11

The extended loop representation of quantum gravity

11.1 Introduction

In chapter 2 we saw that the extended loops arise as natural extensions of the group of loops into a Lie structure. We also saw in chapter 4 that the use of extended loops provided a natural framework for the regularization for Maxwell theory. The intention in this chapter is to explore to what extent they can be useful for addressing regularization issues in quantum gravity. As an important by-product we will find that they are also an efficient computational tool for discussing several issues related to the solution space of quantum gravity and the action of the constraints.

Regularization issues in quantum gravity are considerably more involved than those of Maxwell theory. It is therefore remarkable that there is a formal similarity with the case of Maxwell theory. In that case one of the regularization difficulties that we confronted in the loop representation was that the vacuum of the theory,

$$\Psi_0(\gamma) = \exp(-\tfrac{1}{2} \oint_\gamma dx^a \oint_\gamma dy^b K_{ab}(x-y)), \tag{11.1}$$

where $K_{ab}(x-y)$ was the (distributional) Feynman propagator, was an ill defined quantity. Apart from this difficulty in the definition of the wavefunctions one also had the expected regularization problems of the Hamiltonian, which was quadratic in momenta.

The ill definition of the vacuum in Maxwell theory appears remarkably similar to the problem of framing that we confronted in the loop representation of quantum gravity in the previous chapter. As we saw there, the exponential of the self-linking number,

$$\Psi_0(\gamma) = \exp\left(-\frac{\Lambda}{4} \oint_\gamma dx^a \oint_\gamma dy^b g_{ax\,by}\right) \tag{11.2}$$

where $g_{ax\,by}$ is the (distributional) propagator of Chern–Simons theory,

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was a solution to all the constraints and embodied all the framing ambiguities that are present in the Kauffman bracket. The similarity of the two expressions, the one corresponding to Maxwell theory and the one corresponding to gravity is quite striking.

A word of caution should be said about jumping to the conclusion that the similarity of these two problems necessarily implies their solutions should be the same. It is true that going to extended loops fixes the regularization problems of Maxwell theory and allows us to recover the Fock structure of the theory. However, one expects that in quantum gravity, due to the diffeomorphism invariance, the structure of the theory will be quite different from a Fock structure. Intuitively, one expects diffeomorphism invariance will yield some sort of discrete structure, possibly better suited for a description in terms of loops, which are essentially discrete, than extended loops, which are inherently continuous. At the moment, however, the picture is far from clear and the attitude should be to explore all possible avenues to regularize the theory in order to be able to decide which is the better strategy. Because of its natural formulation in terms of objects to which we can apply the usual rules of functional calculus, the extended loop representation presents an attractive formulation in which we can set many of the unsolved questions about regularization raised in the previous chapter.

Another issue related to the use of extended loops is that part of the geometric flavor that representations in terms of loops have is lost. For instance, we saw in chapter 8 how the diffeomorphism invariance of general relativity was naturally coded in the ideas of knot theory. In the extended representation this connection is lost and the diffeomorphism constraint has to be treated as a functional equation. Not everything is lost, since as we will see, several of the ideas of knot theory can be generalized to the extended representation. These issues, connected with the problem that extended holonomies may have convergence problems, have led to a general feeling that some intermediate avenue between ordinary loops and extended loops could be the genuine framework for quantizing gravity. At present, however, such a framework has not been developed.

The proposal to use extended loops to build a representation for quantum gravity was first advanced in references [224, 225].

The structure of this chapter is as follows. We start with a discussion of wavefunctions and their identities in terms of extended loops. We then write the constraints in terms of the extended representation via the loop transform. We then proceed to find the extended version of the solutions to the constraints that we discussed in chapter 10. The usual loop representation is then obtained as a limit of the extended representation. We end with a discussion of the regularization of constraints and solutions in terms of this representation.

11.2 Wavefunctions

We start by discussing general properties that wavefunctions in the extended representations must satisfy. Wavefunctions are related to those in the connection representation by the extended transform,

$$\Psi(\mathbf{X}) = \int DA\, \Psi[A]\, W_{\mathbf{X}}[A], \tag{11.3}$$

with $W_{\mathbf{X}}[A]$ the extended Wilson loop,

$$W_{\mathbf{X}}[A] = \mathrm{Tr}(H_A[\mathbf{X}]) = \mathrm{Tr}[A_{\underset{\sim}{\mu}}]\, X^{\underset{\sim}{\mu}}, \tag{11.4}$$

where the notation is as usual, indices with tildes represent sets of pairs of vector indices and space points $\underset{\sim}{\mu} = (a_1\, x_1 \ldots a_n\, x_n)$ and repeated indices with tildes imply integrations over the x_is, Einstein convention summations on the a_i's and a summation on n from zero to infinity. The notation $A_{\underset{\sim}{\mu}}$ denotes the product $A_{a_1}(x_1) \cdots A_{a_n}(x_n)$.

In order to have a gauge invariant Wilson loop, the multitensors $\mathbf{X}$ must satisfy the differential constraint,

$$\frac{\partial}{\partial x_i^{a_i}} X^{a_1 x_1 \ldots a_i x_i \ldots a_n x_n} = \\ \left(\delta(x_i - x_{i-1}) - \delta(x_i - x_{i+1}) \right) X^{a_1 x_1 \ldots a_{i-1} x_{i-1}\, a_{i+1} x_{i+1} \ldots a_n x_n}, \tag{11.5}$$

and we call the space of such multitensors $\mathcal{D}_o$. Notice that we do not require the algebraic constraints that we introduced in chapter 2. At this point one has a choice of which precise kind of extended representation one wants to consider. The choice to ignore the algebraic constraint has the payoff that the resulting representation is simpler, because one avoids dealing with non-linear constraints. The price is that the degree of redundancy in the description is higher.

