2

The effect of Lorentz and discrete transformations on helicity states, fields and wave functions

In discussing experiments it is often necessary to refer a given physical situation to different reference frames, e.g. to the laboratory or centre-of-mass system. Thus we need to understand how helicity states are affected by Lorentz transformations. The approach is quite similar to the discussion of rotations in Section 1.1 and we seek the analogue of eqn (1.1.17). However, because sequences of Lorentz transformations are more complicated than sequences of rotations the result will look a little less simple. We shall compare and contrast this situation with the transformation properties of fields and wave functions.

2.1 Particles with non-zero mass

Let us suppose that in a given reference system S an observer O sees a particle A in motion with momentum $\mathbf{p}$ and helicity λ, i.e. the observer reports a state of motion specified by $|\mathbf{p};\lambda\rangle$.

Let S^l be a reference frame obtained by carrying out a physical Lorentz transformation l on S. We wish to know how observer O^l describes the motion of A.

By analogy with the rotational case (see eqn (1.1.21)) O^l will describe the state as

$$|\mathbf{p};\lambda\rangle_{S^l} = U(l^{-1})|\mathbf{p};\lambda\rangle \tag{2.1.1}$$

when $U(l)$ is the operator effecting a Lorentz transformation l.

Let us denote by $\mathbf{p}'$ the momentum vector that O^l attributes to A, i.e. $\mathbf{p}' = l^{-1}\mathbf{p}$. Its components p'^{μ} are clearly the components of $\mathbf{p}$ as seen by O^l, i.e. (see eqns (1.2.14), (1.2.15))

$$p'^{\mu} \equiv (p^{\mu})_{S^l} = \Lambda^{\mu}{}_{\nu}(l^{-1})p^{\nu}. \tag{2.1.2}$$

It is obvious that we must expect to find that $|\mathbf{p};\lambda\rangle_{S^l} = |\mathbf{p}';\lambda'\rangle$ with $\mathbf{p}'$ given by (2.1.2). The only question is what values of λ' should appear. To answer this one writes

$$U(l^{-1})|\mathbf{p};\lambda\rangle = U(l^{-1})U[h(\mathbf{p})]|\mathring{p};\lambda\rangle \tag{2.1.3}$$

using the definition of helicity states (1.2.25). One then invokes the brilliant stratagem of multiplying eqn (2.1.3) by unity in the form

$$U[h(\mathbf{p}')]U^{-1}[h(\mathbf{p}')]$$

where $h(\mathbf{p}')$ is the helicity transformation that would be used to define a state $|\mathbf{p}';\lambda\rangle$, i.e. $h(\mathbf{p}')$ is such that

$$|\mathbf{p}';\lambda\rangle = U[h(\mathbf{p}')]|\mathring{p};\lambda\rangle. \tag{2.1.4}$$

One can now write (2.1.3) in the form

$$U(l^{-1})|\mathbf{p};\lambda\rangle = U[h(\mathbf{p}')]\mathscr{R}|\mathring{p};\lambda\rangle \tag{2.1.5}$$

where $\mathscr{R}$ is short for the product $U^{-1}[h(\mathbf{p}')]U(l^{-1})U[h(\mathbf{p})]$. Since the operators U *represent* the various physical operations we can simplify and write

$$\mathscr{R} = U[h^{-1}(\mathbf{p}')l^{-1}h(\mathbf{p})] \equiv U[h^{-1}(l^{-1}\mathbf{p})l^{-1}h(\mathbf{p})]. \tag{2.1.6}$$

The crucial observation is that the sequence of physical operations in U is just a *rotation* no matter what l is. The simplest way to see this is to study the effect of the sequence of operations $h^{-1}(\mathbf{p}')l^{-1}h(\mathbf{p})$ on the 4-vector $\mathring{p} = (m, 0, 0, 0)$. We have

(1) $h(\mathbf{p}) : \mathring{p} \to p$
(2) $l^{-1} : p \to p'$
(3) $h(\mathbf{p}')$ is such that it takes $\mathring{p} \to p'$, thus $h^{-1}(\mathbf{p}') : p' \to \mathring{p}$.

Hence the sequence (1), (2), (3) takes $\mathring{p} \to \mathring{p}$. From the form of $\mathring{p}$ it is clear that only a rotation could have this property. Hence $\mathscr{R}$ represents a rotation no matter what l is. Let us label this physical rotation as $r(l, \mathbf{p})$, i.e.

$$r(l,\mathbf{p}) \equiv h^{-1}(l^{-1}\mathbf{p})l^{-1}h(\mathbf{p}). \tag{2.1.7}$$

We shall refer to this as the *Wick helicity rotation* for the transformation l of *axes* that takes $\mathbf{p}$ to $\mathbf{p}' = l^{-1}\mathbf{p}$. (It is not the same as the Wigner rotation, as will be explained later.)

Once it is recognized that $\mathscr{R}$ corresponds to a rotation the completion of the evaluation of $|\mathbf{p};\lambda\rangle_{S^l}$ becomes simple. From (1.1.18) and (1.1.19) we know what rotations do to particles at rest. Thus

$$\mathscr{R}|\mathring{p};\lambda\rangle = \mathscr{D}^{(s)}_{\lambda'\lambda}[r(l,\mathbf{p})]|\mathring{p};\lambda'\rangle \tag{2.1.8}$$

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and since the $\mathscr{D}_{\lambda'\lambda}$ are just numbers, substituting back into (2.1.5) and (2.1.1) and then using (2.1.4) gives

$$|\mathbf{p};\lambda\rangle_{S^l} = \mathscr{D}^{(s)}_{\lambda'\lambda}[r(l,\mathbf{p})]U[h(\mathbf{p}')]|\overset{\circ}{p};\lambda'\rangle = \mathscr{D}^{(s)}_{\lambda'\lambda}[r(l,\mathbf{p})]|\mathbf{p}';\lambda'\rangle \qquad (2.1.9)$$

with $\mathbf{p}' = l^{-1}\mathbf{p}$.

This is the desired relationship between the description used in frames S^l and S for the motion of the particle. In the above form it is valid for an arbitrary Lorentz transformation from S to S^l. The reason why $|\mathbf{p};\lambda\rangle_{S^l}$ and $|\mathbf{p}';\lambda'\rangle$ are related by a rotation is that the helicity rest frame of the particle reached from S is not the same as the one reached from S^l. Indeed if we call these helicity rest frames S_A and S'_A respectively, then one can show that

$$S_A = r(l,\mathbf{p})S'_A \qquad (2.1.10)$$

It should be clear that for canonical states we have a result analogous to (2.1.9). The only difference is that $r(l,\mathbf{p})$ is replaced by

$$r_{\text{Wig}}(l,\mathbf{p}) \equiv l^{-1}(\mathbf{v}')l^{-1}l(\mathbf{v}) \qquad (2.1.11)$$

where $l(\mathbf{v})$ and $l(\mathbf{v}')$ are pure boosts corresponding to the momenta $\mathbf{p}$ and $\mathbf{p}' = l^{-1}\mathbf{p}$. The rotation in (2.1.11) is known as the Wigner spin rotation. If S^0 and $S^{0'}$ are the *canonical* rest frames reached from S and S^l respectively, then analogously to (2.1.10) one finds

$$S^0 = r_{\text{Wig}}(l,\mathbf{p})S^{0'}. \qquad (2.1.12)$$

To gain some physical intuition for the rotations involved we shall look at a few cases of practical interest.

2.2 Examples of Wick and Wigner rotations

We here derive explicit expressions for these rotations for several cases of practical interest and we end with a discussion of the Thomas precession.

