LINEARIZATION OF HOLOMORPHIC MAPPINGS ON FULLY NUCLEAR SPACES WITH A BASIS

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In [**13**] Mazet proved the following result.

If U is an open subset of a locally convex space E then there exists a complete locally convex space $\mathscr{G}(U)$ and a holomorphic mapping $\delta_U : U \to \mathscr{G}(U)$ such that for any complete locally convex space F and any $f \in \mathscr{H}(U; F)$, the space of holomorphic mappings from U to F, there exists a unique linear mapping $T_f : \mathscr{G}(U) \to F$ such that the following diagram commutes;

$$\begin{array}{ccc} U & \xrightarrow{f} & F \\ {\scriptstyle\delta_U}\downarrow & \nearrow_{T_f} & \\ \mathscr{G}(U) & & \end{array}$$

The space $\mathscr{G}(U)$ is unique up to a linear topological isomorphism. Previously, similar but less general constructions, have been considered by Ryan [**16**] and Schottenloher [**17**].

Recently Mujica and Nachbin [**15**] obtained a new proof of Mazet's result and gave several consequences. They also constructed a dense subspace $\mathscr{G}_0(U)$ of $\mathscr{G}(U)$ in the following manner; if M is a finite dimensional subspace of E and i_M is the canonical embedding of $U \cap M$ in U, then $\delta_U \circ i_M \in \mathscr{H}(U \cap M; \mathscr{G}(U))$ and hence there exists a unique linear mapping Π_M such that the following diagram commutes.

$$\begin{array}{ccc} U \cap M & \xrightarrow{\delta_U \circ i_M} & G(U) \\ {\scriptstyle\delta_{U\cap M}}\downarrow & \nearrow_{\pi_M} & \\ G(U \cap M) & & \end{array}$$

The same construction yields a canonical method of identifying $\mathscr{G}(U \cap M)$ with a subspace of $\mathscr{G}(U \cap N)$ for M and N finite dimensional subspaces of E with $M \subset N$ and hence $\bigcup_M \Pi_M(\mathscr{G}(U \cap M))$ is a subspace of $\mathscr{G}(U)$. Mujica and Nachbin called this subspace $\mathscr{G}_0(U)$ and obtained a number of results connecting linear properties of $\mathscr{G}_0(U)$ and holomorphic properties of U. In the process they posed two problems (see Example 10) which we show to have negative answers. Our method is to restrict ourselves to the study of holomorphic functions on fully nuclear spaces with a basis and to use the known structure of such spaces (see [**4, 5, 9, 11, 12**] and [**10**, Chapters 5 and 6]). In this way we obtain a positive result in the spirit of Mujica and Nachbin [**15**] but with a strictly weaker hypothesis on more specialized spaces and this immediately leads to the desired counterexamples.

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We refer to Dineen [**10**] for basic results in infinite dimensional holomorphy and to Horvath [**12**] for the theory of locally convex spaces. A recent comprehensive study of barrelled spaces can be found in [**6**] and Chapter 12 of this book discusses the various holomorphic properties we study in Section 2.

In Section 1 we give the required background information and prove some results about holomorphic functions that may be of independent interest. In Section 2 we prove our main results.

1. Let U denote an open subset of a locally convex space E over the complex numbers $\mathbf{C}$, and let $\mathcal{H}(U)$ denote the space of $\mathbf{C}$-valued holomorphic functions on U. A subset $\mathcal{F}$ of $\mathcal{H}(U)$ is said to be *locally bounded* if for each point x in U there exists a neighbourhood V_x of x, $V_x \subset U$, such that

$$\sup_{f\in\mathcal{F}} \|f\|_{V_x} < \infty \quad \Big(\|f\|_{V_x} = \sup_{y\in V_x} |f(y)|\Big).$$

