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A NOTE ON UNDIRECTED GRAPHS REALIZABLE AS P.O. SETS

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1. **Introduction.** Let $(P, \geq)$ be a p.o. set. The *basis graph* of $(P, \geq)$ is defined to be the directed graph whose vertex set is P and in which the ordered pair $\langle a, b\rangle$ is an edge if and only if b covers a in $(P, \geq)$.

Let D be a directed graph. All graphs considered in this note are finite and are free of loops and multiple edges. Let C be a circuit in D and let C^+ denote the set of edges of C oriented in a given sense and C^- denote the set of edges oriented in the opposite sense. Then the circuit C is called an *admissible* circuit of D if

$$(1) \qquad |C^+| \geq 2 \quad \text{and} \quad |C^-| \geq 2$$

A directed graph D is a basis graph (by which we mean that it is the basis graph of some p.o. set) if and only if every one of its circuits is admissible.

We shall say that an (undirected) graph is *realizable* as a p.o. set if it can be oriented so that the resulting directed graph is a basis graph. An orientation of a graph G that 'turns' it into a basis graph is called an *admissible orientation* of G. Ore [2, p. 155] posed the problem of characterizing graphs which are realizable as p.o. sets; or, in other words, characterizing those which have an admissible orientation. Let $\gamma(G)$ denote the *girth* of G, the length of the smallest circuit in G. If G has no circuits we write $\gamma(G)=\infty$. Then an obviously necessary condition for a graph G to have an admissible orientation can be inferred from (1) as

$$(2) \qquad \gamma(G) \geq 4.$$

There does not seem to exist in the literature an example of a graph satisfying (2) which does not have an admissible orientation. In §2 we present such an example. In §3 we give a simple sufficient condition for a graph G to possess an admissible orientation. We show by means of an example that this sufficient condition is not a necessary condition. We also prove that in case of planar graphs (2) is a sufficient condition.

Alvarez [1] considered the analogous problem for modular and distributive lattices. We know of no characterization of graphs realizable as lattices. There is a multitude of examples of graphs that are not realizable as lattices.

2. **A counter example.** The graph G of Figure 1 is constructed as follows: We take a K_5 (complete 5-graph) on $a_1, a_2, \ldots, a_5$. We then subdivide each of the edges of the K_5 by inserting a new vertex $(b_1, b_2, \ldots, b_5; c_1, c_2, \ldots, c_5)$. We join

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two of the new vertices if and only if they are subdividers of two edges of the K_5 that do not have a common end vertex in the K_5. The graph H of Figure 2 is obtained by deleting the five 'spoke' edges of G.

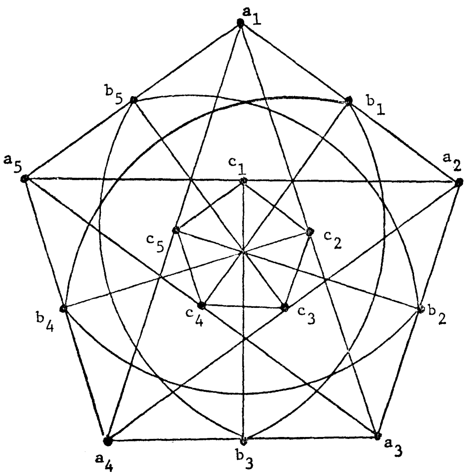

Figure 1 (G).

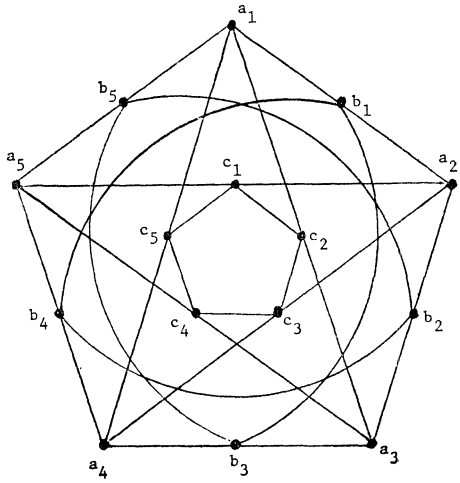

Figure 2 (H).

We observe that $\gamma(G)=\gamma(H)=4$.

PROPOSITION 1. *The graph G has no admissible orientation.*

Proof. We shall only sketch the proof. If possible let there exist an admissible orientation of G and let D be a basis graph obtained by orienting G. We can choose D so that three of the edges of the interior pentagon $(c_1, c_2, \ldots, c_5, c_1)$ of D are oriented in the clockwise direction. For if D does not have this property, D^*, the directed graph obtained by reversing orientations on all the edges of D will serve the purpose. Further, it is possible to choose D in such a way that either

(I) $$\{\langle c_1, c_2\rangle, \langle c_2, c_3\rangle, \langle c_3, c_4\rangle\} \subset E(D)$$

or

(II) $$\{\langle c_1, c_2\rangle, \langle c_2, c_3\rangle, \langle c_4, c_5\rangle\} \subset E(D)$$

For, otherwise, we can suitably rename the vertices of D to get a basis graph D' in which either (I) or (II) holds. (This renaming would correspond to an automorphism of G)

The rest of the proof hinges on the following observation: the choice of orientations on a certain subset of $E(G)$ determines the orientations on a certain other (possibly null) subset of $E(G)$ if we insist that the orientation be an admissible orientation. In other words, if we assume the existence of certain ordered pairs in $E(D)$, then (1) implies the existence of certain other ordered pairs in $E(D)$. For example,

$$\{\langle c_1, c_2\rangle, \langle c_2, c_3\rangle\} \subseteq E(D) \Rightarrow \{\langle c_1, a_2\rangle, \langle a_2, c_3\rangle\} \subseteq E(D)$$

by (1) as applied to the quadrilateral $(c_1, c_2, c_3, a_2, c_1)$.