2.2.1 Pure rotation of axes

In frame S let $\mathbf{p}$ lie in the XZ-plane, $\mathbf{p} = (p,\theta,0)$. Apply a rotation through angle β about OY to the frame S such that $l = r_y(\beta)$. Then in S^r we have $l^{-1}\mathbf{p} = (p,\theta-\beta,0)$. One finds trivially $r(l,\mathbf{p}) = 1$, i.e. there is no Wick helicity rotation. Thus, in this case,

$$|\mathbf{p};\lambda\rangle_{S^R} = |\mathbf{p}';\lambda\rangle \qquad \text{with} \qquad \mathbf{p}' = r_y^{-1}\mathbf{p}. \qquad (2.2.1)$$

For a general rotation $r(\alpha,\beta,\gamma)$ of S, with $\mathbf{p} = (p,\theta,\varphi)$ and $r^{-1}\mathbf{p} = (p,\theta',\varphi')$, one finds

$$|\mathbf{p};\lambda\rangle_{S^r} = e^{i\lambda\zeta}|\mathbf{p}';\lambda\rangle \qquad \text{with} \qquad \mathbf{p}' = r^{-1}\mathbf{p} \qquad (2.2.2)$$

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where

$$\cos\zeta = \frac{\cos\beta - \cos\theta\cos\theta'}{\sin\theta\sin\theta'} \tag{2.2.3}$$

(In the event that $\cos\zeta$ appears indeterminate it is simpler to use eqn (2.1.7) to determine the rotation involved.)

Both the above results are in accord with the fact that λ is invariant under rotations.

For the canonical spin states for $l = r_y(\beta)$, $\mathbf{p}$ in the XZ-plane, one finds $r_{\text{Wig}}(l, \mathbf{p}) = r_y(-\beta)$. Here the spin transforms just as it would non-relativistically (see eqn (1.1.17)).

2.2.2 Pure Lorentz boost of axes

To begin with, take the boost velocity $\boldsymbol{\beta}$ to lie along OZ so that $l = l_z(\beta)$. In the original and boosted frames we have:

$$\begin{aligned} S&: \quad \mathbf{p} = (p, \theta, \varphi), E \quad \text{speed } v \\ S^l&: \quad \mathbf{p}' = l^{-1}\mathbf{p} = (p', \theta', \varphi), E' \quad \text{speed } v' \end{aligned}$$

and from eqn (1.2.22)

$$\begin{aligned} h(\mathbf{p}) &= r(\varphi, \theta, 0) l_z(v) \\ h(\mathbf{p}') &= r(\varphi, \theta', 0) l_z(v'). \end{aligned}$$

The Wick helicity rotation is now

$$r\,[l_z(\beta), \mathbf{p}] = h^{-1}(\mathbf{p}') l_z^{-1}(\beta) h(\mathbf{p}). \tag{2.2.4}$$

It is easy to see that this is just a rotation about the Y-axis: simply examine the effect of the sequence of operations in (2.2.4) on the unit vector in the Y-direction $\mathbf{e}_{(y)} = (0, 0, 1, 0)$. It remains unchanged. Thus

$$r\,[l_z(\beta), \mathbf{p}] = r_y(\theta_{\text{Wick}})$$

so that

$$\mathscr{D}^{(s)}_{\lambda'\lambda}[r_{\text{Wick}}] = d^s_{\lambda'\lambda}(\theta_{\text{Wick}}) \tag{2.2.5}$$

and the angle θ_{Wick} can be found most easily by checking the effect of r_{Wick} upon the unit vector $\mathbf{e}_{(x)} = (0, 1, 0, 0)$. Carrying out the sequence of operations one ends up with

$$\mathbf{e}'_{(x)} = \left\{0,\ \cos\theta\cos\theta' + \gamma\sin\theta\sin\theta',\ 0,\ -\frac{m}{E}(\sin\theta\cos\theta' - \gamma\cos\theta\sin\theta')\right\}$$

where $\gamma = (1 - \beta^2)^{-1/2}$.

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Comparing with (1.1.6) and (1.1.7) and using the relation between θ and θ' we end up with

$$\begin{aligned} \cos\theta_{\text{Wick}} &= \frac{\gamma}{p'}(p - \beta E\cos\delta) \\ \sin\theta_{\text{Wick}} &= -\frac{m}{p'}\gamma\beta\sin\delta \end{aligned} \tag{2.2.6}$$

where δ $(0 \le \delta \le \pi)$ is the angle between $\boldsymbol{\beta}$ and $\mathbf{p}$. (In *this* case $\delta = \theta$.)

For the general case of a boost $l(\boldsymbol{\beta})$ of the axes, with $\boldsymbol{\beta} = (\beta, \theta_\beta, \varphi_\beta)$, one has

$$|\mathbf{p}; \lambda\rangle_{S^{l(\boldsymbol{\beta})}} = e^{i\eta(\lambda-\lambda')} d_{\lambda'\lambda}(\theta_{\text{Wick}}) |l^{-1}\mathbf{p}; \lambda'\rangle \tag{2.2.7}$$

corresponding to $r_{\text{Wick}} = r(\eta, \theta_{\text{Wick}}, -\eta)$, where η is given by

$$\begin{aligned} \cos\eta &= \frac{\sin\theta\cos\theta_\beta - \cos\theta\sin\theta_\beta\cos(\varphi - \varphi_\beta)}{\sin\delta} \\ \sin\eta &= \frac{\sin\theta_\beta\sin(\varphi - \varphi_\beta)}{\sin\delta}. \end{aligned} \tag{2.2.8}$$

As in (2.2.6), δ is the angle between $\boldsymbol{\beta}$ and $\mathbf{p}$, $0 \le \delta \le \pi$.

When both $\mathbf{p}$ and $\boldsymbol{\beta}$ lie in the XZ-plane the general result simplifies to

$$|\mathbf{p}; \lambda\rangle_{S^{l(\boldsymbol{\beta})}} = d_{\lambda'\lambda}(\pm\theta_{\text{Wick}}) |l^{-1}\mathbf{p}; \lambda'\rangle \tag{2.2.9}$$

with θ_{Wick} given by (2.2.6); the $\pm$ correspond to $\boldsymbol{\beta} \times \mathbf{p}$ being along or opposite to OY respectively.

2.2.3 Boost along or opposite to p

It is clear that if S^l is boosted from S in a direction *opposite* to the momentum of $\mathbf{p}$ of the particle then

$$|\mathbf{p}; \lambda\rangle_{S^l} = |l^{-1}\mathbf{p}; \lambda\rangle. \tag{2.2.10}$$

This holds also for boosts *along* $\mathbf{p}$ provided that the boost speed v satisfies $v < p/E$. For higher boost speeds along $\mathbf{p}$ the particle direction will have reversed in S^l and one finds

$$|\mathbf{p}; \lambda\rangle_{S^l} = (-1)^{s+\lambda} |l^{-1}\mathbf{p}; -\lambda\rangle. \tag{2.2.11}$$

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2.2.4 Transformation from CM to Lab

A case of practical importance is the transformation from centre-of-mass frame (CM) to laboratory frame (Lab). Let the particle, mass m, have momentum $\mathbf{p} = (p, \theta, \varphi)$ in the CM and $l^{-1}\mathbf{p} \equiv \mathbf{p}_\mathrm{L} = (p_\mathrm{L}, \theta_\mathrm{L}, \varphi)$ in the Lab. The boost is along the negative Z-axis with speed β_Lab (i.e. the speed of the Lab as seen in the CM frame).