If $\mathcal{V} = (V_n)_{n=1}^{\infty}$ is an increasing open cover of U we let

$$\mathcal{H}_{\mathcal{V}}(U) = \{f \in \mathcal{H}(U) : \|f\|_{V_n} < \infty \text{ for all } n\}$$

and we endow $\mathcal{H}_{\mathcal{V}}(U)$ with the topology generated by the semi-norms $\|\,.\,\|_{V_n}$. Each $\mathcal{H}_{\mathcal{V}}(U)$ is a Fréchet space and, since holomorphic functions are locally bounded, we have

$$\mathcal{H}(U) = \bigcup_{\mathcal{V}} \mathcal{H}_{\mathcal{V}}(U).$$

We denote by τ_δ the inductive limit topology on $\mathcal{H}(U)$ generated by $\mathcal{H}_{\mathcal{V}}(U)$ as $\mathcal{V}$ ranges over all possible increasing countable open covers of U. Hence

$$(\mathcal{H}(U), \tau_\delta) = \varinjlim_{\mathcal{V}} \mathcal{H}_{\mathcal{V}}(U).$$

Since each locally bounded subset of $\mathcal{H}(U)$ is contained and bounded in some $\mathcal{H}_{\mathcal{V}}(U)$ it follows that the locally bounded subsets of $\mathcal{H}(U)$ are τ_δ-bounded. The τ_δ-bounded subsets of $\mathcal{H}(U)$ are locally bounded if U is an open subset of a Fréchet space, a $\mathcal{DFM}$ space, a strict inductive limit of Fréchet–Montel spaces which admits a continuous norm or an open compact surjective limit of $\mathcal{DFM}$ spaces [**1, 5, 7, 8, 9, 10, 14**]. The converse is not true in general, ([**1, 5, 8, 14**]). The following proposition gives necessary and sufficient conditions for this to be the case.

Proposition 1. *The following are equivalent*:

(a) *the* τ_δ *bounded subsets of* $\mathcal{H}(U)$ *are locally bounded*,

(b) $\varinjlim_{\mathcal{V}} \mathcal{H}_{\mathcal{V}}(U)$ *is a regular inductive limit*,

(c) *there exists a Hausdorff locally convex topology* τ *on* $\mathcal{H}(U)$ *such that the* τ*-bounded subsets of* $\mathcal{H}(U)$ *are precisely the locally bounded sets.*

Proof. Clearly by the definitions of the τ_δ topology and regular inductive limit we have (a)$\Leftrightarrow$(b) and (a)$\Rightarrow$(c). Thus to complete the proof it suffices to show (c)$\Rightarrow$(a). Suppose (c) is satisfied. Let τ_b denote the bornological topology associated with τ. By our hypothesis the τ_b bounded subsets of $\mathcal{H}(U)$ are locally bounded and hence $\tau_b \geqslant \tau_\delta$. If $\mathcal{V}$ is an increasing countable open cover of U then the bounded subsets of $\mathcal{H}_{\mathcal{V}}(U)$ are locally bounded and, since $\mathcal{H}_V(U)$ is a Fréchet space, the natural inclusion $\mathcal{H}_V(U) \to (\mathcal{H}(U), \tau_b)$ is continuous. By the definition of the inductive limit topology it follows that the identity mapping $(\mathcal{H}(U), \tau_\delta) \to (\mathcal{H}(U), \tau_b)$ is continuous. Hence $\tau_\delta \geq \tau_b$ and thus $\tau_b = \tau_\delta$. Hence (a) is satisfied and this completes the proof.

DEFINITION 2 [**4**]. A locally convex space E is *fully nuclear* if E and its strong dual E'_β are both complete infrabarrelled nuclear spaces.