As in the case of loops, the structure of the particular gauge group imprints on the wavefunctions in the extended representation a series of relations, the Mandelstam identities. When we introduced the Mandelstam identities in chapter 3 for usual loops we did it by considering the properties of the traces of products of group elements, which in that case were the holonomies. In the extended case, this is not possible, since the holonomies no longer belong to the gauge group, as we discussed in chapter 2. It turns out that the Mandelstam identities in the extended case arise as a consequence of the properties of the traces of products of the connections $\mathrm{Tr}(A_{a_1}(x_1) \cdots A_{a_n}(x_n))$ combined with the linearity of the extended holonomies in terms of the multitensors. Their explicit form

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is

$$W_{\mathbf{X}_1\times\mathbf{X}_2}[A]=W_{\mathbf{X}_2\times\mathbf{X}_1}[A], \tag{11.6}$$

$$W_{\mathbf{X}}[A]=W_{\bar{\mathbf{X}}}[A], \tag{11.7}$$

$$W_{\mathbf{X}_1}[A]W_{\mathbf{X}_2}[A]=W_{\mathbf{X}_1\times\mathbf{X}_2}[A]+W_{\mathbf{X}_1\times\bar{\mathbf{X}}_2}[A]. \tag{11.8}$$

The first identity corresponds to the usual cyclic property of traces. The second one corresponds to the inversion of loops $W_\gamma[A]=W_{\gamma^{-1}}[A]$ which in terms of extended loops corresponds to inversion of the indices,

$$\overline{X}^{\mu_1\ldots\mu_n}\equiv(-1)^n X^{\mu_n\cdots\mu_1}. \tag{11.9}$$

This equation corresponds (when particularized to loops and making use in that case of the algebraic constraint) to the expression for the inverse of a multitensor that we introduced in chapter 2. Notice that in general it is *not* the inverse multitensor.

In terms of wavefunctions the identities translate into

$$\Psi(\mathbf{X}_1\times\mathbf{X}_2)=\Psi(\mathbf{X}_2\times\mathbf{X}_1), \tag{11.10}$$

$$\Psi(\mathbf{X})=\Psi(\overline{\mathbf{X}}), \tag{11.11}$$

$$\begin{aligned}\Psi(\mathbf{X}_1\times\mathbf{X}_2\times\mathbf{X}_3)+\Psi(\mathbf{X}_1\times\mathbf{X}_2\times\overline{\mathbf{X}}_3)=&\\ \Psi(\mathbf{X}_2\times\mathbf{X}_1\times\mathbf{X}_3)+\Psi(\mathbf{X}_2\times\mathbf{X}_1\times\overline{\mathbf{X}}_3).&\end{aligned} \tag{11.12}$$

The identity corresponding to loop inversions (11.11) implies in the extended representations that wavefunctions must depend on the extended coordinates through the combination

$$R^{\mu_1\ldots\mu_n}=\frac{1}{2}\left[X^{\mu_1\ldots\mu_n}+(-1)^n X^{\mu_n\cdots\mu_1}\right], \tag{11.13}$$

where the **R**s satisfy the following symmetry property under the inversion of the indices

$$R^{\mu_1\ldots\mu_n}=(-1)^n R^{\mu_n\cdots\mu_1}. \tag{11.14}$$

An important property of the wavefunctions in the extended representation is that they are *linear* functions of the extended coordinates. This is due to the fact that the extended Wilson loop is also a linear function of the extended coordinates. The general form of a wavefunction in the extended representation is therefore given by

$$\Psi(\mathbf{X})=D_{\underset{\sim}{\mu}}X^{\underset{\sim}{\mu}}, \tag{11.15}$$

and all the information of the particular wavefunction is coded in the coefficients D. In turn, the properties that the wavefunctions have as a consequence of the Mandelstam identities are translated into properties of the coefficients D,

$$D_{\mu_1\ldots\mu_n}=D_{(\mu_1\ldots\mu_n)_c}, \tag{11.16}$$

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$$D_{\mu_1\ldots\mu_n} = (-1)^n D_{\mu_n\ldots\mu_1}, \tag{11.17}$$

$$D_{\mu_1\ldots\mu_k\mu_{k+1}\ldots\mu_n} + (-1)^k D_{\mu_k\ldots\mu_1\mu_{k+1}\ldots\mu_n} =$$
$$\tfrac{1}{k} D_{(\mu_1\ldots\mu_k)_c\,\mu_{k+1}\ldots\mu_n} + (-1)^k \tfrac{1}{k} D_{(\mu_k\ldots\mu_1)_c\,\mu_{k+1}\ldots\mu_n} \;\; \forall k, \tag{11.18}$$

where c indicates the cyclic combination of indices,

$$D_{(\mu_1\ldots\mu_k)_c} = D_{(\mu_1\,\mu_2\ldots\mu_k)} + D_{(\mu_2\,\mu_3\ldots\mu_k\,\mu_1)} + \ldots D_{(\mu_k\,\mu_1\ldots\mu_{k-1})}. \tag{11.19}$$

The linearity is a remarkable property of the wavefunctions in the extended representation. Notice that all the wavefunctions explicitly known in the loop representation for quantum gravity have this property when they are written in terms of the multitangent fields. Moreover, this property will also be inherited by the operators that we can construct in the extended representation. In general, the linearity of the wavefunctions could be imposed by means of the "linearity constraint" $\mathcal{L}$

$$\mathcal{L}(\mathbf{X}')\,\Psi(\mathbf{X}) \equiv X'^{\underset{\sim}{\mu}\underset{\sim}{\nu}} \frac{\delta^2}{\delta X^{\underset{\sim}{\mu}} \delta X^{\underset{\sim}{\nu}}}\,\Psi(\mathbf{X}) = 0, \tag{11.20}$$

where $\mathbf{X}'$ is any object that satisfies the differential constraints. The functional derivatives produce elements of the extended group of loops and therefore the second functional derivative is the group product of the resulting elements. The addition of the element $\mathbf{X}'$ is to ensure that the result is a function of multitensors that satisfy the differential constraint (i.e., it makes the linearity constraint a well defined operator on the space of wavefunctions with support on $\mathcal{D}_o$).