In (2.2.8) and (2.2.7) we have $\delta = \pi - \theta$, $\varphi_\beta = \theta_\beta = \pi$ and thus $\eta = \pi$ so that

$$
\begin{aligned}
|\mathbf{p};\lambda\rangle_\mathrm{Lab} &= (-1)^{\lambda-\lambda'} d_{\lambda'\lambda}(\theta_\mathrm{Wick})|\mathbf{p}_\mathrm{L};\lambda'\rangle \\
&= d_{\lambda'\lambda}(\alpha)|\mathbf{p}_\mathrm{L};\lambda'\rangle
\end{aligned}
\tag{2.2.12}
$$

where

$$
\begin{aligned}
\cos\alpha &= \frac{\gamma_\mathrm{Lab}}{p_\mathrm{L}}(p + \beta_\mathrm{Lab} E \cos\theta) \\
\sin\alpha &= \frac{m\gamma_\mathrm{Lab}\beta_\mathrm{Lab}}{p_\mathrm{L}} \sin\theta
\end{aligned}
\tag{2.2.13}
$$

Another convenient expression for $\sin\alpha$ is

$$
\sin\alpha = \frac{m}{E_\mathrm{L}}(\sin\theta\cos\theta_\mathrm{L} - \gamma_\mathrm{Lab}\cos\theta\sin\theta_\mathrm{L}) \tag{2.2.14}
$$

For an elastic reaction $A + B \to A + B$, with B the target in the Lab, one finds for the final state B particle

$$
\alpha_B = \theta_\mathrm{R} \equiv \text{Lab recoil angle.} \tag{2.2.15}
$$

For elastic scattering of equal-mass particles, e.g. $pp \to pp$, in addition one finds for the final state A particle, which is scattered through θ_L in the Lab frame,

$$
\alpha_A = \theta_\mathrm{L} \equiv \text{Lab scattering angle.} \tag{2.2.16}
$$

2.2.5 Non-relativistic limit of CM to Lab transformation

For a non-relativistic collision we have $\gamma_\mathrm{Lab} \to 1$, $E_\mathrm{L} \to m$, and from (2.2.14) we find

$$
\alpha = \theta - \theta_\mathrm{L}, \tag{2.2.17}
$$

which is what we would expect non-relativistically given that the helicity is the spin projection along the direction of motion.

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2.2.6 Ultra high energy collisions

Consider a very high energy collision in the Lab, which produces particles all of which are highly relativistic in the CM. Then $\beta_{\text{Lab}} \approx 1$, $E \approx p$ and $p_{\text{L}} \approx \gamma_{\text{Lab}} E(1 + \cos\theta)$, provided that $\theta \neq 180°$. Then from (2.2.13)

$$\sin\alpha \approx \frac{m}{E}\left(\frac{\sin\theta}{1+\cos\theta}\right) = \frac{m}{E}\tan\left(\frac{\theta}{2}\right). \tag{2.2.18}$$

For a two-body reaction $A + B \to C + D$ we have $E_C \approx E_D \approx \sqrt{s}/2$ where $\sqrt{s}$ is the total CM energy. Thus

$$\sin\alpha \approx \frac{2m}{\sqrt{s}}\tan\left(\frac{\theta}{2}\right), \tag{2.2.19}$$

showing that $\alpha \to 0$ as $s \to \infty$ at fixed θ or at fixed momentum transfer to the scattered particle.

Hence follows the important result that a particle for which $m/\sqrt{s} \to 0$ does not undergo a Wick helicity rotation in the transformation CM to Lab.

2.2.7 Massless particles

The transformation of the helicity state for a massless particle can be deduced from the previous results by putting $m = 0$. Thus, under an arbitrary rotation, (2.2.2) continues to hold but under an arbitrary boost $l(\boldsymbol{\beta})$, $\theta_{\text{Wick}} = 0$ and instead of (2.2.7) we have

$$|\mathbf{p};\lambda\rangle_{S^{l(\boldsymbol{\beta})}} = |l^{-1}\mathbf{p};\lambda\rangle, \tag{2.2.20}$$

so that the helicity label is unaltered by a boost.

2.2.8 The Thomas precession

We shall give what we hope is an intelligible derivation of this famous effect, which so baffled physicists at the time of the discovery of intrinsic spin.

Let $\mathbf{s}$ be the expectation or mean value of the spin operator $\hat{\mathbf{s}}$ for an electron of charge $-e$. The electron's intrinsic magnetic moment $\boldsymbol{\mu}$ is given in Gaussian units by

$$\boldsymbol{\mu} = -\frac{ge}{2mc}\mathbf{s}, \tag{2.2.21}$$

where g is the gyromagnetic factor, which is very nearly equal to 2. For non-relativistic motion we expect $\mathbf{s}$ to obey a classical equation of motion. In particular, for a magnetic field in the rest frame of the particle, $\mathring{\mathbf{B}}$, we expect to have

$$\frac{d\mathbf{s}}{dt} = \boldsymbol{\mu} \times \mathring{\mathbf{B}} = -\frac{ge}{2mc}\mathbf{s} \times \mathring{\mathbf{B}}. \tag{2.2.22}$$

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Consider an electron that at time t has velocity $\mathbf{v}$ in some fixed reference frame, in which there is an electric field $\mathbf{E}$. If we Lorentz-transform to the electron's comoving canonical rest frame S_t^0 at that instant we shall find a magnetic field $\overset{\circ}{\mathbf{B}}$ that, to order v/c, is given by

$$\overset{\circ}{\mathbf{B}} = -\frac{\mathbf{v}}{c} \times \mathbf{E}. \tag{2.2.23}$$

It was originally supposed that a correct description of the motion of $\mathbf{s}$ was thus given by

$$\frac{d\mathbf{s}}{dt} = -\frac{\mu}{c}\mathbf{s} \times (\mathbf{v} \times \mathbf{E}) = \frac{ge}{2mc^2}\mathbf{s} \times (\mathbf{v} \times \mathbf{E}), \tag{2.2.24}$$

but this leads, in hydrogenic-type atoms, to a spin–orbit interaction that is too large by a factor of 2.

To see that (2.2.24) is incorrect, imagine a situation in which there is *no* torque acting on $\boldsymbol{\mu}$ or $\mathbf{s}$ in the canonical rest frame. We shall use the canonical definition of the spin, so that $\mathbf{s}(t)$ is the non-relativistic spin vector in the canonical rest frame S_t^0 reached from our reference frame, the Lab S_L, say, at time t when the electron has velocity $\mathbf{v}$. Thus $\mathbf{s}(t)$ is the spin vector in

$$S_t^0 = l(\mathbf{v})S_\mathrm{L}. \tag{2.2.25}$$

In the following we ignore time dilatations since they turn out to be irrelevant to our accuracy.

As viewed from the canonical rest frame S_t^0, the electron is at rest at time t but has accelerated to some infinitesimal velocity $d\overset{\circ}{\mathbf{v}}$ at time $t + dt$. The motion is wholly non-relativistic and there is no physical torque, so the mean spin vector in S_t^0 at time $t + dt$ should still be $\mathbf{s}(t)$. But this is equivalent to saying that $\mathbf{s}(t)$ is the mean spin vector in the canonical rest frame $S_{t+dt}^{0'}$ reached from S_t^0 by the infinitesimal boost $l(d\overset{\circ}{\mathbf{v}})$ (see Fig. 2.1),

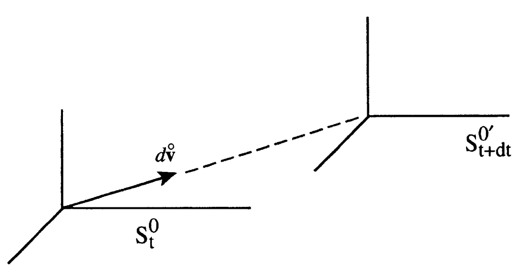

Fig. 2.1. Boost from S_t^0 to $S_{t+dt}^{0'}$.