PROPOSITION 3 [**11**]. *If E is a fully nuclear space with basis then the monomials $\{z^m\}_{m \in \mathbf{N}^{(\mathbf{N})}}$ form an unconditional equicontinuous basis for $(\mathcal{H}(E), \tau_\delta)$.*

PROPOSITION 4. *If E is a fully nuclear space with basis and the τ_δ bounded subsets of $\mathcal{H}(E)$ are locally bounded, then $(\mathcal{H}(E), \tau_\delta)$ is a Montel space.*

Proof. As $(\mathcal{H}(E), \tau_\delta)$ is infrabarrelled it is necessary only to show that a locally bounded set of holomorphic functions on E is τ_δ relatively compact. Let $(f_\alpha)_{\alpha \in \Gamma}$ denote a locally bounded net of holomorphic functions on E. By Montel's theorem $(f_\alpha)_{\alpha \in \Gamma}$ contains a subnet, which we may suppose is the original set, and which converges uniformly on compact subsets of E to a Gâteaux holomorphic function f. By locally boundedness it follows that $f \in \mathcal{H}(E)$. By [**10**, Lemma 3.28] and the definition of the τ_ω topology it suffices to show that $f_\alpha \to f$ as $\alpha \to \infty$ uniformly on a neighbourhood of the origin.

Choose V an open polydisc neighbourhood of the origin such that $\sup_\alpha \|f_\alpha\|_V < \infty$. By [**10**, Lemma 5.18] there exists an open polydisc neighbourhood W of the origin and a sequence of positive real numbers $\delta = (\delta_n)_n$, $\delta_n > 1$, $\sum_1^\infty \frac{1}{\delta_n} < \infty$ such that $\delta W \subset V$. Let

$$f_\alpha(z) = \sum_{m \in \mathbf{N}^{(\mathbf{N})}} a_m^\alpha z^m \quad \text{and} \quad f(z) = \sum_{m \in \mathbf{N}^{(\mathbf{N})}} a_m z^m.$$

By the above $\|a_m^\alpha z^m\|_W \leq M/\delta^m$ for all α and all $m \in \mathbf{N}^{(\mathbf{N})}$. Since $f_\alpha \to f$ as $\alpha \to \infty$ it follows that $a_m^\alpha \to a_m$ as $\alpha \to \infty$ for each m in $\mathbf{N}^{(\mathbf{N})}$ and so $\|a_m z^m\|_W \leq M/\delta^m$ for all m in $\mathbf{N}^{(\mathbf{N})}$.

Let $\epsilon > 0$ be arbitrary. Choose J, a finite subset of $\mathbf{N}^{(\mathbf{N})}$, such that

$$2M \sum_{\mathbf{N}^{(\mathbf{N})} \setminus J} \frac{1}{\delta^m} < \epsilon$$

and then choose α_0 such that $\|a_m^\alpha z^m - a_m z^m\|_W \leq \epsilon/|J|$ for $m \in J$ and $\alpha \geq \alpha_0$. For $\alpha \geq \alpha_0$ we have

$$\|f_\alpha - f\|_W \leq \sum_{m \in J} \|a_m^\alpha z^m - a_m z^m\|_W + \sum_{m \in \mathbf{N}^{(\mathbf{N})} \setminus J} \|a_m^\alpha z^m\|_W + \sum_{m \in \mathbf{N}^{(\mathbf{N})} \setminus J} \|a_m z^m\|_W$$

$$\leq |J| \frac{\epsilon}{|J|} + 2 \sum_{m \in \mathbf{N}^{(\mathbf{N})} \setminus J} M/\delta^m \leq 2\epsilon.$$

This completes the proof.

LEMMA 5. *If E is a fully nuclear space with basis and $B = \left\{ \sum_{m \in \mathbf{N}^{(\mathbf{N})}} a_m^\alpha z^m \right\}_{\alpha \in \Gamma}$ is a locally bounded set of elements of $\mathcal{H}(E)$, then*

$$\tilde{B} := \left\{ \sum_{m \in \mathbf{N}^{(\mathbf{N})}} b_m z^m ; |b_m| \leq \sup_\alpha |a_m^\alpha| \right\}$$

is also locally bounded.