Any observable of the theory has to commute with the linearity constraint. This means that the action of any quantum observable on a wavefunction reduces to a shift in the argument of the wavefunction. The linearity in the wavefunctions is in correspondence with the proliferation of arguments. One trades the non-linearity of the wavefunctions in terms of a connection for an increased number of arguments in the extended representation. This is a technique that is applied in constructive quantum field theories for non-linear theories, where non-linearities are traded for an increase in the number of variables.

An example that clarifies these issues of linearity and proliferation of variables is given by the usual Fourier representation of the quantum mechanics of a free particle in one dimension. The usual theory has wavefunctions in the position representation $\Psi(x)$ and momentum representation $\Psi(k)$ related by the usual Fourier transform. The idea of extended representation is to substitute the basis of the Fourier transform by an infinite parameter basis,

$$\exp(ikx) \rightarrow k_0 + k_1 x + k_2 x^2 + k_3 x^3 + \cdots \tag{11.21}$$

and the resulting wavefunctions in the "extended" representation are

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given by linear functions of an infinite tower of ks, $\Psi(\vec{k})$. The linearity is imposed by a linearity constraint $\partial^2/\partial k_i \partial k_j \Psi(\vec{k}) = 0$

One can write the physical operators of the theory in terms of such a representation, and they all become linear operators,

$$\hat{x} = \sum_{n=0}^{\infty} k_{n-1} \frac{\partial}{\partial k_n}, \tag{11.22}$$

$$\hat{p} = \sum_{n=0}^{\infty} (n+1) k_{n+1} \frac{\partial}{\partial k_n}, \tag{11.23}$$

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} = \frac{1}{2m} \sum_{n=0}^{\infty} (n+1)(n+2) k_{n+2} \frac{\partial}{\partial k_n}, \tag{11.24}$$

which commute with the linearity constraint.

How is the usual theory recovered? Since one has first class constraints (the linearity constraints), one can fix the gauge generated by them. In particular one can choose $k_n = k_1^n/n!$ and one recovers the usual theory free of constraints. If one decides to quantize the theory before fixing the gauge, the usual theory is recovered by considering analytic functions of the tower of ks and introducing an inner product that implements a gauge fixing similar to the one discussed.

At the moment this seems like a futile exercise: we have converted the simplest quantum mechanical problem into a field theory with an infinite number of variables and constraints. It is true that for the example of a free particle nothing is gained in solving the theory in this way. In the case of gauge theories, however, one knows that fixing the gauge is not necessarily the easiest way of solving a theory. The attractiveness of having a theory cast in terms of linear functions and first order differential operators may well compensate for the proliferation of variables (a less obvious problem in a theory that from the outset has an infinite number of degrees of freedom).

An intriguing point is that the resulting quantum theory with linear wavefunctions and first order operators could, in principle, be obtained as the canonical quantization of a classical theory with constraints and operators linear in momenta. The classical theory involved has an infinite number of degrees of freedom and the linearity implies the use of Grassmann variables in its formulation. These classical theories have not been studied in detail at present.

11.3 The constraints

We now proceed to write the constraints of quantum gravity in terms of the extended representation. We will proceed formally via the loop transform exactly as we did in chapter 8. We could also proceed via the construction of a non-canonical algebra which is the natural generalization of the T algebra to the extended case. As we argued at length, the results one gets are equivalent to those of a loop transform and involve a similar number of formal manipulations. We will therefore concentrate on the loop transform approach.

11.3.1 The diffeomorphism constraint

We start with the diffeomorphism constraint. The action of this constraint on the wavefunctions $\Psi(\mathbf{R})$ is defined by

$$\hat{\mathcal{C}}_{ax}\Psi(\mathbf{R}) = \int DA\, W_{\mathbf{R}}[A]\,[\hat{\mathcal{C}}_{ax}\Psi(A)\,]. \tag{11.25}$$

The constraint acting on $\Psi(A)$ can be applied on the generalized Wilson functional integrating (formally) by parts. As a result we get

$$\hat{\mathcal{C}}_{ax}\Psi(\mathbf{R}) = \int DA\, \Psi[A]\,[F^i_{ab}(x)\frac{\delta}{\delta A^i_{bx}} W_{\mathbf{R}}[A]]. \tag{11.26}$$

At this point it is useful to introduce some notation that will prove beneficial in the calculations. Let $\delta^{\underset{\sim}{\alpha}}_{\underset{\sim}{\beta}}$ be defined as

$$\delta^{\underset{\sim}{\alpha}}_{\underset{\sim}{\beta}} = \begin{cases} \delta^{\alpha_1}_{\beta_1}\cdots\delta^{\alpha_n}_{\beta_n}\,, & \text{if } n(\underset{\sim}{\alpha}) = n(\underset{\sim}{\beta}) = n \geq 1 \\ 1\,, & \text{if } n(\underset{\sim}{\alpha}) = n(\underset{\sim}{\beta}) = 0 \\ 0\,, & \text{otherwise,} \end{cases} \tag{11.27}$$

where $n(\underset{\sim}{\alpha})$ is the number of indices of the set $\underset{\sim}{\alpha}$. The δ matrix allows us to write the group product defined in chapter 2 as

$$(\mathbf{E}_1 \times \mathbf{E}_2)^{\underset{\sim}{\rho}} = \delta^{\underset{\sim}{\rho}}_{\underset{\sim}{\nu}\underset{\sim}{\beta}}\, E_1^{\underset{\sim}{\nu}} E_2^{\underset{\sim}{\beta}}. \tag{11.28}$$

Notice that in particular

$$(\delta_{\underset{\sim}{\nu}} \times \delta_{\underset{\sim}{\beta}})^{\underset{\sim}{\rho}} = \delta^{\underset{\sim}{\rho}}_{\underset{\sim}{\nu}\underset{\sim}{\beta}}, \tag{11.29}$$

where $\delta_{\underset{\sim}{\alpha}}$ are the "vectors" with components $(\delta_{\underset{\sim}{\alpha}})^{\underset{\sim}{\mu}} = \delta^{\underset{\sim}{\mu}}_{\underset{\sim}{\alpha}}$.