We consider the following six cases

(i) $\{\langle c_1, c_2\rangle, \langle c_2, c_3\rangle, \langle c_3, c_4\rangle, \langle b_2, a_2\rangle\} \subset E(D)$

(ii) $\{\langle c_1, c_2\rangle, \langle c_2, c_3\rangle, \langle c_3, c_4\rangle, \langle a_2, b_2\rangle\} \subset E(D)$

(iii) $\{\langle c_1, c_2\rangle, \langle c_2, c_3\rangle, \langle c_4, c_5\rangle, \langle a_2, b_2\rangle\} \subset E(D)$

(iv) $\{\langle c_1, c_2\rangle, \langle c_2, c_3\rangle, \langle c_4, c_5\rangle, \langle b_2, a_2\rangle, \langle b_1, a_2\rangle\} \subset E(D)$

(v) $\{\langle c_1, c_2\rangle, \langle c_2, c_3\rangle, \langle c_4, c_5\rangle, \langle b_2, a_2\rangle, \langle a_2, b_1\rangle, \langle a_3, c_2\rangle\} \subset E(D)$

(vi) $\{\langle c_1, c_2\rangle, \langle c_2, c_3\rangle, \langle c_4, c_5\rangle, \langle b_2, a_2\rangle, \langle a_2, b_1\rangle, \langle c_2, a_3\rangle\} \subset E(D)$

In these cases we arrive at the following implications through a sequence of intermediate implications which we omit.

(i) $\Rightarrow \{\langle b_2, b_5\rangle, \langle b_5, a_5\rangle, \langle a_5, b_4\rangle, \langle b_2, b_4\rangle\} \subset E(D)$
(ii) $\Rightarrow \{\langle b_3, b_1\rangle, \langle b_1, a_1\rangle, \langle a_1, b_5\rangle, \langle b_3, b_5\rangle\} \subset E(D)$
(iii) $\Rightarrow \{\langle a_2, b_1\rangle, \langle b_1, b_4\rangle, \langle b_4, b_2\rangle, \langle a_2, b_2\rangle\} \subset E(D)$
(iv) $\Rightarrow \{\langle c_2, a_1\rangle, \langle a_1, b_5\rangle, \langle b_5, c_3\rangle, \langle c_2, c_3\rangle\} \subset E(D)$
(v) $\Rightarrow \{\langle c_4, a_5\rangle, \langle a_5, b_5\rangle, \langle b_5, c_3\rangle, \langle c_4, c_3\rangle\} \subset E(D)$
(vi) $\Rightarrow \{\langle b_2, b_5\rangle, \langle b_5, a_5\rangle, \langle a_5, b_4\rangle, \langle b_2, b_4\rangle\} \subset E(D)$

These implications show that each of the six cases is incompatible with (1). Also, the cases (i) and (ii) show that (I) is, and the cases (iii)–(vi) show that (II) is incompatible with (1). This completes the proof of the proposition.

3. **A sufficient condition.** A k-coloration of a graph G is a function f with domain $V(G)$ and range $\{1, 2, \ldots, k\}$ which satisfies the condition that $f(a) \neq f(b)$ if a and b are joined in G. We write $I_j = \{v \mid v \in V(G), f(v) = j\}$. Then I_j is an independent set of G for all j, $1 \leq j \leq k$. The least value of k for which there is a k-coloration of G is called the chromatic number of G and is denoted by $\kappa(G)$. Given a k-coloration of G, there is an orientation of G, called the orientation *associated* with the coloration, in which an edge with its end vertices in I_i and I_j, $i \neq j$, $i < j$, is oriented away from I_i.

PROPOSITION 2. *A sufficient condition for G to have an admissible orientation is*

$$\gamma(G) > \kappa(G). \tag{3}$$

Proof. If possible let G satisfy (3) and not have an admissible orientation. If $\gamma(G) = 3$ then (3) cannot hold. Therefore $\gamma(G) \geq 4$. Consider the orientation of G associated with a $\kappa(G)$-coloration of G. Let D denote the resulting directed graph and let C be any circuit of D. If possible let $|C^-| \leq 1$. Then $|C^+| \geq \kappa(G)$ implying the existence of a directed path of length $\geq \kappa(G)$ in D, which is impossible. Hence $|C^-| \geq 2$. Similarly $|C^+| \geq 2$. Therefore D is a basis graph. This completes the proof.

For the graph G considered in §2, $\gamma(G) = \kappa(G) = 4$.

That (3) is not a necessary condition for G to possess an admissible orientation can be shown by considering H (§2), for which $\gamma(H)=\kappa(H)=4$. Consider the 5-coloration of H which partitions $V(H)$ into the following five independent sets:

$$I_1 = \{b_2, b_3\}, \qquad I_2 = \{a_2, a_3, b_4, b_5\},$$
$$I_3 = \{a_4, a_5, b_1, b_2\}, \qquad I_4 = \{a_1, c_1, c_4\}$$

and

$$I_5 = \{c_3, c_5\}.$$

The orientation of H associated with this 5-coloration can be easily checked to be an admissible orientation.

PROPOSITION 3. *If G is a planar graph, then* (2) *is necessary and sufficient for G to have an admissible orientation.*

Proof. By Grötzsch's theorem (see [3, p. 229]) it follows that G has a 3-coloration. Hence $\gamma(G) > \kappa(G)$. The proof now follows from Proposition 2.

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