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i.e.

$$\Big(\mathbf{s}(t+dt)\Big)_{S^{0'}_{t+dt}} = \mathbf{s}(t) \tag{2.2.26}$$

We now have the following situation at time $t+dt$. The mean spin vector is $\mathbf{s}(t+dt)$ in the canonical rest frame S^0_{t+dt} reached from S_L; it is $\mathbf{s}(t)$ in the canonical rest frame $S^{0'}_{t+dt}$ reached from the Lorentz-transformed frame $S^0_t = l(\mathbf{v})S_\mathrm{L}$.

From our earlier discussion we know that S^0_{t+dt} and $S^{0'}_{t+dt}$ are not generally the same rest frame and are related by a Wigner rotation. From (2.1.12)

$$S^0_{t+dt} = r_\mathrm{Wig}[l(\mathbf{v}), \mathbf{v}+d\mathbf{v}]S^{0'}_{t+dt}. \tag{2.2.27}$$

It follows that

$$\mathbf{s}(t+dt) = r^{-1}_\mathrm{Wig}\,(\mathbf{s}(t+dt))_{S^{0'}_{t+dt}} = r^{-1}_\mathrm{Wig}\mathbf{s}(t). \tag{2.2.28}$$

Thus, even in the absence of a physical torque, $\mathbf{s}(t+dt) \neq \mathbf{s}(t)$. To find the intrinsic rate of change of $\mathbf{s}$ we study the Wigner rotation, taking into account that $d\mathbf{v}$ is infinitesimal.

We have from (2.1.11), since $d\overset{\circ}{\mathbf{v}} = \left[l^{-1}(\mathbf{v})\right](\mathbf{v}+d\mathbf{v})$,

$$r_\mathrm{Wig}[l(\mathbf{v}), \mathbf{v}+d\mathbf{v}] = l^{-1}(d\overset{\circ}{\mathbf{v}})l^{-1}(\mathbf{v})l(\mathbf{v}+d\mathbf{v}). \tag{2.2.29}$$

To identify the rotation involved we evaluate the matrix $\Lambda(r_\mathrm{Wig})$, using (1.2.13) and working to first order in $d\mathbf{v}$. Note that to this order

$$d\overset{\circ}{\mathbf{v}} = \gamma^2 d\mathbf{v}_\parallel + \gamma d\mathbf{v}_\perp \tag{2.2.30}$$

where $\parallel$ and $\perp$ are relative to the direction of $\mathbf{v}$ and $\gamma = (1-v^2/c^2)^{-1/2}$.

We find eventually

$$\mathbf{s}(t+dt) = \left[r^{-1}(d\boldsymbol{\vartheta})\right]\mathbf{s}(t) \tag{2.2.31}$$

where

$$d\boldsymbol{\vartheta} = \frac{\gamma^2}{1+\gamma}\left(\frac{\mathbf{v}\times d\mathbf{v}}{c^2}\right). \tag{2.2.32}$$

From this follows

$$\frac{d\mathbf{s}}{dt} = \boldsymbol{\omega}_T \times \mathbf{s}, \tag{2.2.33}$$

where the Thomas angular velocity is

$$\boldsymbol{\omega}_\mathrm{T} = \frac{\gamma^2}{1+\gamma}\left(\frac{\mathbf{a}\times\mathbf{v}}{c^2}\right) \approx \frac{1}{2}\left(\frac{\mathbf{a}\times\mathbf{v}}{c^2}\right), \tag{2.2.34}$$

$\mathbf{a} = d\mathbf{v}/dt$ being the electron's acceleration at time t.

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Thus owing to the interpretation of $\mathbf{s}(t)$ as a vector in the canonical rest frame we find that $\mathbf{s}(t)$ rotates even when no physical torque acts on it in the rest frame. Clearly, then, in the presence of a magnetic torque (2.2.24) should be modified to

$$\frac{d\mathbf{s}}{dt} = \frac{ge}{2mc^2}\mathbf{s} \times (\mathbf{v} \times \mathbf{E}) + \boldsymbol{\omega}_{\mathrm{T}} \times \mathbf{s}. \tag{2.2.35}$$

For a one-electron Coulombic atom, with potential $V(r)$,

$$(-e)\mathbf{E} = -\frac{1}{r}\frac{dV}{dr}\mathbf{r}$$

and

$$\mathbf{a} = -\frac{e\mathbf{E}}{m},$$

leading, via (2.2.34), to

$$\frac{d\mathbf{s}}{dt} = \frac{g-1}{2m^2c^2}\left(\frac{1}{r}\frac{dV}{dr}\right)\mathbf{L} \times \mathbf{s}. \tag{2.2.36}$$

We see that for $g = 2$ the Thomas term just halves the strength of the spin–orbit interaction.

In Section 3.4 we shall introduce a *covariant* mean spin 4-vector and in subsection 6.3.1 derive relativistically covariant equations for its motion. They will offer a more direct derivation of the above results.

2.3 The discrete transformations

We now consider how helicity states transform under space inversion and time reversal. These results are crucial to an understanding of the physical consequences of these symmetries in specific reactions. We also briefly discuss charge conjugation.

2.3.1 Parity

Under space inversion, $S \to S^{\mathscr{P}} = l_{\mathscr{P}}S$ such that $x \to x' = (t, -\mathbf{x})$. The Hilbert space operator $U(l_{\mathscr{P}}^{-1})$ is usually written as $\mathscr{P}$ and has the following effect on the Lorentz generators $\hat{\mathbf{J}} = \left\{\hat{J}_i\right\}, \hat{\mathbf{K}} = \left\{\hat{K}_i\right\}$, see (1.2.1):

$$\mathscr{P}^{-1}\hat{\mathbf{J}}\mathscr{P} = \hat{\mathbf{J}} \tag{2.3.1}$$

$$\mathscr{P}^{-1}\hat{\mathbf{K}}\mathscr{P} = -\hat{\mathbf{K}}. \tag{2.3.2}$$

The operator $\mathscr{P}$ is unitary and taken to satisfy $\mathscr{P}^2 = 1$. Under $S \to S^{\mathscr{P}}$ we have, as in (2.1.1),

$$|\mathbf{p};\lambda\rangle \to U(l_{\mathscr{P}}^{-1})|\mathbf{p};\lambda\rangle \equiv \mathscr{P}|\mathbf{p};\lambda\rangle \tag{2.3.3}$$

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Consider the action of $\mathscr{P}$ on the helicity state of a massive particle with spin s

$$|\mathbf{p};\lambda\rangle \equiv |p,\theta,\varphi;\lambda\rangle = U[h(\mathbf{p})]|\overset{\circ}{p};\lambda\rangle$$
$$= U[r(\varphi,\theta,0)l_z(v)]|\overset{\circ}{p};\lambda\rangle; \tag{2.3.4}$$

we have

$$\mathscr{P}|p,\theta,\varphi;\lambda\rangle = U[r(\varphi,\theta,0)l_z(-v)]\mathscr{P}|\overset{\circ}{p};\lambda\rangle. \tag{2.3.5}$$

The intrinsic parity $\eta_{\mathscr{P}}$ is defined by

$$\mathscr{P}|\overset{\circ}{p};\lambda\rangle = \eta_{\mathscr{P}}|\overset{\circ}{p};\lambda\rangle \tag{2.3.6}$$

with $\eta_{\mathscr{P}}^2 = 1$. After some manipulation, using

$$l_z(-v) = r_y(-\pi)l_z(v)r_y(\pi)$$

we find

$$\mathscr{P}|p,\theta,\varphi;\lambda\rangle = \eta_{\mathscr{P}}e^{-i\pi s}|p,\pi-\theta,\varphi+\pi;-\lambda\rangle.^1 \tag{2.3.7}$$

For massless particles we have already defined the intrinsic parity in (1.2.31). For the operator for reflections in the XZ-plane, $\mathscr{Y} = r_y(\pi)\mathscr{P}$, we have

$$\mathscr{Y}|p,\theta,\varphi;\lambda\rangle = \eta_{\mathscr{P}}(-1)^{s-\lambda}|p,\theta,-\varphi;-\lambda\rangle \tag{2.3.8}$$

which is consistent with (1.2.31) since there $\lambda = s$ and $\varphi = 0$.

