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GENERAL CONVERGENCE IN QUASI-NORMAL FAMILIES

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Abstract Montel introduced the concept of quasi-normal families $f : \Omega \to \mathbb{C}$ in 1922: $\mathcal{F}$ is quasi-normal of order N if every sequence $\{f_n\}$ from $\mathcal{F}$ has a subsequence which converges uniformly on compact subsets of $\Omega \setminus Z^\dagger$, where $Z^\dagger \subset \Omega$ contains at most $N \in \mathbb{N}$ elements. ($\mathcal{F}$ is of order $N := \infty$ if every such exceptional set $Z^\dagger$ is finite.) The problem is that $Z^\dagger$ normally depends on the subsequence. So even if every sequence has a subsequence which converges to a given function f in Ω except at N points, the sequence itself may not converge in any domain $D \subseteq \Omega$.

In this paper we introduce the concept of *general convergence*. Indeed, $\{f_n\}$ above converges generally to f. We also introduce a related concept, *restrained sequences*, and study some of their properties. The definitions extend earlier concepts introduced for sequences of linear fractional transformations.

Keywords: restrained sequences; N-valent functions; general convergence; quasi-normal families

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1. Introduction

The concept of *quasi-normal families* for holomorphic functions and for meromorphic functions in a domain $\Omega \subseteq \mathbb{C}$ was introduced by Montel [**5**, **6**]. The purpose of the present paper is to introduce a convergence concept for quasi-normal families, which we call *general convergence* (Definition 2.6). It generalizes normal convergence (uniform convergence on compact subsets) in the sense that if $\{f_n\}$ converges normally in a domain D, then it converges generally to f in D, whereas the converse does not hold in general. It is related to convergence in measure, but it is more precise. In particular it allows us to approximate constant limits numerically in an easy way; a fact which is important in continued fraction theory and dynamical systems (see Remark 3.4). We shall also introduce what we call *restrained sequences* in a quasi-normal family (Definition 2.9). More generally, one can talk about mappings from one metric space (Ω_1, m_1) to another (Ω_2, m_2), but we shall mainly stay on the Riemann sphere $\hat{\mathbb{C}}$ equipped with the chordal metric (spherical metric)

$$d(z, w) := \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} \quad \text{for } z, w \in \hat{\mathbb{C}}.$$

We shall use the notation $\bar{A}$, A° and ∂A to denote the closure, the interior and the boundary of a set A in $\hat{\mathbb{C}}$, or, more generally, in a metric space (Ω, m). Moreover, $\sigma(z, A)$ denotes the Euclidean distance and $d(z, A)$ denotes the chordal distance between a point z and a set A in $\hat{\mathbb{C}}$ and $d(A, D)$ the chordal distance between two subsets of $\hat{\mathbb{C}}$, $B(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$ and $B_d(\hat{a}, r) := \{z \in \hat{\mathbb{C}} : d(z, \hat{a}) < r\}$ for some $a \in \mathbb{C}$, $\hat{a} \in \hat{\mathbb{C}}$ and $r > 0$. Similarly, $B_m(a, r)$ is an r-neighbourhood of a point a in a metric space (Ω, m) and $B_m(A, r) := \cup_{a \in A} B_m(a, r)$ for a set $A \in \Omega$. In particular, $B_m(A, r) = \emptyset$ if $A = \emptyset$. Finally, $\sup\{m(p, q) : p, q \in \emptyset\} := 0$ and $\hat{\mathbb{N}} := \mathbb{N} \cup \{0\} \cup \{\infty\}$.

In § 2 we set the scene and introduce the two concepts. In § 3 we give some useful characterizations of these concepts. In § 4 we study families of $\leqslant N$-valent meromorphic functions, and in § 5 we look at families of univalent functions. Finally, in § 6 we connect this to generalized iteration of univalent meromorphic functions.

2. Definitions and examples

Montel restricted his definitions to holomorphic or meromorphic functions in planar domains. It is useful to have a broader perspective. So, we change Montel's definition slightly to concern mappings between metric spaces.

Definition 2.1. We say that a family $\mathcal{F}$ of mappings from the metric space (Ω_1, m_1) into the metric space (Ω_2, m_2) is quasi-normal if every sequence from $\mathcal{F}$ has a subsequence which converges uniformly on compact subsets of $\Omega_1 \setminus Z^\dagger$, where $Z^\dagger \subseteq \Omega_1$ is a finite set, possibly depending on the subsequence.

Remark 2.2. If $(\Omega_1, m_1) = (\Omega_2, m_2) = (\hat{\mathbb{C}}, d)$, then the functions meromorphic in Ω_1 are the rational functions, the polynomials and the constant functions.

If Ω_1 is a proper subset of $\hat{\mathbb{C}}$ in this setting, then we may without loss of generality assume that $\Omega_1 \subseteq \mathbb{C}$, since a rotation φ of the Riemann sphere (i.e. φ is a linear fractional transformation, isometric with respect to the chordal metric) can be chosen such that $\varphi(u) = \infty$ for a $u \notin \Omega_1$. If f is meromorphic in Ω_1, then $f \circ \varphi^{-1}$ is meromorphic in $\tilde{\Omega}_1 := \varphi(\Omega_1) \subseteq \mathbb{C}$, and the classical work by Montel and others is valid.

Example 2.3 (see p. 67 of [6]). Let $P(z)$ be a fixed complex polynomial of degree greater than or equal to 1, and let $\mathcal{F}$ be the family of functions of the form $f(z) := aP(z)$, $a \in \mathbb{C}$. Then $\mathcal{F}$ consists of functions holomorphic in $\Omega_1 = \mathbb{C}$. Let $f_n(z) := a_n P(z)$ be a sequence from $\mathcal{F}$. Let $\{f_{n_k}\}$ be a subsequence such that $a_{n_k} \to a \in \hat{\mathbb{C}}$.

If $a \neq \infty$, then $\{f_{n_k}(z)\}$ converges spherically uniformly to $aP(z)$ in compact subsets of $\mathbb{C}$. If $a = \infty$, then $\{f_{n_k}(z)\}$ converges spherically uniformly to ∞ in compact subsets of $\mathbb{C} \setminus Z^\dagger$, where $Z^\dagger$ is the set of zeros of $P(z)$. It does not converge spherically uniformly in any open set containing points from $Z^\dagger$ since the limit function is discontinuous at $Z^\dagger$. Hence $\mathcal{F}$ is quasi-normal in $\mathbb{C}$.

Example 2.4. Let $P(z)$ be a fixed complex polynomial of degree greater than or equal to 1, and let $\mathcal{F}$ be the family of functions $f(z) := a/P(z)$, $a \in \mathbb{C} \setminus \{0\}$. Then $\mathcal{F}$ consists of functions meromorphic in $\Omega_1 = \hat{\mathbb{C}}$. Let $f_n(z) := a_n/P(z)$ be a sequence from $\mathcal{F}$. If $a_n \to a \neq 0, \infty$, then $\{f_n\}$ converges spherically uniformly in $\hat{\mathbb{C}}$ to an $f \in \mathcal{F}$. If $a_n \to 0$,

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then $\{f_n\}$ converges spherically uniformly to 0 on compact subsets of $\hat{\mathbb{C}} \setminus Z^\dagger$, where $Z^\dagger$ is the set of zeros of P. Finally, if $a_n \to \infty$, then $\{f_n\}$ converges spherically uniformly to ∞ on compact subsets of $\hat{\mathbb{C}} \setminus \{\infty\} = \mathbb{C}$. Hence $\mathcal{F}$ is quasi-normal in $\hat{\mathbb{C}}$.