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Proof. Let K denote an arbitrary compact polydisc in E. Let W denote an open subset of E such that $K \subset W$ and $\sup_{f \in B} \|f\|_W = M(W) < \infty$. By [**10**, Lemma 5.18] there exists an open polydisc neighbourhood V of 0 in E and a sequence of positive real numbers $\delta = (\delta_n)_n$, $\delta_n > 1$ and $\sum_1^\infty \frac{1}{\delta_n} < \infty$, such that $\delta(K+V) \subset W$. Hence, for all $\alpha \in \Gamma$ and all $m \in \mathbf{N}^{(\mathbf{N})}$

$$\|a_m^\alpha z^m\|_{K+V} \leq \frac{M(W)}{\delta^m}.$$

Therefore, if $|b_m| \leq \sup_\alpha |a_m^\alpha|$, we have

$$\sum_{m \in \mathbf{N}^{(\mathbf{N})}} \|b_m z^m\|_{K+V} \leq M(W) \, . \sum_{m \in \mathbf{N}^{(\mathbf{N})}} \frac{1}{\delta^m} < \infty.$$

Since K was arbitrary this implies that the elements of $\tilde{B}$ are elements of $\mathscr{H}(E)$ and moreover are locally bounded. This completes the proof.

PROPOSITION 6. *Let E denote a fully nuclear space with basis and suppose the τ_δ bounded subsets of $\mathscr{H}(E)$ are locally bounded. Then $(\mathscr{H}(E), \tau_\delta)'_\beta$ has an absolute basis and is a bornological Montel space.*

Proof. By Proposition 4, $(\mathscr{H}(E), \tau_\delta)'_\beta$ is a Montel space. If $f \in \mathscr{H}(E)$, $f = \sum_{m \in \mathbf{N}^{(\mathbf{N})}} a_m z^m$, we let $w^m(f) = a_m$ for all $m \in \mathbf{N}^{(\mathbf{N})}$. If $T \in (\mathscr{H}(E), \tau_\delta)'$ then, by Proposition 3, we have

$$T(f) = \sum_{m \in \mathbf{N}^{(\mathbf{N})}} w^m(f) T(z^m) = \sum_{m \in \mathbf{N}^{(\mathbf{N})}} T(z^m) w^m(f)$$

and $\{w^m\}_{m \in \mathbf{N}^{(\mathbf{N})}}$ is a weak basis for $(\mathscr{H}(E), \tau_\delta)'$. If $B = \left\{ \sum_{m \in \mathbf{N}^{(\mathbf{N})}} a_m^\alpha z^m \right\}_{\alpha \in \Gamma}$ is a τ_δ bounded subset of $\mathscr{H}(E)$ and

$$\tilde{B} = \left\{ \sum_{m \in \mathbf{N}^{(\mathbf{N})}} b_m z^m ; |b_m| \leq \sup_\alpha |a_m^\alpha| \text{ for } m \in \mathbf{N}^{(\mathbf{N})} \right\}$$

then, by Lemma 5, $\tilde{B}$ is also τ_δ bounded and $B \subset \tilde{B}$. Hence, if $T \in (\mathscr{H}(E), \tau_\delta)'$, then

$$\begin{aligned} \|T\|_B \leq \|T\|_{\tilde{B}} &= \sup_{f \in \tilde{B}} \left| \sum_{m \in \mathbf{N}^{(\mathbf{N})}} T(z^m) w^m(f) \right| \\ &= \sum_{m \in \mathbf{N}^{(\mathbf{N})}} |T(z^m)| \, \|w^m\|_{\tilde{B}}. \end{aligned}$$

This implies that

$$T = \sum_{m \in \mathbf{N}^{(\mathbf{N})}} T(z^m) w^m$$

is in $(H(E), \tau_\delta)'_\beta$ and that, moreover, semi-norms of the form

$$\sum_{m \in \mathbf{N}^{(\mathbf{N})}} |T(z^m)| \, \|w^m\|_B$$

generate the topology of $(H(E), \tau_\delta)'_\beta$. Hence $\{w^m\}_{m\in\mathbf{N}^{(\mathbf{N})}}$ is an absolute basis for $(\mathcal{H}(E), \tau_\delta)'_\beta$. To complete the proof it suffices, since $(\mathcal{H}(E), \tau_\delta)'_\beta$ is infrabarrelled, to show that any mapping $T:(\mathcal{H}(E), \tau_\delta)'_\beta \to \mathbf{C}$, which is bounded on bounded sets, is continuous. Since $(\mathcal{H}(E), \tau_\delta)'_\beta$ is quasicomplete and has an absolute basis it follow that if