The functional derivative of any product of As can be written with the help of the δ matrix as

$$\frac{\delta}{\delta A^i_{bx}}(A_{\underset{\sim}{\alpha}}) = A_{\underset{\sim}{\mu}}\tau^i A_{\underset{\sim}{\nu}}\, \delta^{\underset{\sim}{\mu}\, bx\, \underset{\sim}{\nu}}_{\underset{\sim}{\alpha}}, \tag{11.30}$$

where the τs are the generators of the SU(2) algebra with the conventions of chapter 8. Taking the trace of the above expression we get

$$\frac{\delta}{\delta A^i_{bx}}\mathrm{Tr}(A_{\underset{\sim}{\alpha}}) = \mathrm{Tr}(\tau^i A_{\underset{\sim}{\beta}})\, \delta^{\underset{\sim}{\beta}}_{\underset{\sim}{\nu\mu}}\, \delta_{\underset{\sim}{\alpha}}^{\underset{\sim}{\mu}\, bx\, \underset{\sim}{\nu}} = \mathrm{Tr}(\tau^i A_{\underset{\sim}{\beta}})\, \delta_{\underset{\sim}{\alpha}}^{(bx\, \underset{\sim}{\beta})_c}. \tag{11.31}$$

The curvature tensor involved in the definition of the constraint can be written as

$$F_{ab}(x) = A_{\underset{\sim}{\nu}} \mathcal{F}_{ab}{}^{\underset{\sim}{\nu}}(x), \tag{11.32}$$

where $\mathcal{F}_{ab}$ represents the element of the algebra of the extended loop group with non-vanishing components,

$$\mathcal{F}_{ab}{}^{a_1 x_1}(x) = \delta^{a_1}_{a}{}^{d}_{b}\, \partial_d\, \delta(x_1 - x), \tag{11.33}$$

$$\mathcal{F}_{ab}{}^{a_1 x_1, a_2 x_2}(x) = \delta^{a_1}_{a}{}^{a_2}_{b}\, \delta(x_1 - x)\, \delta(x_2 - x). \tag{11.34}$$

Using (11.31) and (11.32) we obtain the following expression for the action of the diffeomorphism constraint on the generalized Wilson functional:

$$\begin{aligned} F^i_{ab}(x)\frac{\delta}{\delta A^i_{bx}}\mathrm{Tr}(A_{\underset{\sim}{\alpha}})\, R^{\underset{\sim}{\alpha}} &= \mathrm{Tr}(F_{ab}(x)\, A_{\underset{\sim}{\beta}})\, \delta_{\underset{\sim}{\alpha}}^{(ax\, \underset{\sim}{\beta})_c}\, R^{\underset{\sim}{\alpha}} \\ &= \mathrm{Tr}(A_{\underset{\sim}{\rho}})\, \delta^{\underset{\sim}{\rho}}_{\underset{\sim}{\nu\beta}}\, \mathcal{F}_{ab}(x)^{\underset{\sim}{\nu}} R^{(bx\, \underset{\sim}{\beta})_c}. \end{aligned} \tag{11.35}$$

Putting expression (11.35) in the expression of the differential constraint and using (11.28) we obtain

$$\begin{aligned} \mathcal{C}_{ax}\Psi(\mathbf{R}) &= \int DA\,\Psi[A]\,\mathrm{Tr}(A_{\underset{\sim}{\rho}}) \left[\mathcal{F}_{ab}(x) \times \mathbf{R}^{(bx)}\right]^{\underset{\sim}{\rho}} \\ &= \Psi(\mathcal{F}_{ab}(x) \times \mathbf{R}^{(bx)}), \end{aligned} \tag{11.36}$$

where we have introduced the element of the group $\mathbf{R}^{(bx)}$ which has components defined by

$$[\mathbf{R}^{(bx)}]^{\underset{\sim}{\mu}} = R^{(bx)\underset{\sim}{\mu}} \equiv R^{(bx\, \underset{\sim}{\mu})_c} \tag{11.37}$$

and satisfies the differential constraint (on the $\underset{\sim}{\mu}$ indices) basepointed at x.

We therefore see that the action of the diffeomorphism constraint reduces to a shift in the argument of the wavefunction, as we suggested, due to the linearity of the operator. The operator can, of course, be written as a first order differential operator,

$$\mathcal{C}_{ax}\Psi(\mathbf{R}) = [\mathcal{F}_{ab}(x) \times \mathbf{R}^{(bx)}]^{\underset{\sim}{\mu}} \frac{\delta}{\delta \mathbf{R}^{\underset{\sim}{\mu}}}\Psi(\mathbf{R}). \tag{11.38}$$

11.3.2 The Hamiltonian constraint

Let us now consider the construction of the Hamiltonian constraint in the extended representation.