In Example 2.3 we found that a convergent subsequence either converges in $\mathbb{C}$ to a holomorphic function, or to infinity in a smaller domain. This is the typical situation for holomorphic functions [**1**, p. 131] but not for meromorphic functions.

Example 2.5 (see p. 213 of [7]). Let $P(z)$ be a fixed complex polynomial of degree greater than or equal to 1, let $g(z)$ and $h(z)$ be two distinct holomorphic functions in $\mathbb{C}$, and let $\mathcal{F}$ be the family of functions of the form

$$f(z) := g(z) + \frac{h(z) - g(z)}{1 + aP(z)} \quad \text{for } a \in \mathbb{C}.$$

Let $f_n \in \mathcal{F}$ be chosen such that $a_n \to \infty$. Then $f_n(z) \to g(z)$ for $z \in \mathbb{C} \setminus Z^\dagger$, where $Z^\dagger$ is the set of zeros of P, whereas $f_n(z) \to h(z)$ for $z \in Z^\dagger$. That is, $\{f_n\}$ converges spherically uniformly on compact subsets of $\mathbb{C} \setminus Z^\dagger$ to a holomorphic function, but not on all of $\mathbb{C}$.

We shall say that a sequence $\{f_n\}$ of mappings from (Ω_1, m_1) into (Ω_2, m_2) converges *N-quasi-normally* (or just *quasi-normally*) to f in the space Ω_1 for an $N \in \hat{\mathbb{N}}$ if it converges uniformly on compact subsets of $\Omega_1 \setminus Z^\dagger$ for some finite set $Z^\dagger \subseteq \Omega_1$ with less than or equal to N elements. Following Montel we say that $Z^\dagger$ is the *set of irregular points* for a quasi-normally convergent sequence if $Z^\dagger$ is minimal. Moreover, the *order* of a quasi-normal family $\mathcal{F}$ is the smallest $N \in \hat{\mathbb{N}}$ such that every sequence from $\mathcal{F}$ has a p-quasi-normally convergent subsequence for some $p \leqslant N$. Hence, if $P(z)$ in Examples 2.3, 2.4 or 2.5 has N distinct zeros, then the families in those examples are quasi-normal of order N.

Definition 2.6. We say that a sequence $\{f_n\}$ of mappings from a metric space (Ω_1, m_1) into a metric space (Ω_2, m_2) converges N-generally (or just generally) in Ω_1 to the mapping f for some $N \in \hat{\mathbb{N}}$ if every subsequence of $\{f_n\}$ has a subsequence which converges N-quasi-normally to f in Ω_1.

Remark 2.7. If there is a point $z_0 \in \Omega_1$ which is an irregular point for every quasi-normally convergent subsequence of a generally convergent sequence $\{f_n\}$, then the limit function f is not specified at z_0. However, this is no problem if the singularity for f at z_0 is removable. Otherwise, it would be more correct to say that $\{f_n\}$ converges generally to the restriction of f to $\Omega \setminus \{z_0\}$, but we shall not make this distinction in this paper.

In the next example, the sequence $\{f_n\}$ in question does not converge spherically uniformly on any open set in $\hat{\mathbb{C}}$. Still, it converges 1-generally to 1 in $\hat{\mathbb{C}}$.

Example 2.8. Let $\{a_n\}$ be a sequence of complex numbers dense in $\mathbb{C}$. For each fixed $n \in \mathbb{N}$, let $\{b_{n,m}\}_{m=1}^{\infty}$ be a sequence of complex numbers not equal to a_n, but converging to a_n, uniformly with respect to n in the Euclidean metric. Finally, let

$f_{n,m}(z) := (z - a_n)/(z - b_{n,m})$ for all m and n, and construct the sequence $\{\tilde{f}_n\}$ of meromorphic mappings from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ by setting

$$\left.\begin{array}{llll} \tilde{f}_1 := f_{1,1}, & & & \\ \tilde{f}_2 := f_{1,2}, & \tilde{f}_3 := f_{2,2}, & & \\ \tilde{f}_4 := f_{1,3}, & \tilde{f}_5 := f_{2,3}, & \tilde{f}_6 := f_{3,3}, & \\ \tilde{f}_7 := f_{1,4}, & \tilde{f}_8 := f_{2,4}, & \tilde{f}_9 := f_{3,4}, & \tilde{f}_{10} := f_{4,4}, \\ & & & \qquad \text{etc.}\ldots. \end{array}\right\} \tag{2.1}$$

Then $\{\tilde{f}_n\}$ does not converge spherically uniformly on any open set in $\hat{\mathbb{C}}$. However, we shall see that it converges 1-generally to 1 in $\hat{\mathbb{C}}$. Let $\{f_k\}$ be a subsequence of $\{\tilde{f}_n\}$. Then $f_k = f_{n_k,m_k}$, where $n_k \leqslant m_k \to \infty$. Without loss of generality (taking subsequences) we may assume that $n_k \to N \in \hat{\mathbb{N}}$, and that $a_{n_k} \to a \in \hat{\mathbb{C}}$. Assume first that $a \neq \infty$.

Case 1 ($N \neq \infty$). Then $n_k = N$ from some k onwards, and thus we assume that $f_k = f_{N,m_k}$ which converges 1-quasi-normally to 1 with irregular point $a = a_N$.

Case 2 ($N = \infty$). Without loss of generality (taking subsequences) $|a_{n_k} - a| \leqslant 1/k$ and $|a_{n_k} - b_{n_k,m_k}| \leqslant 1/k$ for all k. Let E be a compact subset of $\hat{\mathbb{C}} \setminus \{a\}$, and let $k_0 \geqslant 4/\sigma(a,E)$. For $z \in E$ and $k \geqslant k_0$ we then have

$$|f_k(z) - 1| = \left|\frac{a_{n_k} - b_{n_k,m_k}}{z - b_{n_k,m_k}}\right| \leqslant \frac{1/k}{|z-a| - |a_{n_k} - a| - |a_{n_k} - b_{n_k,m_k}|} \leqslant \frac{2/k}{\sigma(a,E)} \to 0.$$

So, again, f_k converges 1-quasi-normally to 1 with irregular point a.

Assume next that $a = \infty$. Then $N = \infty$. We shall prove that $f_k(z) \to 1$ uniformly on every compact subset $E \subseteq \hat{\mathbb{C}} \setminus \{a\} = \mathbb{C}$. Since $z \in E$ means that z is bounded, whereas $b_{n_k,m_k} \to \infty$, we have from some k onwards that $|z| < \frac{1}{2}|b_{n_k,m_k}|$. Since $|a_{n_k} - b_{n_k,m_k}| \leqslant (1/k) \to 0$, we thus have

$$|f_k(z) - 1| = \left|\frac{a_{n_k} - b_{n_k,m_k}}{z - b_{n_k,m_k}}\right| \leqslant \frac{2/k}{|b_{n_k,m_k}|} \to 0.$$

This example shows that general convergence is a natural concept of convergence in quasi-normal families. We can find the convergence hidden in the sequence $\{\tilde{f}_n\}$ by allowing the argument z to vary with n to avoid the irregular points. (For a more precise statement, see Theorem 3.5. For approximating the limit, see Remarks 3.4 and 3.6.)