$$\sum_{m\in\mathbf{N}^{(\mathbf{N})}} b_m w^m \in (\mathcal{H}(E), \tau_\delta)'_\beta,$$

then $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} e^{i\theta_m} b_m w^m$ also belongs to $(\mathcal{H}(E), \tau_\delta)'_\beta$ for any choice of real scalars $\{\theta_m\}_{m\in\mathbf{N}^{(\mathbf{N})}}$. Hence $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} |b_m T(w^m)| < \infty$ for any $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} b_m w^m \in (\mathcal{H}(E), \tau_\delta)'_\beta$. Let

$$V = \left\{ \sum_{m\in\mathbf{N}^{(\mathbf{N})}} b_m w^m \in (\mathcal{H}(E), \tau_\delta)'_\beta;\ \sum_{m\in\mathbf{N}^{(\mathbf{N})}} |b_m T(w^m)| \le 1 \right\}.$$

By the above V is absorbing and it is clearly convex and balanced. Since the mappings

$$\sum_{m\in\mathbf{N}^{(\mathbf{N})}} b_m w^m \to b_m$$

are continuous it follows that V is also closed. Since a Montel space is barrelled it follows that V is a neighbourhood of zero in $(\mathcal{H}(E), \tau_\delta)'_\beta$ and so the semi-norm

$$p\left(\sum_{m\in\mathbf{N}^{(\mathbf{N})}} b_m w^m\right) := \sum_{m\in\mathbf{N}^{(\mathbf{N})}} |b_m T(w^m)|$$

is continuous on $(\mathcal{H}(E), \tau_\delta)'_\beta$. Since

$$\left|T\left(\sum_{m\in\mathbf{N}^{(\mathbf{N})}} b_m w^m\right)\right| \le p\left(\sum_{m\in\mathbf{N}^{(\mathbf{N})}} b_m w^m\right)$$

it follows that T is continuous and this completes the proof.

We now recall some properties of $\mathcal{G}(U)$ from Mujica and Nachbin [**15**]. By definition

$$\mathcal{G}(U) = \{\phi \in \mathcal{H}(U)' : \phi \text{ is } \tau_0 \text{ continuous on the locally bounded subsets of } \mathcal{H}(U)\}.$$

Hence $(\mathcal{H}(U), \tau_0)' \subset \mathcal{G}(U)$. If $\mathcal{V}$ is an increasing countable open cover of U then the bounded subsets of $\mathcal{H}_\mathcal{V}(U)$ are locally bounded and, moreover, by Montel's theorem are τ_0-relatively compact. Hence, if $\phi \in \mathcal{G}(U)$, then ϕ is bounded on the bounded subsets of $\mathcal{H}_\mathcal{V}(U)$ and hence is continuous when restricted to $\mathcal{H}_\mathcal{V}(U)$. By the definition of the inductive limit topology it follows that ϕ is τ_δ continuous and so

$$(\mathcal{H}(U), \tau_0)' \subset \mathcal{G}(U) \subset (\mathcal{H}(U), \tau_\delta)'.$$

$\mathcal{G}(U)$ is endowed with the topology τ_l of uniform convergence on the locally bounded subsets of $\mathcal{H}(U)$. If we let i denote the inductive dual topology (see [**3**]), then the mapping

$$f \in \mathcal{H}(U) \to T_f \in \mathcal{G}(U)'$$

in Mazet's theorem gives the following topological isomorphism.