In this case we have to use the properties of the $SU(2)$ algebra in order to take into account the two derivatives that appear in $\hat{\mathcal{H}}(x)$. We have now

$$\hat{\mathcal{H}}(x)\,\Psi(\mathbf{R}) = \int DA\,\Psi[A]\,\epsilon^{ijk}\left[F^i_{ba}(x)\frac{\delta}{\delta A^j_{bx}}\frac{\delta}{\delta A^k_{ax}}W_A(\mathbf{R})\right]. \qquad (11.39)$$

From (11.31) we get the following expression for the second functional derivative

$$\frac{\delta}{\delta A^j_{bx}}\frac{\delta}{\delta A^k_{ax}}\mathrm{Tr}(A_{\underset{\sim}{\alpha}}) = \mathrm{Tr}(\tau^k\frac{\delta}{\delta A^j_{bx}}A_{\underset{\sim}{\beta}})\,\delta_{\underset{\sim}{\alpha}}^{(ax\,\underset{\sim}{\beta})_c} =$$
$$\mathrm{Tr}(\tau^k A_{\underset{\sim}{\mu}}\tau^j A_{\underset{\sim}{\nu}})\,\delta_{\underset{\sim}{\beta}}^{\underset{\sim}{\mu}\,bx\,\underset{\sim}{\nu}}\,\delta_{\underset{\sim}{\alpha}}^{(ax\,\underset{\sim}{\beta})_c} = \mathrm{Tr}(\tau^k A_{\underset{\sim}{\mu}}\tau^j A_{\underset{\sim}{\nu}})\,\delta_{\underset{\sim}{\alpha}}^{(ax\,\underset{\sim}{\mu}\,bx\,\underset{\sim}{\nu})_c}. \qquad (11.40)$$

To put this result in a useful form we need the following well known property of the $SU(2)$ matrices

$$\epsilon^{ijk}\mathrm{Tr}(\tau^k A_{\underset{\sim}{\mu}}\tau^j A_{\underset{\sim}{\nu}}) = \mathrm{Tr}(\tau^i A_{\underset{\sim}{\nu}})\,\mathrm{Tr}(A_{\underset{\sim}{\mu}}) - \mathrm{Tr}(A_{\underset{\sim}{\nu}})\,\mathrm{Tr}(\tau^i A_{\underset{\sim}{\mu}}), \qquad (11.41)$$

which allows us to write the product between traces of $SU(2)$ matrices as a combination of traces in the following way:

$$\mathrm{Tr}(A_{\underset{\sim}{\mu}})\,\mathrm{Tr}(A_{\underset{\sim}{\nu}}) = \mathrm{Tr}(A_{\underset{\sim}{\mu}}A_{\underset{\sim}{\nu}}) + (-1)^{n(\underset{\sim}{\nu})}\mathrm{Tr}(A_{\underset{\sim}{\mu}}A_{\underset{\sim}{\nu}^{-1}}), \qquad (11.42)$$

where if $\underset{\sim}{\nu} = (\nu_1,\ldots,\nu_n)$, then $\underset{\sim}{\nu}^{-1} = (\nu_n,\ldots,\nu_1)$. This allows us to rearrange the expression of interest as

$$\epsilon^{ijk}\mathrm{Tr}(\tau^k A_{\underset{\sim}{\mu}}\tau^j A_{\underset{\sim}{\nu}}) = (-1)^{n(\underset{\sim}{\mu})}\mathrm{Tr}(\tau^i A_{\underset{\sim}{\nu}}A_{\underset{\sim}{\mu}^{-1}}) - (-1)^{n(\underset{\sim}{\nu})}\mathrm{Tr}(\tau^i A_{\underset{\sim}{\nu}^{-1}}A_{\underset{\sim}{\mu}}). \qquad (11.43)$$

We then have for the action of the constraint on the product of connections,

$$\epsilon^{ijk}F^i_{ba}(x)\frac{\delta}{\delta A^j_{bx}}\frac{\delta}{\delta A^k_{ax}}\,\mathrm{Tr}(A_{\underset{\sim}{\alpha}}) =$$
$$(-1)^{n(\underset{\sim}{\mu})}\,\mathrm{Tr}(F_{ba}(x)A_{\underset{\sim}{\nu}\underset{\sim}{\mu}})\left\{\delta_{\underset{\sim}{\alpha}}^{(ax\,\underset{\sim}{\mu}^{-1}\,bx\,\underset{\sim}{\nu})_c} - (-1)^{n(\underset{\sim}{\mu}+\underset{\sim}{\nu})}\,\delta_{\underset{\sim}{\alpha}}^{(ax\,\underset{\sim}{\mu}\,bx\,\underset{\sim}{\nu}^{-1})_c}\right\} =$$
$$(-1)^{n(\underset{\sim}{\mu})}\,\mathrm{Tr}(F_{ba}(x)A_{\underset{\sim}{\nu}\underset{\sim}{\mu}})\left\{\delta_{\underset{\sim}{\alpha}}^{(bx\,\underset{\sim}{\nu}\,ax\,\underset{\sim}{\mu}^{-1})_c} + (-1)^{n(\underset{\sim}{\mu}+\underset{\sim}{\nu})}\,\delta_{\underset{\sim}{\alpha}}^{(\underset{\sim}{\mu}\,ax\,\underset{\sim}{\nu}^{-1}\,bx)_c}\right\} =$$
$$(-1)^{n(\underset{\sim}{\mu})}\,\mathrm{Tr}(A_{\underset{\sim}{\beta}\underset{\sim}{\nu}\underset{\sim}{\mu}})\,\mathcal{F}_{ab}{}^{\underset{\sim}{\beta}}(x)\,\delta_{\underset{\sim}{\chi}}^{(ax\,\underset{\sim}{\nu}\,bx\,\underset{\sim}{\mu}^{-1})_c}\left\{\delta_{\underset{\sim}{\alpha}}^{\underset{\sim}{\chi}} + (-1)^{n(\underset{\sim}{\chi})}\,\delta_{\underset{\sim}{\alpha}}^{\underset{\sim}{\chi}^{-1}}\right\}, \qquad (11.44)$$

where the combination that arises in curly braces gives rise exactly to the element $\mathbf{R}$ that we introduced before when contracted with $\mathbf{X}$. This