The second concept is particularly useful in dynamical systems. It says that the asymptotic behaviour of $\{f_n(z)\}$ is *essentially* independent of z. (See also Theorem 3.1 and Remark 3.2.)

Definition 2.9. We say that a sequence $\{f_n\}$ of mappings from a metric space (Ω_1, m_1) into a metric space (Ω_2, m_2) is N-restrained (or just restrained) in Ω_1 for some $N \in \hat{\mathbb{N}}$ if every subsequence of $\{f_n\}$ has a subsequence which converges N-quasi-normally in Ω_1 to some constant (which may depend on the subsequence).

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Example 2.10. Let $\{a_n\}$ and $\{b_{n,m}\}$ be as in Example 2.8, let $f_{n,m}(z) := a_n(z-a_n)/(z-b_{n,m})$, and let $\{\tilde{f}_n\}$ be given by (2.1). Let $\{\tilde{f}_{n_k}\}$ be a subsequence of $\{\tilde{f}_n\}$. Then there is a subsequence of $\{a_{n_k}\}$ converging to some $a \in \hat{\mathbb{C}}$. By the arguments in the previous example, the corresponding subsequence $\{\tilde{f}_{n_k}\}$ has a subsequence converging 1-quasi-normally to a in $\hat{\mathbb{C}}$ with irregular point a. Hence, $\{\tilde{f}_n\}$ is 1-restrained in $\hat{\mathbb{C}}$.

3. Some useful characterizations

In this section we restrict our attention to N-generally convergent and N-restrained sequences with *finite* N.

Theorem 3.1. *The sequence $\{f_n\}$ of mappings from (Ω_1, m_1) into (Ω_2, m_2) is N-restrained for some $N \in \mathbb{N} \cup \{0\}$ if and only if there exists a sequence $\{Z_n^\dagger\}$ of sets of N elements from Ω_1 such that*

$$\lim_{n\to\infty} \{\sup\{m_2(f_n(u), f_n(v)) : u, v \in C \setminus B_{m_1}(Z_n^\dagger, \delta)\}\} = 0 \tag{3.1}$$

for every compact set $C \subseteq \Omega_1$ and every $\delta > 0$.

Proof. First let $\{f_n\}$ be N-restrained in Ω_1. Then every subsequence of $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ converging uniformly on compact subsets of $\Omega_1 \setminus Z^\dagger$ to some constant c, for some $Z^\dagger \subseteq \Omega_1$ which contains N elements, where $Z^\dagger$ and c in general depend on the subsequence. That is,

$$\lim_{k\to\infty} \{\sup\{m_2(f_{n_k}(u), f_{n_k}(v)) : u, v \in C \setminus B_{m_1}(Z^\dagger, \delta)\}\} = 0 \tag{3.2}$$

for every compact set $C \subseteq \Omega_1$ and every $\delta > 0$. Assume that there exists no sequence $\{Z_n^\dagger\}$ such that (3.1) holds for every C and δ. Then there exist a subsequence $\{n_k\}$ of $\mathbb{N}$, a compact set $C \subseteq \Omega_1$ and a $\delta > 0$ such that

$$\liminf_{k\to\infty} \{\sup\{m_2(f_{n_k}(u), f_{n_k}(v)) : u, v \in C \setminus B_{m_1}(Z_{n_k}^\dagger, \delta)\}\} > 0 \tag{3.3}$$

for every sequence of sets $Z_{n_k}^\dagger \subseteq \Omega_1$ with N elements. However, this subsequence must have a subsequence satisfying (3.2), which contradicts (3.3). Hence (3.1) holds.

Next, let there exist a sequence $\{Z_n^\dagger\}$ of sets of N elements from Ω_1 such that (3.1) holds for every compact set $C \subseteq \Omega_1$ and every $\delta > 0$. Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$. For each $k \in \mathbb{N}$, we number the points $z_{n_k}^{(j)\dagger} \in Z_{n_k}^\dagger$, $j = 1, 2, \ldots, N$, arbitrarily. Now, a sequence $\{a_n\}$ from Ω_1 must have one of the following two properties.

(i) $\{a_n\}$ has a limit point in $a \in \Omega_1$, in which case $\{a_n\}$ has a subsequence converging to a.

(ii) $\{a_n\}$ has no limit point in Ω_1, in which case $\liminf m_1(a_n, C) > 0$ for every compact $C \subseteq \Omega_1$.

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For each $j \in \{1, 2, \dots, N\}$, $\{z_{n_k}^{(j)\dagger}\}$ is a sequence from Ω_1. Hence we may assume (taking subsequences) that either $z_{n_k}^{(j)\dagger} \to z^{(j)\dagger} \in \Omega_1$ or $\liminf m_1(z_{n_k}^{(j)\dagger}, C) > 0$ for every compact $C \subseteq \Omega_1$.

Let $Z^\dagger$ be the set of points $z^{(j)\dagger} \in \Omega_1$. We want to prove that $\{f_{n_k}\}$ has a subsequence converging normally to a constant in $\Omega_1 \setminus Z^\dagger$. Let C_1 be an arbitrarily chosen compact subset of $\Omega_1 \setminus Z^\dagger$. Then there exist a compact $C \subseteq \Omega_1$ and a $\delta > 0$ such that $C_1 \subseteq C \setminus B_{m_1}(Z^\dagger, 2\delta)$. Further, let $k_0 \in \mathbb{N}$ be chosen so large that

$$d(z_{n_k}^{(j)\dagger}, z^{(j)\dagger}) < \delta \quad \text{for all } j \text{ such that } z_{n_k}^{(j)\dagger} \to z^{(j)\dagger} \in Z^\dagger,$$
$$z_{n_k}^{(j)\dagger} \in \Omega_1 \setminus C \quad \text{otherwise}$$

for all $k \geqslant k_0$. Then $B_{m_1}(z_{n_k}^{(j)\dagger}, \delta) \subseteq B_{m_1}(Z^\dagger, 2\delta)$ for $k \geqslant k_0$ for each j such that $z_{n_k}^{(j)\dagger} \to Z^\dagger$, and thus $C_1 \subseteq C \setminus B_{m_1}(Z_{n_k}^\dagger, \delta)$. It follows therefore from (3.1) that $m_2(f_{n_k}(u), f_{n_k}(v)) \to 0$ uniformly with respect to $u, v \in C_1$. Without loss of generality (taking subsequences) we may assume that $f_{n_k}(z_0) \to c$ for some constant c at some given point $z_0 \in C_1$. Then $\{f_{n_k}\}$ converges uniformly to c in C_1. This proves that $\{f_{n_k}\}$ converges p-quasi-normally to a constant with all irregular points $\subseteq Z^\dagger$, where $0 \leqslant p \leqslant N$ is the number of points in $Z^\dagger$. Hence $\{f_n\}$ is N-restrained in Ω_1.