2.3.2 Time reversal

The time-reversal operator $\mathscr{T}$ is an anti-unitary operator (i.e. $\mathscr{T}$ is anti-linear with $\mathscr{T}^{-1} = \mathscr{T}^\dagger$), which has the following action on the Lorentz generators:

$$\begin{aligned}\mathscr{T}^{-1}\hat{\mathbf{J}}\mathscr{T} &= -\hat{\mathbf{J}}\\ \mathscr{T}^{-1}\hat{\mathbf{K}}\mathscr{T} &= \hat{\mathbf{K}}.\end{aligned} \tag{2.3.9}$$

Because of the anti-linearity these imply

$$\begin{aligned}\mathscr{T}^{-1}r\mathscr{T} &= r\\ \mathscr{T}^{-1}l\mathscr{T} &= l^{-1}\end{aligned} \tag{2.3.10}$$

for any rotation r and pure boost l.

[1] Of course the vector $(p,\pi-\theta,\varphi+\pi)$ is just $-\mathbf{p}$, but we are loth to use that notation since e.g. $|-(-\mathbf{p});\lambda\rangle \neq |\mathbf{p};\lambda\rangle$. Indeed, with $-\mathbf{p} = (p,\pi-\theta,\varphi+\pi)$

$$\begin{aligned}|-(-\mathbf{p});\lambda\rangle &= |p,\theta,\varphi+2\pi;\lambda\rangle = (-1)^{2s}|p,\theta,\varphi;\lambda\rangle\\ &= \pm|\mathbf{p};\lambda\rangle,\end{aligned}$$

the plus sign corresponding to bosons and the minus sign to fermions.

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Because of its anti-linearity care must be exercised when using $\mathscr{T}$ inside matrix elements, and it is safer to revert to a Hilbert-space notation for these rather than the Dirac notation. We recall that for any operator O

$$\langle\beta|O|\alpha\rangle \equiv (\beta, O\alpha). \tag{2.3.11}$$

For a linear operator $\hat{L}$ the hermitian conjugate $\hat{L}^\dagger$ is defined by

$$(\beta, \hat{L}\alpha) = (\hat{L}^\dagger\beta, \alpha) = (\alpha, L^\dagger\beta)^*, \tag{2.3.12}$$

so that, as usual,

$$\langle\beta|\hat{L}|\alpha\rangle = \langle\alpha|\hat{L}^\dagger|\beta\rangle^*. \tag{2.3.13}$$

For the anti-linear operator $\mathscr{T}$ the hermitian conjugate $\mathscr{T}^\dagger$ has to be defined by

$$(\beta, \mathscr{T}\alpha) = (\mathscr{T}^\dagger\beta, \alpha)^* = (\alpha, \mathscr{T}^\dagger\beta). \tag{2.3.14}$$

It is therefore safer to use the notation $|\mathscr{T}\alpha\rangle$ rather than $\mathscr{T}|\alpha\rangle$ for the time-reversed state of $|\alpha\rangle$. Thus, under $S \to S^{\mathscr{T}} = l_{\mathscr{T}}S$ such that $x' = l_{\mathscr{T}}^{-1}x = (-t, \mathbf{x})$,

$$|\mathbf{p};\lambda\rangle \to |\mathscr{T}(\mathbf{p},\lambda)\rangle \tag{2.3.15}$$

We follow the convention used by Jacob and Wick (1959) and take, for a particle at rest,

$$|\mathscr{T}(\mathring{p},\lambda)\rangle = (-1)^{s-\lambda}|\mathring{p};-\lambda\rangle. \tag{2.3.16}$$

Note that with this convention $\mathscr{T}^2 = (-1)^{2s}$.

It follows from (2.3.16) and (2.3.10) that

$$|\mathscr{T}(p,\theta,\varphi;\lambda)\rangle = e^{-i\pi\lambda}|p,\pi-\theta,\varphi+\pi;\lambda\rangle \tag{2.3.17}$$

and the same result holds for massless particles.

Note that for any linear operator $\hat{L}$ one has

$$\begin{aligned}\langle\mathscr{T}\alpha|\hat{L}|\mathscr{T}\beta\rangle &= (\mathscr{T}\alpha, \hat{L}\mathscr{T}\beta) = (\alpha, \mathscr{T}^\dagger\hat{L}\mathscr{T}\beta)^* \\ &= \langle\alpha|\mathscr{T}^\dagger\hat{L}\mathscr{T}|\beta\rangle^* \\ &= \langle\beta|\mathscr{T}^\dagger\hat{L}^\dagger\mathscr{T}|\alpha\rangle,\end{aligned} \tag{2.3.18}$$

the last step following since $\mathscr{T}^\dagger\hat{L}\mathscr{T}$ is a linear operator.

Time-reversal invariance is usually taken to mean that, for transition amplitudes or S-matrix elements,

$$\langle\mathscr{T}\alpha|S|\mathscr{T}\beta\rangle = \langle\beta|S|\alpha\rangle. \tag{2.3.19}$$

From (2.3.18) we see that time-reversal invariance implies

$$\mathscr{T}^{-1}S\mathscr{T} = S^\dagger \tag{2.3.20}$$

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in contrast to all linear invariances, where there would be no dagger symbol on the right-hand side.

2.3.3 Charge conjugation

The charge conjugation operator $\mathscr{C}$ ($\mathscr{C}^2 = 1$) changes particles into anti-particles and vice versa. For a particle A at rest

$$\mathscr{C}|A; \mathring{p}, \lambda\rangle = \eta_{\mathscr{C}}|\overline{A}; \mathring{p}, \lambda\rangle \tag{2.3.21}$$

where $\eta_{\mathscr{C}} = \pm 1$ is the *charge parity* of the particle. Since $\mathscr{C}$ has no effect on the kinematic variables, we have also

$$\mathscr{C}|A; \mathbf{p}, \lambda\rangle = \eta_{\mathscr{C}}|\overline{A}; \mathbf{p}, \lambda\rangle. \tag{2.3.22}$$

Note that $\eta_{\mathscr{C}} = +1$ for pions and nucleons, -1 for photons.

We remind the reader that some care must be exercised when dealing with multiplets of an internal symmetry. For example, if protons and neutrons are regarded as forming an isotopic spin doublet of the nucleon N, so that

$$|N; I_z = 1/2\rangle = |p\rangle, \qquad |N; I_z = -1/2\rangle = |n\rangle, \tag{2.3.23}$$

then the antinucleon multiplet that transforms like an isospin doublet is

$$|\overline{N}; I_z = 1/2\rangle = -|\overline{n}\rangle, \qquad |\overline{N}; I_z = -1/2\rangle = |\overline{p}\rangle. \tag{2.3.24}$$

This is explained in subsection 2.4.2.

2.4 Fields and wave functions

On the one had we saw in Section 2.1 that under Lorentz transformations the state vector in a relativistic theory transforms in a complicated way, the transformation matrix depending upon the Wick helicity rotation or the Wigner rotation.

On the other hand, in setting up a field theory it is customary to use fields that transform simply under Lorentz transformations. Thus if a Lorentz transformation l acting on the reference frame S takes it to S^l,

$$S \xrightarrow{l} S^l,$$

so that $x \to x' = l^{-1}x$, then the fields $\phi_n(x), n = 1, \ldots, N$, are taken to undergo the transformation $\phi_n(x) \to \phi'_n(x')$ where

$$\phi'_n(x) = U(l)\phi_n(x)U(l^{-1}) = D_{nm}(l^{-1})\phi_m(lx). \tag{2.4.1}$$

here D_{nm} is an N-dimensional representation of the homogeneous Lorentz group (see Appendix 2). Note that the matrices depend *only* on l.