Proposition 7 [**15**]. *$(\mathcal{H}(U), \tau_\delta) \cong \mathcal{G}(U)'_i$ and locally bounded subsets of $\mathcal{H}(U)$ correspond to the equicontinuous subsets of both $\mathcal{G}(U)'$ and $\mathcal{G}_0(U)'$.*

2. Main results.

THEOREM 8. *Let E be a fully nuclear space with basis. The following are equivalent.*
(1) *The τ_δ bounded subsets of $\mathcal{H}(E)$ are locally bounded.*
(2) *$\mathcal{G}(E)$ is bornological.*
(3) *$\mathcal{G}(E)$ is barrelled.*
(4) *$\mathcal{G}_0(E)$ is bornological.*
(5) *$\mathcal{G}_0(E)$ is infrabarrelled.*

Proof. If any of the conditions (2), (3), (4) or (5) are satisfied for E an arbitrary locally convex space then, by Proposition 7 and the definition of infrabarrelledness, the locally bounded subsets of $\mathcal{H}(E)$ are the bounded subsets of a locally convex topology on $\mathcal{H}(E)$. By Proposition 1 this implies that (1) is satisfied.

Since $\mathcal{G}(E)$ is complete we always have (2)$\Rightarrow$(3) and (4)$\Rightarrow$(5)$\Rightarrow$(3). To complete the proof we show (1)$\Rightarrow$(2) and (1)$\Rightarrow$(4).

Now suppose (1) is satisfied. By Proposition 4, $(\mathcal{H}(E), \tau_\delta)$ is a Montel space and hence the closed locally bounded sets are τ_δ-compact. Since $\tau_\delta \gtrsim \tau_0$ it follows that the τ_δ and τ_0 topologies agree on the locally bounded subsets of $\mathcal{H}(E)$. Hence $\mathcal{G}(E) = (\mathcal{H}(E), \tau_\delta)'$ and τ_l is the topology of uniform convergence on the τ_δ bounded subsets of $\mathcal{H}(E)$. Hence $(\mathcal{G}(E), \tau_l) \cong (\mathcal{H}(E), \tau_\delta)'_\beta$. By Proposition 6, $(\mathcal{H}(E), \tau_\delta)'_\beta$ is bornological and so (1)$\Rightarrow$(2).

If $m \in \mathbf{N}^{(\mathbf{N})}$, then the linear functional w^m on $\mathcal{H}(E)$ is supported by a finite dimensional compact subset of E and hence belongs to $\mathcal{G}_0(E)$. Hence $\{w^m\}_{m\in\mathbf{N}^{(\mathbf{N})}}$ is an absolute basis for $(\mathcal{G}_0(E), \tau_l)$. If $B = \left\{\sum_{m\in\mathbf{N}^{(\mathbf{N})}} b_m^\lambda w^m\right\}_{\lambda\in\Gamma}$ is a bounded subset of $\mathcal{G}(E)$, let

$$\tilde{B} = \left\{\sum_{m\in J} e^{i\theta_m} b_m^\lambda w^m;\ |J| < \infty, J \subset \mathbf{N}^{(\mathbf{N})}, \lambda \in \Gamma, \theta_m \in \mathbf{R}\right\}.$$

Then $\tilde{B}$ is a bounded subset of $\mathcal{G}_0(E)$ and $B \subset \bar{\tilde{B}}^{\mathcal{G}(E)}$. Moreover, if $f = \sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m \in \mathcal{H}(E) = (\mathcal{G}(E), \tau_l)'$, then

$$\begin{aligned}\|f\|_B = \sup_{\lambda\in\Gamma}\left|\sum_{m\in\mathbf{N}^{(\mathbf{N})}} b_m^\lambda a_m\right| &\le \sup_{\lambda\in\Gamma}\sum_{m\in\mathbf{N}^{(\mathbf{N})}} |b_m^\lambda| \cdot |a_m| \\ &= \|f\|_{\tilde{B}} \qquad (*)\end{aligned}$$

and so $(\mathcal{G}(E), \tau_l)'_\beta \cong (\mathcal{G}_0(E), \tau_l)'_\beta$. Since $\mathcal{G}(E)$ is barrelled, Proposition 7 now implies that $\mathcal{G}_0(E)$ is infrabarrelled. Let $T: \mathcal{G}_0(E) \to \mathbf{C}$ denote a linear functional which is bounded on bounded sets. If $B = \left\{\sum_{m\in\mathbf{N}^{(\mathbf{N})}} b_m^\lambda w^m\right\}_{\lambda\in\Gamma}$ is a bounded subset of $\mathcal{G}(E)$, then

$$\|T\|_{\tilde{B}} = \sup_{\lambda\in\Gamma}\sum_{m\in\mathbf{N}^{(\mathbf{N})}} |b_m^\lambda|\,|T(w^m)| < \infty.$$