□

If all $Z_n^\dagger$ are identical, i.e. $Z_n^\dagger = Z^\dagger$ for all n, then (3.1) means that the diameter of $f_n(C)$ shrinks to 0 as $n \to \infty$ for compact subsets C of $\Omega_1 \setminus Z^\dagger$. If we choose one $\ell_n \in f_n(C)$ for each n, then the asymptotic behaviour of $\{\ell_n\}$ is independent of C, and is called the *limiting structure* of $\{f_n\}$. Clearly, $m_2(f_n(z), \ell_n) \to 0$ uniformly on compact subsets of $\Omega_1 \setminus Z^\dagger$.

Remark 3.2. The important thing about N-restrained sequences is that $Z_n^\dagger$ may vary with n. Then the compact subsets of Ω_1 on which $m_2(f_n(z), \ell_n) \to 0$ uniformly also vary with n: for each n we have to make sure that we stay away from $Z_n^\dagger$ as described in (3.1). The typical case is that $\{Z_n^\dagger\}$ is more or less unknown. Still one may be able to detect the main stream asymptotic behaviour, i.e. the limiting structure, of a 1-restrained sequence $\{f_n\}$ in the following way: choose 3 distinct points z_k from Ω_1. For each n pick out the two function values $f_n(z_{j_n})$ and $f_n(z_{k_n})$ such that

$$m_2(f_n(z_{j_n}), f_n(z_{k_n})) \leqslant m_2(f_n(z_j), f_n(z_k)) \quad \text{for all } z_j, z_k.$$

Let ℓ_n be the intermediate value of these two. Then the asymptotic behaviour of $\{\ell_n\}$ describes the limiting structure of $\{f_n\}$. In other words, if N is equal to 1, we have a simple way of determining such a sequence $\{\ell_n\}$. This idea may be adapted for larger values of N under proper conditions. This makes the concept particularly useful in practical applications.

Theorem 3.1 also allows for a wider definition of N-restrained sequences for finite N.

Definition 3.3. Let (Ω_1, m_1) and (Ω_2, m_2) be two metric spaces, and let $\Omega_{1,n} \subseteq \Omega_1$ for all n. We say that a sequence $\{f_n\}$ of mappings f_n from $(\Omega_{1,n}, m_1)$ into (Ω_2, m_2) is N-restrained in $\{\Omega_{1,n}\}$ for some $N \in \mathbb{N} \cup \{0\}$ if there exists a sequence $\{Z_n^\dagger\}$ of sets of N elements from Ω_1 such that

$$\lim_{n\to\infty} \{\sup\{m_2(f_n(u), f_n(v)) : u, v \in (C \cap \Omega_{1,n}) \setminus B_{m_1}(Z_n^\dagger, \delta)\}\} = 0 \tag{3.4}$$

for every compact subset C of Ω_1 and every $\delta > 0$.

Of course, we could have made this definition even more general, but this form is what we shall use in § 5.

Remark 3.4. If $\{f_n\}$ converges N-generally to a constant c, then $\{f_n\}$ is in particular N-restrained. Hence $\{f_n\}$ converges N-generally to a constant c for some $N \in \mathbb{N} \cup \{0\}$ if and only if there exists a sequence $\{Z_n^\dagger\}$ of sets with N elements from Ω_1 such that

$$\lim_{n\to\infty} \{\sup\{m_2(f_n(v), c) : v \in C \setminus B_{m_1}(Z_n^\dagger, \delta)\}\} = 0 \tag{3.5}$$

for every compact $C \subseteq \Omega_1$ and every $\delta > 0$. Moreover, we can approximate c by the intermediate value ℓ_n in Remark 3.2. This simple observation is the reason why general convergence is so important in continued fraction theory. There are also methods to obtain bounds for the error in the approximation $\ell_n \approx c$.

More generally, we have the following characterization of generally convergent sequences.

Theorem 3.5. *The sequence $\{f_n\}$ of mappings from (Ω_1, m_1) into (Ω_2, m_2) converges N-generally in Ω_1 to the mapping f for some $N \in \mathbb{N} \cup \{0\}$ if and only if there exists a sequence $\{Z_n^\dagger\}$ of sets of N elements from Ω_1 such that*

$$\lim_{n\to\infty} \{\sup\{m_2(f_n(z), f(z)) : z \in C \setminus B_{m_1}(Z_n^\dagger, \delta)\}\} = 0 \tag{3.6}$$

for every compact set $C \subseteq \Omega_1$ and every $\delta > 0$.

Proof. The proof is similar to the proof of Theorem 3.1. First let $\{f_n\}$ converge N-generally to f. Then every subsequence of $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that

$$\lim_{k\to\infty} \{\sup\{m_2(f_{n_k}(z), f(z)) : z \in C \setminus B_{m_1}(Z^\dagger, \delta)\}\} = 0 \tag{3.7}$$

for every compact $C \subseteq \Omega_1$ and $\delta > 0$, where $Z^\dagger$ is some subset of Ω_1 with N elements. Assume there is no sequence $\{Z_n^\dagger\}$ such that (3.6) holds. Then there exist $\{n_k\}$, C and δ such that

$$\liminf\{\sup\{m_2(f_{n_k}(z), f(z)) : z \in C \setminus B_{m_1}(Z_n^\dagger, \delta)\}\} > 0 \tag{3.8}$$

for every sequence $\{Z_n^\dagger\}$ of sets with N elements. This sequence has no subsequence satisfying (3.7), a contradiction, and thus (3.6) must hold.

To prove the converse, let $\{Z_n^\dagger\}$ be such that (3.6) holds for all C and δ, where each $Z_n^\dagger$ has N elements. Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$. Without loss of generality

$Z^\dagger_{n_k} \to Z^\dagger$ in the same sense as in the proof of Theorem 3.1. Let C_1 be an arbitrarily chosen subset of $\Omega_1 \setminus Z^\dagger$. Then $f_{n_k}(z) \to f(z)$ uniformly in C_1, by the same argument as used in the proof of Theorem 3.1. This proves that $\{f_n\}$ converges N-generally to f. □

Remark 3.6. In Remark 3.4 we described how to approximate a constant limit function for a generally convergent sequence. It is of course harder to approximate the limit function if it is non-constant. Still, the knowledge that there is at most N irregular points for each n may be helpful, even if we know nothing about their location. For instance, let $z_1, \dots, z_{N+M} \in \Omega_1$ be distinct, chosen points, where $M \geqslant 2$, and compute the vectors $\boldsymbol{\varphi}_n = [f_n(z_1), \dots, f_n(z_{N+M})]$. By comparing $\boldsymbol{\varphi}_n$ with previous vectors $\boldsymbol{\varphi}_k$, one may be able to weed out N components of $\boldsymbol{\varphi}_n$ which are less likely to belong to the (at least) M corresponding convergent subsequences of $\{f_n(z_k)\}$, $k = 1, 2, \dots, N+M$. If we have some bounds for the error $m_2(f(z_k), f_n(z_k))$ for $m_1(z_k, Z^\dagger_k) \geqslant \delta > 0$, we may even be able to remove the uncertainty in this procedure.

4. Quasi-normality and valence

In this section we shall see some more examples of quasi-normal families and how they behave. We let (Ω_1, m_1) and (Ω_2, m_2) be subsets of $\hat{\mathbb{C}}$ with the chordal metric d. (For domains $\Omega \subseteq \hat{\mathbb{C}}$ with $\infty \notin \Omega$, the chordal metric is equivalent to the Euclidean metric.) We shall further let our functions belong to the following family.