We consider here some aspects of the relationship between the two approaches.

The fields $\phi_n(x)$ are generally not irreducible, in the sense that they have more components (N) than are needed to describe quanta of some given spin s, i.e. $N > 2s + 1$. As a consequence the representation $D^{(N)}$ may be reducible under pure rotations, as, for example, when massive spin-1 quanta are described by a Lorentz 4-vector, or they may even be reducible under all homogeneous Lorentz transformations, as in the case when spin-1/2 quanta are described by a four-component Dirac field. (In the latter case the representation becomes irreducible if the operation of space inversion is included.)

In order to construct Lorentz-invariant lagrangians etc. it is useful to deal with conjugate fields $\overline{\phi}_n(x)$. These may be just the hermitian conjugate fields $\phi_n^\dagger(x)$ or some fixed linear combination of these (e.g. $\overline{\Psi}(x) = \Psi^\dagger(x)\beta$ in the Dirac theory) so designed that $\overline{\phi}$ transforms contra-grediently to ϕ, i.e. under $S \xrightarrow{l} S^l$, $\overline{\phi}_n(x) \to \overline{\phi}'_n(x')$ where

$$\overline{\phi}'_n(x) = U(l)\overline{\phi}_n(x)U(l^{-1}) = \overline{\phi}_m(lx)D_{mn}(l). \tag{2.4.2}$$

Thus in matrix notation, regarding ϕ as a column vector and $\overline{\phi}$ as a row vector:

$$\begin{aligned} \phi'(x') &= D^{-1}(l)\phi(x) \\ \overline{\phi}'(x') &= \overline{\phi}(x)D(l), \end{aligned} \tag{2.4.3}$$

so that $\overline{\phi}\phi$ is a scalar, i.e.

$$\overline{\phi}'(x')\phi'(x') = \overline{\phi}(x)\phi(x). \tag{2.4.4}$$

The use of $\overline{\phi}$ and ϕ makes it quite simple to construct quantities with definite transformation properties under Lorentz transformations. But some price has to be paid for the redundant components; this price is the existence of field equations that must be satisfied even by non-interacting fields. These equations are nothing more than invariant conditions of constraint upon the unwanted components. In a series of elegant papers Weinberg (1964a, 1964b) showed how one may construct irreducible fields ϕ_λ with only $2s + 1$ components. These satisfy no field equations (other than the Klein–Gordon equation, which just imposes the correct relation between energy and momentum) but they do not transform simply under Lorentz transformations. They shed an interesting light upon the whole question of fields and field equations and we therefore give a brief discussion of this approach in Appendix 3. Here we continue to deal with the usual fields $\phi_n(x)$.

The fields $\phi_n(x)$, $\overline{\phi}_n(x)$ are Fourier expanded in terms of creation and annihilation operators ($a^\dagger$, a for particles and $b^\dagger$, b for antiparticles), which create and annihilate quanta of spin s with definite momenta and helicity.

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Thus one writes

$$\phi_n(x) = \sum_\lambda \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2} 2p^0} \left[u_n(\mathbf{p},\lambda) a(\mathbf{p},\lambda) e^{-ip\cdot x} + v_n(\mathbf{p},\lambda) b^\dagger(\mathbf{p},\lambda) e^{ip\cdot x} \right] \quad (2.4.5)$$

$$\overline{\phi}(x) = \sum_\lambda \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2} 2p^0} \left[\overline{u}_n(\mathbf{p},\lambda) a^\dagger(\mathbf{p},\lambda) e^{ip\cdot x} + \overline{v}_n(\mathbf{p},\lambda) b(\mathbf{p},\lambda) e^{-ip\cdot x} \right] \quad (2.4.6)$$

where the u and v are 'wave functions' for the quanta (in the Dirac case they just correspond to the Dirac 4-spinors u, v).

Since $a^\dagger(\mathbf{p},\lambda)$ creates the state $|\mathbf{p};\lambda\rangle$ from the Lorentz invariant vacuum, it follows from eqn (2.1.9) and the unitarity of the representations of the rotation group that

$$U(l) a(\mathbf{p},\lambda) U(l^{-1}) = \mathscr{D}^{(s)}_{\lambda\lambda'}(r) a(l\mathbf{p},\lambda'), \quad (2.4.7)$$

where $r = r(l,\mathbf{p})$ is the Wick rotation defined in eqn (2.1.7).

For free fields or fields in the interaction representation and with particle states such that

$$|\mathbf{p};\lambda\rangle = a^\dagger(\mathbf{p},\lambda)|0\rangle, \quad (2.4.8)$$

where the operators satisfy commutation or anticommutation relations

$$\left[a(\mathbf{p},\lambda), a^\dagger(\mathbf{p}',\lambda') \right]_\pm = 2p^0 \delta(\mathbf{p}' - \mathbf{p}) \delta_{\lambda'\lambda}, \quad (2.4.9)$$

one has

$$\langle 0|\phi_n(x)|\mathbf{p};\lambda\rangle = \frac{u_n(\mathbf{p},\lambda) e^{-ip\cdot x}}{(2\pi)^{3/2}} \quad (2.4.10)$$

and for antiparticles

$$\langle 0|\overline{\phi}_n(x)|\overline{p};\lambda\rangle = \frac{\overline{v}_n(\mathbf{p},\lambda) e^{-ip\cdot x}}{(2\pi)^{3/2}}. \quad (2.4.11)$$

The set of wave functions $u_n(\mathbf{p},\lambda)$ will be said to correspond to the state $|\mathbf{p};\lambda\rangle$:

$$|\mathbf{p};\lambda\rangle \longleftrightarrow u_n(\mathbf{p},\lambda). \quad (2.4.12)$$

Clearly the $u_n(\mathbf{p},\lambda)e^{-ip\cdot x}$ satisfy the same free-field equations as do the $\phi_n(x)$. Thus the u_n are usually obtained by solving those equations, but care must be exercised in order to have consistent phase conventions. Thus if

$$|\mathbf{p};\lambda\rangle \longleftrightarrow u_n(\mathbf{p},\lambda)$$

and

$$|\mathring{p}; s_z = \lambda\rangle \longleftrightarrow u_n(\mathring{p},\lambda)$$

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then from (1.2.25) and (2.4.1), using the Lorentz invariance of the vacuum,

$$\begin{aligned}\langle 0|\phi_n(x)|\mathbf{p},\lambda\rangle &= \langle 0|\phi_n(x)U[h(\mathbf{p})]|\overset{\circ}{p};\lambda\rangle \\ &= \langle 0|U^{-1}[h(\mathbf{p})]\phi_n(x)U[h(\mathbf{p})]|\overset{\circ}{p};\lambda\rangle \\ &= D_{nm}[h(\mathbf{p})]\langle 0|\phi_m(h^{-1}x)|\overset{\circ}{p};\lambda\rangle, \end{aligned} \tag{2.4.13}$$

which leads, via (2.4.10), to the requirement that

$$u_n(\mathbf{p},\lambda) = D_{nm}[h(\mathbf{p})]u_m(\overset{\circ}{p},\lambda). \tag{2.4.14}$$

A similar argument, for antiparticles, leads to

$$\bar{v}_n(\mathbf{p},\lambda) = \bar{v}_m(\overset{\circ}{p},\lambda)D_{mn}[h^{-1}(\mathbf{p})]. \tag{2.4.15}$$

Consider now the effect of an arbitrary Lorentz transformation $S \xrightarrow{l} S^l$. Using eqns (2.1.3), (2.1.9) and (2.4.1) in (2.4.10), we have the correspondence

$$|\mathbf{p};\lambda\rangle \longleftrightarrow u_n(\mathbf{p},\lambda)$$

and

$$\begin{aligned}U(l^{-1})|\mathbf{p};\lambda\rangle &\longleftrightarrow D_{nm}(l^{-1})u_m(\mathbf{p},\lambda) \\ &= u_n(l^{-1}\mathbf{p},\lambda')\mathscr{D}^{(s)}_{\lambda'\lambda}(r)\end{aligned} \tag{2.4.16}$$

where $r = r(l,\mathbf{p})$.