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contraction is exactly what we need to do to get the expression of the action of the constraint on an extended holonomy,

$$\begin{aligned}\epsilon^{ijk}F^i_{ba}(x)\frac{\delta}{\delta A^j_{bx}}\frac{\delta}{\delta A^k_{ax}}\,W_A(\mathbf{R}) =&\\ &2\,(-1)^{n(\underset{\sim}{\mu})}\,\mathrm{Tr}(A_{\underset{\sim}{\beta}\underset{\sim}{\nu}\underset{\sim}{\mu}})\,\mathcal{F}_{ab}{}^{\underset{\sim}{\beta}}{}_{\underset{\sim}{\rho}}(x)\,\delta_{\underset{\sim}{\chi}}^{(ax\,\underset{\sim}{\nu}\,bx\,\underset{\sim}{\mu}^{-1})_c}\,R^{\underset{\sim}{\chi}} =\\ &2\,(-1)^{n(\underset{\sim}{\mu})}\,\mathrm{Tr}(A_{\underset{\sim}{\alpha}})\,\delta^{\underset{\sim}{\alpha}}_{\underset{\sim}{\beta}\underset{\sim}{\rho}}\,\mathcal{F}_{ab}{}^{\underset{\sim}{\beta}}(x)\,\left[\delta^{\underset{\sim}{\rho}}_{\underset{\sim}{\nu}\underset{\sim}{\mu}}\,R^{(ax\,\underset{\sim}{\nu}\,bx\,\underset{\sim}{\mu}^{-1})_c}\right],\end{aligned} \tag{11.45}$$

where in the first step we have used the symmetry property (11.14) of the Rs under the inversion of the indexes. The expression in square brackets defines a specific combination of Rs, that we denote

$$[\mathbf{R}^{(ax,\,bx)}]^{\underset{\sim}{\rho}} = R^{(ax,\,bx)\underset{\sim}{\rho}} \equiv (\delta_{\underset{\sim}{\nu}} \times \delta_{\underset{\sim}{\mu}})^{\underset{\sim}{\rho}}(-1)^{n(\underset{\sim}{\mu})}\,R^{(ax\,\underset{\sim}{\nu}\,bx\,\underset{\sim}{\mu}^{-1})_c}. \tag{11.46}$$

Explicitly,

$$R^{(ax,\,bx)\rho_1\ldots\rho_n} = \sum_{k=0}^{n}(-1)^{n-k}\,R^{(\,ax\,\rho_1\ldots\rho_k\,bx\,\rho_n\ldots\rho_{k+1})_c}. \tag{11.47}$$

An important fact is that this combination satisfies the differential constraint with respect to the $\underset{\sim}{\rho}$ indices basepointed at x. It also satisfies the following property

$$R^{(ax,\,bx)\underset{\sim}{\rho}^{-1}} = (-1)^{n(\underset{\sim}{\rho})}\,R^{(bx,\,ax)\underset{\sim}{\rho}}. \tag{11.48}$$

Equation (11.45) can then be written

$$\begin{aligned}\epsilon^{ijk}F^i_{ba}(x)\frac{\delta}{\delta A^j_{bx}}\frac{\delta}{\delta A^k_{ax}}\,W_A(\mathbf{R}) &= 2\,\mathrm{Tr}(A_{\underset{\sim}{\alpha}})\,(\delta_{\underset{\sim}{\beta}} \times \delta_{\underset{\sim}{\rho}})^{\underset{\sim}{\alpha}}\,\mathcal{F}_{ab}{}^{\underset{\sim}{\beta}}(x)\,R^{(ax,\,bx)\underset{\sim}{\rho}}\\ &= 2\,\mathrm{Tr}(A_{\underset{\sim}{\alpha}})\,(\mathcal{F}_{ab} \times \mathbf{R}^{(ax,\,bx)})^{\underset{\sim}{\alpha}},\end{aligned} \tag{11.49}$$

and from this we conclude that

$$\hat{\mathcal{H}}(x)\,\Psi(\mathbf{R}) = 2\,\Psi(\mathcal{F}_{ab}(x) \times \mathbf{R}^{(ax,\,bx)}). \tag{11.50}$$

Also in this case the action of the Hamiltonian constraint reduces to evaluating the wavefunction on a new argument. As was already mentioned, this is a general property of the operators in the extended representation due to the linearity of the wavefunctions. In fact, the last expression can be written as the action of a single functional derivative with respect to the $\mathbf{R}$ variables

$$\hat{\mathcal{H}}(x)\,\Psi(\mathbf{R}) = 2\,\left[\mathcal{F}_{ab}(x) \times \mathbf{R}^{(ax,\,bx)}\right]^{\underset{\sim}{\mu}}\,\frac{\delta}{\delta R^{\underset{\sim}{\mu}}}\,\Psi(\mathbf{R}). \tag{11.51}$$

Notice that in order for this expression to be well defined on the space of wavefunctions considered it is necessary that the term contracted with

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the functional derivative satisfies the differential constraint, as is the case here.

The new element of the extended group of loops on which the wavefunction is evaluated involves a combination of multitensor fields with two indices fixed at the point where the Hamiltonian is acting and the other indices having a specific alternating order. We will show in the next section that this alternating order of the indexes is related to the reroutings of a loop when the above expression is particularized to loops. The appearance of a rerouting is typical of the loop representation and plays a crucial role in the quantum gravity case.