Definition 4.1. We say that $f \in \mathcal{V}(\Omega, r_2, r_3, \delta)$, where $r_2, r_3 \in \mathbb{N}$, $\delta > 0$ and Ω is a domain in $\hat{\mathbb{C}}$, if f is a meromorphic function in Ω and there exist three values $w_j \in \hat{\mathbb{C}}$, $j = 1, 2, 3$, such that

(i) $f(z) = w_j$ for at most r_j values of $z \in \Omega$, where $r_1 \leqslant r_2 \leqslant r_3$, and

(ii) $d(w_i, w_j) \geqslant \delta$ for $i \neq j$.

According to Montel [**6**, p. 149], the family $\mathcal{F}$ of meromorphic functions on a domain $\Omega \subseteq \mathbb{C}$ is quasi-normal of order r_2 in Ω if there exist three distinct values $w_1, w_2, w_3 \in \hat{\mathbb{C}}$ such that

$$f(z) = w_j \text{ for at most } r_j \text{ values of } z \in \Omega \text{ for every } f \in \mathcal{F}, \text{ where } r_1 \leqslant r_2 \leqslant r_3.$$

This was extended by Chuang [**1**, Theorem 5.4, p. 147] to allow the points w_j to depend on the function $f \in \mathcal{F}$, as long as $d(w_i, w_j) \geqslant \delta$ for $i \neq j$ for some fixed constant $\delta > 0$. We then have the following theorem.

Theorem 4.2 (see Theorem 5.4 on p. 147 of [1]). *Let $\Omega \subseteq \mathbb{C}$ be a domain, and let $r_2, r_3 \in \mathbb{N}$ and $\delta > 0$. Then $\mathcal{V}(\Omega, r_2, r_3, \delta)$ is a quasi-normal family of order r_2.*

Of course, we may extend this to domains $\Omega \subseteq \hat{\mathbb{C}}$.

Corollary 4.3. *Let $\Omega \subseteq \hat{\mathbb{C}}$ be a domain, and let $r_2, r_3 \in \mathbb{N}$ and $\delta > 0$. Then $\mathcal{V}(\Omega, r_2, r_3, \delta)$ is a quasi-normal family of order r_2.*

Proof. If $\Omega \neq \hat{\mathbb{C}}$, we may without loss of generality (rotation of the Riemann sphere) assume that $\Omega \subseteq \mathbb{C}$, and the result follows from Theorem 4.2.

Let $\Omega = \hat{\mathbb{C}}$. Let $\{f_n\}$ be a sequence from $\mathcal{V}(\hat{\mathbb{C}}, r_2, r_3, \delta)$. Assume first that the restriction f_n^* of f_n to $\mathbb{C}$ has a subsequence belonging to $\mathcal{V}(\mathbb{C}, r_2, r_3, \delta_1)$ for some $0 < \delta_1 \leqslant \delta$. This is a quasi-normal family of order r_2. Let $\{f_{n_k}^*\}$ be a subsequence which converges r_2-quasi-normally in $\mathbb{C}$ with irregular points $Z^\dagger \subset \mathbb{C}$.

Case 1 ($Z^\dagger$ contains less than r_2 points). Then $\{f_{n_k}\}$ converges r_2-normally in $\hat{\mathbb{C}}$ with irregular points $Z^\dagger$ or $Z^\dagger \cup \{\infty\}$.

Case 2 ($Z^\dagger$ contains r_2 points). To see that $\{f_{n_k}\}$ has a subsequence which converges normally in a neighbourhood of ∞, let $z_0 \in \mathbb{C} \setminus Z^\dagger$, and let φ be a rotation of the Riemann sphere such that $\varphi(z_0) = \infty$. Then $\{f_{n_k} \circ \varphi^{-1}\}$ restricted to $\mathbb{C}$ belongs to $\mathcal{V}(\mathbb{C}, r_2, r_3, \delta_1)$ and thus has a subsequence $\{\tilde{f}_{n_k}\}$ converging r_2-quasi-normally in $\mathbb{C}$. $\varphi(Z^\dagger) \subseteq \mathbb{C}$ are r_2 irregular points for this subsequence. Since $\varphi(\infty) \notin \varphi(Z^\dagger)$, it follows that the subsequence converges normally in a neighbourhood of $\varphi(\infty)$, i.e. $\{f_{n_k}\}$ has a subsequence which converges r_2-quasi-normally in $\hat{\mathbb{C}}$ with irregular points $Z^\dagger$.

Assume next that f_n^* has no subsequence belonging to $\mathcal{V}(\mathbb{C}, r_2, r_3, \delta_1)$. (This can only happen if $f_n(\infty) = w_j$ for all but finitely many n, where $r_j = 1$.) Let $z_0 \in \mathbb{C}$ be chosen such that $f_n(z_0) \neq w_j$ for all n and $j = 1, 2, 3$. A rotation of $\hat{\mathbb{C}}$ with $\varphi(z_0) = \infty$ will then give a sequence $\{f_n\} \subseteq \mathcal{V}(\hat{\mathbb{C}}, r_2, r_3, \delta)$ with $\{f_n^*\} \subseteq \mathcal{V}(\mathbb{C}, r_2, r_3, \delta_1)$, and the result follows as above. □

For such functions we also have the following theorem.

Theorem 4.4. *Let $\Omega \subseteq \hat{\mathbb{C}}$ be a domain, let $r_2, r_3 \in \mathbb{N}$ and $\delta > 0$, and let $\{f_n\}$ be a sequence of functions from $\mathcal{V}(\Omega, r_2, r_3, \delta)$ converging uniformly with respect to the chordal metric on compact subsets of Ω. Then the limit function is either constant or a function from $\mathcal{V}(\Omega, r_2, r_3, \delta)$.*

Proof. First let $\Omega \neq \hat{\mathbb{C}}$. Then we may without loss of generality (rotation of the sphere $\hat{\mathbb{C}}$) assume that $\Omega \subseteq \mathbb{C}$. Assume that the limit function f is non-constant. For each n, let $w_{1,n}, w_{2,n}, w_{3,n} \in \hat{\mathbb{C}}$ satisfy (i) and (ii) for f_n in Definition 4.1. Then $\{f_n(z) - w_{j,n}\}$ is a sequence of functions which have at most r_j zeros for $j = 1, 2, 3$. By the Hurwitz theorem generalized to meromorphic functions, it thus follows that $f(z) - w_j$ has at most r_j distinct zeros for every limit point w_j of $\{w_{j,n}\}$. Hence, $f \in \mathcal{V}(\Omega, r_2, r_3, \delta)$.

If $\Omega = \hat{\mathbb{C}}$, the result follows for $\Omega_1 := \hat{\mathbb{C}} \setminus \{z_0\}$ for every $z_0 \in \hat{\mathbb{C}}$, and thus in all of $\hat{\mathbb{C}}$. □

Evidently, the family of N-valent meromorphic functions on Ω is a subclass of $\mathcal{V}(\Omega, N, N, \delta)$ for sufficiently small $\delta > 0$. Indeed, if we say that a function f is $\leqslant N$-*valent* in a set Ω when each value $w \in f(\Omega)$ is taken at most N times in Ω, then we have the following theorem.

