomorphic to $\mathrm{Hom}(P(V)/P(\mathfrak{g}), K^*)$.

If V is irreducible, and if $h(\chi; V)$ is in the centre of $G_K(V)$, then by (2.2) and the above we have $\chi(\mathrm{M}_1) = \chi(\mathrm{M}_2) = \ldots = \chi(\mathrm{M}_n)$. Hence, $h(\chi; V)$ is of the form $\chi(\mathrm{M}_1)I$, where I is the identity matrix.

The statement about the centre of $G_K'(V)$ can be proved similarly.

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5. Algebraic group theoretical aspects. In this section, we shall only state some theorems on the algebraic group theoretical aspects of the group $G_K(V)$. For the proofs, see Chevalley's paper cited in the footnote on the first page of this paper.

Let Ω be a universal domain over the given field K, and simply write $\mathfrak{X}_r(V)$, $\mathfrak{U}(V)$, $\mathfrak{V}(V)$, $G(V)$, $\mathfrak{H}(V)$ for $\mathfrak{X}_{r,\Omega}(V)$, $\mathfrak{U}_\Omega(V)$, $\mathfrak{V}_\Omega(V)$, $G_\Omega(V)$, $\mathfrak{H}_\Omega(V)$, respectively.

(5.1) The groups $\mathfrak{X}_r(V)$, $\mathfrak{U}(V)$, $\mathfrak{V}(V)$, $\mathfrak{H}(V)$, $\mathfrak{U}(V)\mathfrak{H}(V)$, $\mathfrak{V}(V)\mathfrak{H}(V)$ and $G(V)$ are connected algebraic groups defined over the prime field. The groups $\mathfrak{X}_{r,K}(V)$, $\mathfrak{U}_K(V), \ldots, G_K(V)$ are the sets of rational points over K of $\mathfrak{X}_r(V)$, $\mathfrak{U}(V), \ldots, G(V)$, respectively.

(5.2) The group $G(V)$ is semi-simple; $\mathfrak{H}(V)$ is a maximal torus, and $\mathfrak{U}(V)\mathfrak{H}(V)$ is a Borel subgroup of $G(V)$.

(5.3) Let V_1, V_2 be faithful $\mathfrak{g}$-modules such that $P(V_2) \subseteq P(V_1)$. Then the homomorphism $\phi : G(V_1) \to G(V_2)$ given in (3.10) is an isogeny defined over the prime field.

(5.4) Every connected semi-simple algebraic group G is isomorphic (as an algebraic group) to an algebraic group of the form $G(V)$.

References

1. A. Borel, *Groupes linéaires algébriques*, Ann. of Math., *64* (1956), 20–82.
2. E. Cartan, *Oeuvres complètes*, Part 1, Vol. *1* (Paris, 1952).
3. C. Chevalley, *Sur certain groupes simples*, Tôhoku Math. J. (2), *7* (1955), 14–66.
4. ——— *Séminaire: Classification des groupes de Lie algébriques*, Ec. Norm. Sup. (1956–58), Vols. 1, 2.
5. ——— *Théorie des groupes de Lie*, Vol. 2 (Paris, 1951).
6. N. Jacobson, *Lie algebras* (New York, 1962).
7. R. Ree, *On some simple groups defined by C. Chevalley*, Trans. Am. Math. Soc., *84* (1957), 392–400.
8. ——— *A family of simple groups associated with the simple Lie algebra of type* (G_2), Am. J. Math., *83* (1961), 432–462.
9. ——— *A family of simple groups associated with the simple Lie algebra of type* (F_4), Am. J. Math., *83* (1961), 401–420.

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