The presence of a multitensor with two indices evaluated at the same point in the Hamiltonian constraint implies that the resulting expression for the operator is divergent. This is due to the distributional character of the multitensors. A multitensor satisfying the differential constraint (2.11) diverges when two successive indices are evaluated at the same spatial point. This divergence of the formal expression of the constraint will have to be regularized, and we will return to it in detail in section 11.6.2.

11.4 Loops as a particular case

As we discussed in chapter 2, the extended group of loops includes the group of loops as a particular case. We should therefore be able to particularize the extended representation to the loop representation by substituting $\mathbf{R} \to \mathbf{R}(\gamma)$. We analyze here in detail the case of the Hamiltonian constraint.

In order to evaluate $\mathbf{R}^{(ax,\,bx)}(\gamma_o)$ we have to use the explicit expression of this object in terms of the multitangents fields. We have

$$\begin{aligned} R^{(ax,\,bx)\mu_1\ldots\mu_n} = {} & \tfrac{1}{2}\sum_{k=0}^{n}\sum_{l=0}^{k}(-1)^{n-k}\big[X^{\mu_{l+1}\ldots\mu_k\, bx\, \mu_n\ldots\mu_{k+1}\, ax\, \mu_1\ldots\mu_l} \\ & +(-1)^n X^{\mu_l\ldots\mu_1\, ax\, \mu_{k+1}\ldots\mu_n\, bx\, \mu_k\ldots\mu_{l+1}}\big] \\ & +\sum_{k=0}^{n}\sum_{l=k}^{n}(-1)^{n-k}\big[X^{\mu_l\ldots\mu_{k+1}\, ax\, \mu_1\ldots\mu_k\, bx\, \mu_n\ldots\mu_{l+1}} \\ & +(-1)^n X^{\mu_{l+1}\ldots\mu_n\, bx\, \mu_k\ldots\mu_1\, ax\, \mu_{k+1}\ldots\mu_l}\big]. \end{aligned} \tag{11.52}$$

One can write the above expression in a more compact and useful form introducing the following combinations of Xs,

$$X^{(ax,\,\overrightarrow{bx}\,)\underset{\sim}{\mu}} \equiv \sum_{k=0}^{n}(-1)^{n-k}X^{(\,ax\,\mu_1\ldots\mu_k\, bx\,\mu_n\ldots\mu_{k+1})_c}, \tag{11.53}$$

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and

$$\overleftarrow{X}^{(ax,bx)\underset{\sim}{\mu}} \equiv \sum_{k=0}^{n}(-1)^k X^{(\,ax\,\mu_k \cdots \mu_1\, bx\, \mu_{k+1}\cdots \mu_n)_c}. \tag{11.54}$$

These objects have definite symmetry properties under the inversion of the indices, which we will use later. Basically, the inversion of the order of the indices flips the direction of the arrow and multiplies the object by $(-1)^{n(\underset{\sim}{\mu})}$,

$$(-1)^{n(\underset{\sim}{\mu})} \overleftarrow{X}^{(ax,bx)\underset{\sim}{\mu}^{-1}}(\gamma_o) = \overrightarrow{X}^{(ax,bx)\underset{\sim}{\mu}}(\gamma_o). \tag{11.55}$$

In terms of these combinations, $\mathbf{R}^{(ax,bx)}$ simply reads

$$R^{(ax,bx)\underset{\sim}{\mu}} = \tfrac{1}{2}[\overrightarrow{X}^{(ax,bx)\underset{\sim}{\mu}} + (-1)^{n(\underset{\sim}{\mu})} \overleftarrow{X}^{(ax,bx)\underset{\sim}{\mu}^{-1}}]. \tag{11.56}$$

As we discussed extensively in chapter 8, the Hamiltonian constraint in the loop representation has only a non-trivial action on intersecting loops. We suppose then that at the point x the loop γ intersects itself p times; i.e., γ has "multiplicity" p at x. We start with some suitable notation to take this fact into account.

If the loop γ has multiplicity p at x one can write it in the following way

$$\gamma_{xx} = \gamma_{xx}^{(1)} \circ \gamma_{xx}^{(2)} \circ \cdots \circ \gamma_{xx}^{(p)}. \tag{11.57}$$

We denote by $[\gamma_{xx}]_i^{i+j}$ the following composition of loops basepointed at x

$$[\gamma_{xx}]_i^{i+j} = \gamma_{xx}^{(i)} \circ \cdots \circ \gamma_{xx}^{(i+j)}. \tag{11.58}$$

Let us suppose that the loop named $\gamma_{xx}^{(1)}$ contains the origin o of the loops. Then

$$\gamma_o = \gamma^{(1)}{}_o^x \circ [\gamma_{xx}]_2^p \circ \gamma^{(1)}{}_x^o. \tag{11.59}$$

Here, $\gamma^{(1)}{}_o^x$ represents the portion of $\gamma^{(1)}$ from the origin o to the point x. The loop γ_o is completely described by the multitangent fields $X^{\underset{\sim}{\mu}}(\gamma_o)$ of all ranks. As we know, these fields satisfy both algebraic (2.10) and differential (2.11) constraints. Moreover, these objects have another property derived from the fact that one can write a loop as a composition of open paths. This reads

$$X^{\mu_1\cdots\mu_n}(\gamma_o) = \int_{\gamma_o} dz^{a_i}\delta(x_i - z) X^{\mu_1\cdots\mu_{i-1}}(\gamma_o^z)\, X^{\mu_{i+1}\cdots\mu_n}(\gamma_z^o), \tag{11.60}$$

which can be derived simply from the properties of the ordered integrals that appear in the definition of the multitangent.