Theorem 4.5. *Let $N \in \mathbb{N}$, and let $\{f_n\}$ be a sequence of $\leqslant N$-valent meromorphic functions on a domain $\Omega \subseteq \hat{\mathbb{C}}$. Then the following statements are equivalent.*

(A) *$\{f_n\}$ is N-restrained in Ω.*

(B) *There exist a compact subset C of Ω and $2N$ sequences $\{u_n^{(j)}\}_{n=1}^{\infty} \subseteq \Omega$, $j = 1, 2, \dots, 2N$, such that*

$$\left.\begin{aligned} &\liminf_{n\to\infty} d(u_n^{(j)}, u_n^{(k)}) > 0 \quad \text{for } j \neq k, \qquad \lim_{n\to\infty} d(u_n^{(j)}, C) = 0, \\ &\lim_{n\to\infty} d(f_n(u_n^{(j)}), f_n(u_n^{(k)})) = 0 \quad \text{for } j, k \in \{1, 2, \dots, 2N\}. \end{aligned}\right\} \tag{4.1}$$

(C) *There exists a sequence $\{Z_n^{\dagger}\}$ of sets of N elements from Ω such that $\lim d(f_n(u_n), f_n(v_n)) = 0$ whenever $\{u_n\}$ and $\{v_n\}$ are contained in a compact subset of Ω and $\liminf d(u_n, Z_n^{\dagger}) > 0$ and $\liminf d(v_n, Z_n^{\dagger}) > 0$.*

(D) *There exists a sequence $\{Z_n^{\dagger}\}$ of sets of N elements from Ω such that*

$$\lim_{n\to\infty} \{\sup\{d(f_n(u), f_n(v)) : u, v \in C \setminus B_d(Z_n^{\dagger}, \delta)\}\} = 0 \tag{4.2}$$

for every compact set $C \subseteq \Omega$ and every $\delta > 0$.

(E) *The sequence of spherical derivatives*

$$f_n^{\#}(z) := \lim_{\tilde{z}\to z} \frac{d(f_n(\tilde{z}), f_n(z))}{d(\tilde{z}, z)} \tag{4.3}$$

converges N-generally to 0 in Ω.

To prove this result, we shall use the following trivial lemma.

Lemma 4.6. *Let $\{f_n\}$ be a sequence of meromorphic functions converging uniformly with respect to the chordal metric in the closure of an (open) domain $D \subseteq \hat{\mathbb{C}}$ to a non-constant limit function f. Let $z_0 \in D$, and let $w_0 := f(z_0)$. Then there exist an $n_0 \in \mathbb{N}$ and an $r > 0$ such that $B_d(w_0, r) \subseteq f_n(D)$ for all $n \geqslant n_0$.*

Proof. First let $D \neq \hat{\mathbb{C}}$, and assume that no such n_0 and r exist. Then there exists a subsequence $\{n_k\} \subseteq \mathbb{N}$ such that $d(f_{n_k}(z_k), w_0) \to 0$ for some $z_k \in \partial D$. However,

$$d(f_{n_k}(z_k), w_0) \geqslant d(f(z_k), w_0) - d(f_{n_k}(z_k), f(z_k)),$$

where $d(f(z_k), w_0) \geqslant d(w_0, \partial f(D)) > 0$ (since f is an open mapping) and $d(f_{n_k}(z_k), f(z_k)) \to 0$. Hence the result holds for the case $D \neq \hat{\mathbb{C}}$. If $D = \hat{\mathbb{C}}$, then we can prove the result by reducing D. □

Proof of Theorem 4.5. The equivalence A $\Leftrightarrow$ D was proved in Theorem 3.1. Moreover, D $\Rightarrow$ C $\Rightarrow$ B follows trivially. It remains to prove that B $\Rightarrow$ A and A $\Leftrightarrow$ E.

B $\Rightarrow$ A. Let $\{u_n^{(j)}\}_{n=1}^{\infty} \subseteq \Omega$, $j = 1, \dots, 2N$, satisfy (4.1) for some compact subset $C \subseteq \Omega$. Let $\{f_{n_k}\}$ be a subsequence such that $u_{n_k}^{(j)} \to u^{(j)}$ for all j. By (4.1), all $u^{(j)}$ are distinct points in C. Since $\{f_n\}$ is a sequence from an N-quasi-normal family, it follows

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that $\{f_{n_k}\}$ has a subsequence $\{\tilde{f}_{k_m}\}$ which converges uniformly on compact subsets of $\Omega \setminus Z^\dagger$ for some $Z^\dagger \subseteq \Omega$ with less than or equal to N elements to some limit function f. At least N of the $2N$ points $u^{(j)}$ must be in $\Omega \setminus Z^\dagger$, say $u^{(j)} \in \Omega \setminus Z^\dagger$ for $j = 1, 2, \dots, N$. Since $\Omega \setminus Z^\dagger$ is an open set, there exists a $\delta > 0$ with $\delta < \frac{1}{2} d(u^{(j)}, u^{(k)})$ for all $j, k \in \{1, 2, \dots, 2N\}$, such that $D := \cup_{j=1}^N B_d(u^{(j)}, \delta) \subseteq \Omega \setminus Z^\dagger$.

Assume that f is non-constant in $\Omega \setminus Z^\dagger$. Then f is $\leqslant N$-valent (by an argument similar to the proof of Theorem 4.4), and $f(u^{(j)}) = c$ for $j = 1, 2, \dots, N$. Indeed, by Lemma 4.6 there exist an $m_0 \in \mathbb{N}$ and an $r > 0$ such that

$$B_d(c, r) \subseteq \tilde{f}_{k_m}(B_d(u^{(j)}, \delta)) \quad \text{for all } m \geqslant m_0 \text{ and } j = 1, 2, \dots, N.$$

On the other hand, $c_m^{(N+1)} := \tilde{f}_{k_m}(u_{n k_m}^{(N+1)}) \to c$ as $m \to \infty$, where $u_{n k_m}^{(N+1)} \notin D$ from some m onwards. That is, from some m onwards, $\tilde{f}_{k_m}$ takes the value $c_m^{(N+1)}$ at $N + 1$ distinct points. This contradicts that $\tilde{f}_{k_m}$ is $\leqslant N$-valent. Hence f is constant.

A $\Leftrightarrow$ E. First let $\{f_n^\#\}$ converge N-generally to 0 in Ω. Without loss of generality (taking subsequences) we may assume that $f_n^\#(z) \to 0$ normally in $\Omega \setminus Z_1^\dagger$ and $f_n(z) \to f$ normally in $\Omega \setminus Z_2^\dagger$, where $Z_1^\dagger$ and $Z_2^\dagger$ has at most N elements each. Then $f_n^\#(z) \to f^\#(z)$ in $\Omega \setminus (Z_1^\dagger \cup Z_2^\dagger)$ by Weierstrass's theorem. That is, $f^\#(z) \equiv 0$ in $\Omega \setminus (Z_1^\dagger \cup Z_2^\dagger)$, which rules out the possibility that f is $\leqslant N$-valent in this set. Hence f is constant.

Conversely, let $\{f_n\}$ be $\leqslant N$-restrained. Without loss of generality (taking subsequences) we assume that $\{f_n\}$ converges normally in $\Omega \setminus Z^\dagger$ to a constant function f. Then $f_n^\#(z) \to 0$ in $\Omega \setminus Z^\dagger$. Hence, every sequence $\{f_n^\#\}$ has a subsequence converging N-quasi-normally to 0. □

Since $\{f_n\}$ converges generally to a constant function only if $\{f_n\}$ is restrained, we also have the following corollary to Theorem 4.5.