In a similar way one finds for antiparticles

$$|\bar{\mathbf{p}};\lambda\rangle \longleftrightarrow \bar{v}_n(\mathbf{p},\lambda)$$

and

$$\begin{aligned}U(l^{-1})|\bar{\mathbf{p}};\lambda\rangle &\longleftrightarrow \bar{v}_m(\mathbf{p},\lambda)D_{mn}(l) \\ &= \bar{v}_n(l^{-1}\mathbf{p},\lambda')\mathscr{D}^{(s)}_{\lambda'\lambda}(r)\end{aligned} \tag{2.4.17}$$

and, in addition,

$$\begin{aligned}\langle\mathbf{p},\lambda| &\longleftrightarrow \bar{u}_n(\mathbf{p},\lambda) \\ \langle\mathbf{p},\lambda|\,U(l) &\longleftrightarrow \bar{u}_m(\mathbf{p},\lambda)D_{mn}(l) \\ &= \bar{u}_n(l^{-1}\mathbf{p},\lambda')\mathscr{D}^{(s)}_{\lambda\lambda'}(r^{-1})\end{aligned} \tag{2.4.18}$$

and for antiparticles

$$\begin{aligned}\langle\bar{\mathbf{p}},\lambda| &\longleftrightarrow v_n(\mathbf{p},\lambda) \\ \langle\bar{\mathbf{p}},\lambda|\,U(l) &\longleftrightarrow D_{nm}(l^{-1})v_m(\mathbf{p},\lambda) \\ &= v_n(l^{-1}\mathbf{p},\lambda')\mathscr{D}^{(s)}_{\lambda\lambda'}(r^{-1}).\end{aligned} \tag{2.4.19}$$

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2.4.1 The discrete transformations of the fields

Consider the discrete transformations. Under space inversion

$$S \xrightarrow{l_{\mathscr{P}}} S^{\mathscr{P}} = l_{\mathscr{P}} S$$

with $x \to x' = l_{\mathscr{P}} x = (t, -\mathbf{x})$, one takes $\phi_n(x) \to \phi_n^{\mathscr{P}}(x')$ with (see Section 2.3)

$$\phi_n^{\mathscr{P}}(x) = \mathscr{P}^{-1}\phi_n(x)\mathscr{P} = P_{nm}\phi_m(t, -\mathbf{x}) \tag{2.4.20}$$

where P is an $N \times N$ matrix ($P^2 = I$) chosen so that $\phi_n^{\mathscr{P}}(x')$ satisfies the space-inverted field equations. This does not fix the absolute phase of P. However, using eqn (2.3.7) we have for a particle of spin s

$$\begin{aligned} \langle 0|\phi_n(x)\mathscr{P}|p, \theta, \varphi; \lambda\rangle &= \eta_{\mathscr{P}} e^{-i\pi s}\langle 0|\phi_n(x)|p, \pi - \theta, \varphi + \pi; -\lambda\rangle. \\ &= \langle 0|\mathscr{P}^{-1}\phi_n(x)\mathscr{P}|p, \theta, \varphi; \lambda\rangle \\ &= P_{nm}\langle 0|\phi_m(t, -\mathbf{x})|p, \theta, \varphi; \lambda\rangle \end{aligned} \tag{2.4.21}$$

from which, via (2.4.10), we have that P must be chosen such that

$$P_{nm}u_m(p, \theta, \varphi; \lambda) = \eta_{\mathscr{P}} e^{-i\pi s} u_n(p, \pi - \theta, \varphi + \pi; -\lambda). \tag{2.4.22}$$

For antiparticles one has, since $P^2 = I$, i.e. $P^{-1} = P$,

$$\bar{v}_m(p, \theta, \varphi; \lambda)P_{mn} = \bar{\eta}_{\mathscr{P}} e^{-i\pi s}\bar{v}_n(p, \pi - \theta, \varphi + \pi; -\lambda) \tag{2.4.23}$$

where $\bar{\eta}_{\mathscr{P}}$ is the intrinsic parity of the antiparticle.

We also have the following correspondence between states and wave functions:

$$\mathscr{P}|\mathbf{p}; \lambda\rangle \longleftrightarrow P_{nm}u_m(\mathbf{p}, \lambda) \tag{2.4.24}$$

$$\mathscr{P}|\bar{\mathbf{p}}; \lambda\rangle \longleftrightarrow \bar{v}_m(\mathbf{p}, \lambda)P_{mn}. \tag{2.4.25}$$

As an example, in the Dirac case it is conventional to choose $P = \gamma^0$. For the particle at rest, the use of (2.4.24) and (2.4.25) in (2.3.6) and its analogue for antiparticles shows that we must then choose $\eta_{\mathscr{P}} = 1$ and $\bar{\eta}_{\mathscr{P}} = -1$.

Consider now the anti-unitary time-reversal operation

$$S \xrightarrow{l_{\mathscr{T}}} S^{\mathscr{T}} = l_{\mathscr{T}} S$$

(see subsection 2.3.2) with $x \to x' = l_{\mathscr{T}}^{-1} x = (-t, \mathbf{x})$. One takes $\phi_n^{\mathscr{T}}(x) \to \phi_n^{\mathscr{T}}(x')$ with

$$\phi_n^{\mathscr{T}}(x) = \mathscr{T}^{-1}\phi_n(x)\mathscr{T} = T_{nm}\phi_m(-t, \mathbf{x}), \tag{2.4.26}$$

where T is an $N \times N$ matrix with $T^*T = (-1)^{2s}I$, chosen such that $\phi_n^{\mathscr{T}}(x)$ satisfies the time-reversed equations. Its phase is fixed as follows. Using eqns (2.3.16), (2.3.18) and (2.4.26) we find

$$\begin{aligned}\langle 0|\phi_n(x)|\mathscr{T}(p,\theta,\varphi;\lambda)\rangle &= e^{-i\pi\lambda}\langle 0|\phi_n(x)|p,\pi-\theta,\varphi+\pi;\lambda\rangle \\ &= \langle 0|\mathscr{T}^{-1}\phi_n(x)\mathscr{T}|p,\theta,\varphi;\lambda\rangle^* \\ &= T^*_{nm}\langle 0|\phi_m(-t,\mathbf{x})|p,\theta,\varphi;\lambda\rangle^* \end{aligned} \tag{2.4.27}$$

from which we have the requirement

$$T^*_{nm}u^*_m(p,\theta,\varphi;\lambda) = e^{-i\pi\lambda}u_n(p,\pi-\theta,\varphi+\pi;\lambda) \tag{2.4.28}$$

or

$$T_{nm}u_m(p,\theta,\varphi;\lambda) = e^{i\pi\lambda}u^*_n(p,\pi-\theta,\varphi+\pi;\lambda). \tag{2.4.29}$$

Similarly, for antiparticles

$$\bar{v}_m(p,\theta,\varphi;\lambda)T_{mn} = e^{i\pi\lambda}\bar{v}^*_n(p,\pi-\theta,\varphi+\pi;\lambda). \tag{2.4.30}$$

Note that one has the correspondence between states and wave functions

$$|\mathscr{T}(p,\theta,\varphi;\lambda)\rangle \longleftrightarrow e^{-i\pi\lambda}u_n(p,\pi-\theta,\varphi+\pi;\lambda) \tag{2.4.31}$$

and for antiparticles

$$|\mathscr{T}(\overline{p,\theta,\varphi;\lambda})\rangle \longleftrightarrow e^{-i\pi\lambda}\bar{v}_n(p,\pi-\theta,\varphi+\pi;\lambda). \tag{2.4.32}$$

With the conventions (1.2.22), for the Dirac case one has $T = \gamma^3\gamma^1$ if we use the standard representation of the γ-matrices, given for example in Bjorken and Drell (1964), in which γ^3 and γ^1 are real.