Hence $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} |b_m|\,|T(w^m)| < \infty$ for all $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} b_m w^m \in \mathcal{G}(E)$ and so T extends to a linear functional on $\mathcal{G}(E)$. Since $\|T\|_B \le \|T\|_{\tilde{B}}$ the extended mapping T is bounded on bounded subsets of $\mathcal{G}(E)$. Since $\mathcal{G}(E)$ is bornological it follows that T is continuous. Hence $\mathcal{G}_0(E)$ is bornological and (1)$\Rightarrow$(4). This completes the proof.

EXAMPLE 9. Let $E = \prod_{n=1}^{\infty} E_n$, where each E_n is a $\mathcal{DFN}$ space with basis. The space E is a fully nuclear space with basis and by [**10**, Proposition 6.27]) the τ_ω, and hence the τ_δ, bounded subsets of $\mathcal{H}(E)$ are locally bounded. Hence the conditions in Theorem 8 are all satisfied. Two particular cases are of interest.

(a) $E_1 = \mathbf{C}^{(\mathbf{N})}$, $E_i = \mathbf{C}$ for $i \geq 2$; then $E = \mathbf{C}^{(\mathbf{N})} \times \mathbf{C}^{\mathbf{N}}$.

(b) $E_i = s'$, where s is the space of rapidly decreasing sequences; then $E \cong \mathcal{D}'$ (the space of distributions on $\mathbf{R}^n$).

EXAMPLE 10. An open subset U of a locally convex space E is said to be *holomorphically infrabarrelled* if the τ_0 bounded subsets of $\mathcal{H}(U)$ are locally bounded. The set U is said to be *holomorphically bornological* if for any locally convex space F and $f: U \to F$, Gâteaux holomorphic and bounded on compact sets the function f is holomorphic ([**1**]). If U is holomorphically bornological, then U is holomorphically infrabarrelled and Mujica and Nachbin [**15**] show that

$$U \text{ holomorphically infrabarrelled} \Rightarrow \mathcal{G}_0(U) \text{ is infrabarrelled}$$

and

$$U \text{ holomorphically bornological} \Rightarrow \mathcal{G}_0(U) \text{ is bornological.}$$

They ask if the converses are true. If $E = \mathbf{C}^{(\mathbf{N})} \times \mathbf{C}^{\mathbf{N}}$ then Theorem 8 implies that $\mathcal{G}_0(E)$ is bornological. By [**8**, Proposition 3.2] the τ_0 and the τ_ω bounded subsets of $\mathcal{H}(E)$ do not coincide. Hence E is not holomorphically infrabarrelled and a fortiori not holomorphically bornological. Thus the space $\mathbf{C}^{(\mathbf{N})} \times \mathbf{C}^{\mathbf{N}}$ provides negative answers to both questions mentioned above. Note that the space $\mathcal{D}'$ is holomorphically infrabarrelled by [**5**, Corollary 8].

REMARK 11. The space $\mathcal{G}_0(E)$, with E a fully nuclear space with a basis, will not in general be barrelled even if $\mathcal{G}_0(E)$ is bornological. If this were the case then Theorem 4.4 of [**15**] would imply that E is holomorphically barrelled and, in particular, the τ_0 bounded subsets of $\mathcal{H}(E)$ would be locally bounded. By the previous example we see that $\mathcal{G}_0(\mathbf{C}^{(\mathbf{N})} \times \mathbf{C}^{\mathbf{N}})$ is bornological but not barrelled.

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