Theorem 4.7. *Let $N \in \mathbb{N}$, and let $\{f_n\}$ be a sequence of $\leqslant N$-valent meromorphic functions on a domain $\Omega \subseteq \hat{\mathbb{C}}$. Then the following statements are equivalent.*

(A) *$\{f_n\}$ converges N-generally in Ω to a constant c.*

(B) *There exist a compact subset C of Ω and $2N$ sequences $\{u_n^{(j)}\}_{n=1}^\infty \subseteq \Omega$, $j = 1, 2, \dots, 2N$, such that*

$$\liminf_{n\to\infty} d(u_n^{(j)}, u_n^{(k)}) > 0, \quad j \neq k, \qquad \lim_{n\to\infty} d(u_n^{(j)}, C) = 0, \qquad \lim_{n\to\infty} f_n(u_n^{(j)}) = c \tag{4.4}$$

for $j, k \in \{1, 2, \dots, 2N\}$.

(C) *There exists a sequence $\{Z_n^\dagger\}$ of sets of N elements from Ω such that $f_n(u_n) \to c$ whenever $\liminf d(u_n, Z_n^\dagger) > 0$ and $\lim d(u_n, C) = 0$ for some compact subset $C \subseteq \Omega$.*

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(D) *There exists a sequence* $\{Z_n^\dagger\}$ *of sets of* N *elements from* Ω *such that*

$$\lim_{n\to\infty}\{\sup\{d(f_n(u),c) : u \in C \setminus B_d(Z_n^\dagger,\delta)\}\} = 0$$

for every compact set $C \subseteq \Omega$ *and every* $\delta > 0$.

(E) $\{f_n\}$ *is restrained in* Ω, *and there exist* $N+1$ *sequences* $\{u_n^{(j)}\}$ *such that*

$$\liminf_{n\to\infty} d(u_n^{(j)}, u_n^{(k)}) > 0 \quad \text{for } j \neq k,$$
$$\lim_{n\to\infty} f_n(u_n^{(j)}) = c \quad \text{for } j = 1,2,\dots,N+1.$$

Some families of functions have the following property.

Definition 4.8. We say that a family $\mathcal{F}$ of functions from (Ω_1, m_1) to (Ω_1, m_2) is of type C if every sequence from $\mathcal{F}$ which converges quasi-normally in Ω_1 to a non-constant function converges normally in Ω_1.

For such families, we have the following simplification of statement (B) in Theorems 4.5 and 4.7.

Theorem 4.9. *Let* $\mathcal{F}$ *be a family of type* C *of* $\leqslant N$*-valent meromorphic functions in a domain* $\Omega \subseteq \hat{\mathbb{C}}$, *and let* $\{f_n\}$ *be a sequence from* $\mathcal{F}$. *Then the following statements hold.*

(1) $\{f_n\}$ *is* N*-restrained in* Ω *if and only if there exist* $N+1$ *sequences* $\{u_n^{(j)}\} \subseteq \Omega$, $j = 1,2,\dots,N+1$, *such that (4.1) holds for* $j,k \in \{1,2,\dots,N+1\}$.

(2) $\{f_n\}$ *converges* N*-generally in* Ω *to a constant if and only if there exist* $N+1$ *sequences* $\{u_n^{(j)}\} \subseteq \Omega$, $j = 1,2,\dots,N+1$, *such that (4.4) holds for* $j,k \in \{1,2,\dots,N+1\}$.

Proof.

(1) The fact that N-restrained sequences $\{f_n\}$ satisfy (4.1), follows from Theorem 4.5. Let (4.1) hold for $j,k \in \{1,2,\dots,N+1\}$. Then $\{f_n\}$ has no subsequence converging normally in Ω_1 to a $\leqslant N$-valent function. Hence it is restrained.

(2) If $\{f_n\}$ converges N-generally to the constant c, then (4.4) follows from Theorem 4.7. Let (4.4) hold for $j,k \in \{1,2,\dots,N+1\}$. Then $\{f_n\}$ is restrained by part (1). The result follows therefore since $\lim_{n\to\infty} f_n(w_n) = c$ for

$$w_n := u_n^{(j)} \quad \text{where } d(u_n^{(j)}, Z_n^\dagger) \geqslant d(u_n^{(k)}, Z_n^\dagger) \text{ for all } k \neq j,$$

where $\{Z_n^\dagger\}$ is as in parts (C) and (D) of Theorem 4.7.

□

5. Univalent meromorphic mappings

We shall let (Ω_1, m_1) and (Ω_2, m_2) be subsets of $\hat{\mathbb{C}}$ with the chordal metric d in this section also. And again we shall let the functions $f \in \mathcal{F}$ be meromorphic in Ω_1, but now we shall restrict our attention to *univalent* mappings. According to Corollary 4.3, we then have that $\mathcal{F}$ is either normal or 1-quasi-normal, which makes Theorems 4.5 and 4.7 much simpler.

A prominent example of such functions is the family $\mathcal{M}$ of linear fractional transformations. It follows that since these functions are univalent, meromorphic mappings of $(\hat{\mathbb{C}}, d)$ onto $(\hat{\mathbb{C}}, d)$, $\mathcal{M}$ must be either normal or 1-quasi-normal in $\hat{\mathbb{C}}$. It is easy to see that $\mathcal{M}$ is not normal: if $\{T_n\}$ from $\mathcal{M}$ converges to a constant c at two distinct points z_1 and z_2, then $\{T_n\}$ does not converge to a $T \in \mathcal{M}$. However, $T_n(z_n) = b$ for $z_n := T_n^{-1}(b)$, so $\{T_n\}$ cannot converge uniformly in $(\hat{\mathbb{C}}, d)$ to c. Hence, $\mathcal{M}$ is quasi-normal of order 1 in $\hat{\mathbb{C}}$. Moreover, it is of type (C), since if $\{T_n\}$ converges to a non-constant function T, then $T \in \mathcal{M}$ and the convergence is uniform in $\hat{\mathbb{C}}$.

These functions have been extensively studied. Indeed, the concept of general convergence was introduced by the author in 1986 [**2**] for a subclass of $\mathcal{M}$ related to continued fractions. In this setting she pointed out that general convergence to a constant is really the correct type of convergence for continued fractions. This was extended to $\mathcal{M}$ in 1987 [**3**].

Another advantage in working with families $\mathcal{F}$ of univalent functions is that every $f \in \mathcal{F}$ has an inverse function f^{-1}. This was the background for the introduction of the concept of restrained sequences in $\mathcal{M}$, as introduced in [**3**]. If $\{T_n\}$ from $\mathcal{M}$ converges generally in $\hat{\mathbb{C}}$ to a constant c, then $Z_n^\dagger := \{T_n^{-1}(b)\}$ for all n must be a sequence of irregular point sets whenever $b \neq c$. Hence $\{T_n^{-1}\}$ is restrained. Indeed, it was proved in [**3**] that $\{T_n\}$ is restrained if and only if $\{T_n^{-1}\}$ is restrained. This is also true under more general conditions.