Finally, under charge conjugation (see subection 2.3.3) we have from eqns (2.3.22) and (2.4.10)

$$\begin{aligned}\frac{u_n(\mathbf{p},\lambda)}{(2\pi)^{3/2}} &= \langle 0|\phi_n(0)|\mathbf{p};\lambda\rangle \\ &= \eta_{\mathscr{C}}\langle 0|\phi_n(0)\mathscr{C}|\overline{\mathbf{p}};\lambda\rangle \\ &= \eta_{\mathscr{C}}\langle 0|\mathscr{C}^{-1}\phi_n(0)\mathscr{C}|\overline{\mathbf{p}};\lambda\rangle, \end{aligned} \tag{2.4.33}$$

which is only possible, via (2.4.11), if

$$\mathscr{C}^{-1}\phi_n(x)\mathscr{C} = \eta_{\mathscr{C}}C_{nm}\bar{\phi}_m(x), \tag{2.4.34}$$

where $\mathscr{C}^2 = I$.

Substituted into (2.4.33) this implies that

$$u_n(\mathbf{p},\lambda) = C_{nm}\bar{v}_m(\mathbf{p},\lambda). \tag{2.4.35}$$

For the Dirac case, in the standard representation of the γ-matrices one has $C = i\gamma^2\gamma^0$, with $C^2 = -I$.

2.4.2 Isospin multiplets for antiparticles

We mentioned in subsection 2.3.3 that if protons and neutrons are regarded as forming a doublet under isotopic spin rotations,

$$|N; I_z = \tfrac{1}{2}\rangle = |p\rangle \qquad |N; I_z = -\tfrac{1}{2}\rangle = |n, \tag{2.4.36}$$

then the antiparticle doublet that transforms as an isodoublet is

$$|\overline{N}; I_z = \tfrac{1}{2}\rangle = -|\overline{n}\rangle \qquad |\overline{N}; I_z = -\tfrac{1}{2}\rangle = |\overline{p}\rangle\rangle \tag{2.4.37}$$

The source of the minus sign or, for a general isospin multiplet, of certain phase factors can be understood as follows.

Let $|A; I_z\rangle$ be an isospin multiplet of *particles* of type A. Under an isospin rotation r, in complete analogy to ordinary rotations (see (1.1.18) and (1.1.19)) one will have

$$U(r)|A; I_z\rangle = \mathscr{D}^{(I)}_{I'_z I_z}(r)|A; I'_z\rangle \tag{2.4.38}$$

where $U(r)$ is the unitary operator that represents the isotopic spin rotation acting on the state vectors and the $\mathscr{D}^{(I)}$ are the $SU(2)$ representation matrices, whose properties are discussed in Appendix 1.

If the creation operators for the particles are labelled $a^\dagger_{I_z}$ then (2.4.38) is tantamount to having

$$U(r)a^\dagger_{I_z}U^{-1}(r) = \mathscr{D}^{(I)}_{I'_z I_z}(r)a^\dagger_{I'_z} \tag{2.4.39}$$

where we do not display arguments such as momentum, helicity etc. that are irrelevant to the discussion.

Consider now the set of usual fields $\Phi_{I_z}(x)$ corresponding to the set of particles of type A and isospin I. They ought to transform analogously to (2.4.1), except that there is here obviously no effect on the space–time coordinates. So we wish to have

$$U(r)\Phi_{I_z}(x)U^{-1}(r) = \mathscr{D}^{(I)}_{I_z I'_z}(r^{-1})\Phi_{I'_z}(x). \tag{2.4.40}$$

Now the field $\Phi_{I_z}(x)$ contains the *annihilation* operator a_{I_z} as in (2.4.5), so we have to check that (2.4.39) and (2.4.40) are compatible. Indeed they are, since taking the hermitian conjugate of (2.4.39) yields

$$\begin{aligned} U(r)a_{I_z}U^{-1}(r) &= \mathscr{D}^{(I)\,*}_{I'_z I_z}(r)a_{I'_z} \\ &= \left[\mathscr{D}^{(I)\dagger}(r)\right]_{I_z I'_z} a_{I'_z} \end{aligned}$$

which, using the unitarity of the matrices $\mathscr{D}^{(I)}$, gives

$$U(r)a_{I_z}U^{-1}(r) = \mathscr{D}^{(I)}_{I_z I'_z}(r^{-1})a_{I'_z}, \tag{2.4.41}$$

as required for (2.4.40).

However, the field $\Phi_{I_z}(x)$ also contains the *creation* operators $b^\dagger_{I_z}$, which create the states $|\overline{A, I_z}\rangle$ corresponding to the antiparticles of the particles A_{I_z}. For consistency with (2.4.40) they will have to transform as follows:

$$U(r)b^\dagger_{I_z}U^{-1}(r) = \mathscr{D}^{(I)}_{I_z I'_z}(r^{-1})b^\dagger_{I'_z}$$

which, as before, via the unitarity nature of $\mathscr{D}^{(I)}$ gives

$$\begin{aligned} U(r)b^\dagger_{I_z}U^{-1}(r) &= \mathscr{D}^{(I)\dagger}_{I_z I'_z}(r)b^\dagger_{I'_z} \\ &= \mathscr{D}^{(I)*}_{I'_z I_z}(r)b^\dagger_{I'_z}. \end{aligned} \tag{2.4.42}$$

Comparing with (2.4.39) and (2.4.38) we have, for the isospin multiplet made up of particles,

$$U(r)|A;I_z\rangle = \mathscr{D}^{(I)}_{I'_z I_z}(r)|A;I'_z\rangle \tag{2.4.43}$$

and, for their antiparticles,

$$U(r)|\overline{A;I_z}\rangle = \mathscr{D}^{(I)*}_{I'_z I_z}(r)|\overline{A;I'_z}\rangle. \tag{2.4.44}$$

In other words the set of antiparticles states $|\overline{A;I_z}\rangle$ does not transform as a standard isospin multiplet.

However, for the group of isospin rotations $SU(2)$ the representations $\mathscr{D}^{(I)}$ and $\mathscr{D}^{(I)^*}$ are *equivalent*, i.e. there exists a unitary matrix $C^{(I)}$, independent of r, such that

$$\mathscr{D}^{(I)^*}(r) = C^{(I)}\mathscr{D}^{(I)}(r)C^{(I)^{-1}} \tag{2.4.45}$$

for all r.

Then the antiparticle multiplet $|\bar{A};I_z\rangle$ that transforms as a standard isospin multiplet is clearly

$$|\bar{A};I_z\rangle \equiv C^{(I)}_{I'_z I_z}|\overline{A;I'_z}\rangle, \tag{2.4.46}$$

i.e.

$$U(r)|\bar{A};I_z\rangle = \mathscr{D}^{(I)}_{I'_z I_z}(r)|\bar{A};I'_z\rangle. \tag{2.4.47}$$

In fact the matrix $C^{(I)}$ is very simple. It can be taken, conventionally, as

$$C^{(I)}_{ij} = (-1)^{I-i}\delta_{i,-j}. \tag{2.4.48}$$

As an example of (2.4.46) and (2.4.48), for the nucleon isodoublet one finds just the results (2.4.36) and (2.4.37). (Of course the *overall* sign in (2.4.37) is irrelevant and sometimes the opposite convention is used.)