Theorem 5.1. *Let (Ω_1, m_1) and (Ω_2, m_2) be two metric spaces, and let $\Omega_{1,n} \subseteq \Omega_1$ for all n. Furthermore, let $\{f_n\}$ be a sequence of univalent mappings f_n from $(\Omega_{1,n}, m_1)$ to (Ω_2, m_2) with the following two properties.*

(i) *If C is a compact subset of Ω_1, then $\bigcup_{n=1}^{\infty} f_n(\Omega_{1,n} \cap C)$ is contained in a compact subset of Ω_2.*

(ii) *If C is a compact subset of Ω_2, then $\bigcup_{n=1}^{\infty} f_n^{-1}(f_n(\Omega_{1,n}) \cap C)$ is contained in a compact subset of Ω_1.*

Then $\{f_n\}$ is restrained in $\{\Omega_{1,n}\}$ if and only if $\{f_n^{-1}\}$ is restrained in $\{f_n(\Omega_{1,n})\}$.

Proof. Let $\{f_n\}$ be 1-restrained in $\{\Omega_{1,n}\}$. That is, there is a sequence $\{z_n^\dagger\}$ from Ω_1 such that

$$\lim_{n\to\infty} \{\sup\{m_2(f_n(u), f_n(v)) : u, v, \in C_n(C_1, \delta_1)\}\} = 0 \tag{5.1}$$

for every compact $C_1 \subseteq \Omega_1$ and $\delta_1 > 0$, where $C_n(C_1, \delta_1) := (C_1 \cap \Omega_{1,n}) \setminus B_{m_1}(z_n^\dagger, \delta_1)$. (See Definition 3.3.)

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Let $w_n^\dagger := f_n(u_n)$ for some $u_n \in C_n(C_1, \delta_1)$ for some suitable choice of C_1 and δ_1. Then (5.1) is equivalent to

$$\lim_{n\to\infty}\{\sup\{m_2(f_n(u), w_n^\dagger) : u \in C_n(C_1, \delta_1)\}\} = 0. \tag{5.1$'$}$$

Furthermore, let $C_2 \subseteq \Omega_2$ be compact and $\delta_2 > 0$. To see that $\{f_n^{-1}\}$ is 1-restrained in $\{f_n(\Omega_{1,n})\}$, it suffices to prove that

$$\lim_{n\to\infty}\{\sup\{m_1(f_n^{-1}(v), z_n^\dagger) : v \in \tilde{C}_n\}\} = 0, \tag{5.2}$$

where $\tilde{C}_n := (C_2 \cap f_n(\Omega_{1,n})) \setminus B_{m_2}(w_n^\dagger, \delta_2)$. Assume that this does not hold. Then there exist a subsequence $\{f_{n_k}\}$ and points $v_k \in \tilde{C}_{n_k}$ such that $\liminf_{k\to\infty} m_1(f_{n_k}^{-1}(v_k), z_{n_k}^\dagger) > 0$. That is, $f_{n_k}^{-1}(v_k) \in C_{n_k}(C_1, \delta_1)$ from some k onwards for a suitable choice of C_1 and δ_1. But then, by (**??**$'$) and property (ii), $m_1(f_{n_k}(f_{n_k}^{-1}(v_k)), w_{n_k}^\dagger) \to 0$, i.e. $m_1(v_k, w_{n_k}^\dagger) \to 0$, which contradicts the fact that $v_k \in \tilde{C}_{n_k}$. Hence (5.2) holds.

The converse follows similarly from property (i). □

The family $\mathcal{M}$ is a group with respect to compositions. A subgroup G of $\mathcal{M}$ is *discrete* if it contains no sequence $\{T_n\}$ of distinct elements converging to the identity function. Restrained sequences are connected to discrete subgroups of $\mathcal{M}$ in the following way.

Theorem 5.2. *A group $G \subseteq \mathcal{M}$ is discrete if and only if every sequence with distinct elements in G is restrained.*

Proof. Let G be discrete. Then no sequence $\{T_n\}$ with distinct elements from G converges to a Möbius map since $T_n \to T \in \mathcal{M}$ only if $t_{n+1} := T_n^{-1} \circ T_{n+1} \to T^{-1} \circ T = I$, where $t_{n+1} \neq I$ since $T_n \neq T_{n+1}$. Hence, every such sequence is restrained.

Conversely, let every sequence from G with distinct elements be restrained. Then no such sequence from G converges to a Möbius map. But then, since $I \in \mathcal{M}$, no such sequence converges to the identity function. □

This shows that for $n = 2$, every sequence from a quasi-conformal discrete convergence group is restrained.

6. Application to generalized iteration

Let $\mathcal{F}$ be a family of functions which map the metric space (Ω_1, m_1) into itself. To a given sequence $\{f_n\}$ from $\mathcal{F}$ we can then form the compositions

$$F_n := f_1 \circ f_2 \circ \cdots \circ f_n \quad \text{and} \quad G_n := f_n \circ f_{n-1} \circ \cdots \circ f_1 \tag{6.1}$$

for $n = 1, 2, 3, \ldots$, and we ask, for instance, whether $\{F_n\}$ converges generally or whether $\{G_n\}$ is restrained in Ω. This generalizes classical iteration, where all f_n are identical. It also generalizes continued fractions, where all f_n have the form $f_n(z) = s_n(z) = a_n/(b_n + z)$ in the expression for $F_n = S_n$ and the form $f_n = s_n^{-1}$ in the expression for $G_n = S_n^{-1}$. The following theorem is easily obtained.

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Theorem 6.1. *Let $\mathcal{F}$ be a family of univalent meromorphic mappings from a domain $\Omega \subseteq \hat{\mathbb{C}}$ into itself. Let $\{f_n\}$ be a sequence from $\mathcal{F}$. Then $\{F_n\}$ converges to an $F \in \mathcal{F}$ or $\{G_n\}$ converges to a $G \in \mathcal{F}$ only if $\{f_n\}$ converges to the identity function $I(z) \equiv z$ in Ω.*

Proof. Let $\{F_n\}$ converge to $F \in \mathcal{F}$. Then $f_n = F_{n-1}^{-1} \circ F_n$ converges to $F^{-1} \circ F(z) \equiv z$. Similarly, let $\{G_n\}$ converge to $G \in \mathcal{F}$. Then $f_n := G_n \circ G_{n-1}^{-1}$ converges to $G \circ G^{-1}(z) \equiv z$. □

In continued fraction theory and in dynamical systems it is important to establish sufficient conditions for F_n and G_n to be restrained or generally convergent. Evidently, we have the following corollary.

Corollary 6.2. *Let $\mathcal{F}$ be a family of univalent meromorphic mappings from a domain $\Omega \subseteq \hat{\mathbb{C}}$ into itself. Let $\{f_n\}$ be a sequence from $\mathcal{F}$ such that no limit function of $\{f_n\}$ is the identity function. Then $\{F_n\}$ and $\{G_n\}$ given by (5.1) are restrained.*

This generalizes a result from [**5**] saying that if V is an open set and $\{f_n\}$ is a sequence of functions from $\mathcal{M}_\varepsilon(V) := \{f \in \mathcal{M} : f(V) \subseteq V \setminus B(\alpha, \varepsilon)$ for some $\alpha \in \bar{V}$, α depending on $f\}$ for some $\varepsilon > 0$, then $\{F_n\}$ does not converge to a function from $\mathcal{M}$.

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