Suppose now that the index μ_i is fixed at the point x. Then

$$X^{\mu_1\cdots\mu_i\, ax\, \mu_{i+1}\cdots\mu_n}(\gamma_o) =$$
$$\sum_{m=1}^{p} X^{\mu_1\cdots\mu_i}(\gamma^{(1)}{}_o^x \circ [\gamma_{xx}]_2^m)\, X_m^{ax}(\gamma)\, X^{\mu_{i+1}\cdots\mu_n}([\gamma_{xx}]_{m+1}^p \circ \gamma^{(1)}{}_x^o), \tag{11.61}$$

where $X_m^{ax}(\gamma)$ is the tangent at x when the loop goes through that point on the mth occasion. The following convention is assumed: $[\gamma_{xx}]_{m+1}^m \approx \iota_{xx}$, with ι_{xx} the null path. The above expression can easily be generalized to the case of any number of indices fixed at x. The above two expressions are exactly the same, except that in the second one we have written explicitly the case in which the point x is at an intersection, partitioning the integral of the first formula in a summation on the different petals of the loop with the intersection at x.

We are now ready to compute $\mathbf{R}^{(ax,\,bx)}(\gamma_o)$. We have

$$X^{\overrightarrow{(ax,\,bx)}\underset{\sim}{\mu}}(\gamma_o) =$$
$$\sum_{m=1}^{p-1}\sum_{q=m+1}^{p} [X_m^{bx}(\gamma)\, X_q^{ax}(\gamma)\, X^{\underset{\sim}{\mu}}([\gamma_{xx}]_1^m \circ \overline{[\gamma_{xx}]}_{m+1}^q \circ [\gamma_{xx}]_{q+1}^p)$$
$$+(-1)^{n(\underset{\sim}{\mu})} X_m^{ax}(\gamma)\, X_q^{bx}(\gamma)\, X^{\underset{\sim}{\mu}^{-1}}([\gamma_{xx}]_1^m \circ \overline{[\gamma_{xx}]}_{m+1}^q \circ [\gamma_{xx}]_{q+1}^p)], \tag{11.62}$$

where $\overline{[\gamma_{xx}]}_{m+1}^q = \overline{\gamma}_{xx}^{(q)} \circ \cdots \circ \overline{\gamma}_{xx}^{(m+1)}$ and $\overline{\gamma}$ denotes the loop γ with opposite orientation. The inversion of the orientation of the loop (rerouting) in (11.62) comes from the property (11.9) of the multitangent fields. We then use the properties of the arrowed objects under inversion of the indices (11.55) and obtain for the action of the Hamiltonian,

$$\hat{\mathcal{H}}(x)\Psi(\gamma_o) = 2\Psi[\mathcal{F}_{ab}(x) \times \mathbf{R}^{(ax,\,bx)}(\gamma_o)] =$$
$$2\int DA\, \Psi(A)\, \mathrm{Tr}(A_{\underset{\sim}{\alpha\mu}})\, \mathcal{F}_{ab}(x)^{\underset{\sim}{\alpha}} X^{\overrightarrow{(ax,\,bx)}\underset{\sim}{\mu}}(\gamma_o) =$$
$$4\sum_{m=1}^{p-1}\sum_{q=m+1}^{p} X_m^{[bx,}(\gamma) X_q^{ax]}(\gamma)\int DA\Psi(A)$$
$$\times \mathrm{Tr}\Big[F_{ab}(x) H_A\Big\{\mathbf{R}([\gamma_{xx}]_1^m \circ \overline{[\gamma_{xx}]}_{m+1}^q \circ [\gamma_{xx}]_{q+1}^p)\Big\}\Big], \tag{11.63}$$

where we have arranged the product of connections contracted with the multitangents as the holonomy, and its contraction with $\mathcal{F}_{ab}$ as the field tensor F_{ab} using formulae we introduced at the beginning of this chapter.

We can now recover the loop derivative through the usual expression,

$$\mathrm{Tr}(F_{ab}(x)H_A\{\mathbf{R}(\gamma_{xx})\}) = \Delta_{ab}(x)\mathrm{Tr}(H_A\{\mathbf{R}(\gamma_{xx})\}). \tag{11.64}$$

The final result is

$$\hat{\mathcal{H}}(x)\Psi(\gamma_o) = 4\sum_{m=1}^{p-1}\sum_{q=m+1}^{p} X_m^{[bx,}(\gamma)X_q^{ax]}(\gamma) \times \Delta_{ab}(x)\,\Psi\left([\gamma_{xx}]_1^m \circ \overline{[\gamma_{xx}]}_{m+1}^q \circ [\gamma_{xx}]_{q+1}^p\right). \tag{11.65}$$

This expression corresponds to the usual Hamiltonian constraint of quantum gravity in the loop representation introduced in chapter 8. For the diffeomorphism constraint we obtain a similar result. Equation (11.36) reduces to the usual expression of the diffeomorphism constraint in the loop representation when one particularizes this constraint to the case of loops.

It is important to stress the relationship between the solutions of the constraints in both representations. Since loops are a particular case of multitensors, any solution found in the extended representation can be particularized to loops and would yield in the limit a solution to the usual constraints of quantum gravity in the loop representation. The converse is not necessarily true. Given a solution in the loop representation, it may not generalize to a solution in the extended representation. An example is the solutions to the Hamiltonian based on smooth non-intersecting loops, which find no analogue in the extended representation.

The process by which one obtains a solution in the loop representation from a solution in the extended representation may be ill defined. In that sense, one can always obtain a solution in terms of loops from the extended representation only at a formal level. In particular we will see that the solutions we find in the next section are only well defined in the extended space if one excises from it certain multitensors, including those which correspond to loops. Therefore such solutions do not have a rigorous meaning in terms of loops, only a formal one, which corresponds to the level of discussion of the solutions that we have maintained up to now.

The fact that the solutions we will present do not include loops as a particular case does not preclude obtaining them through a suitable limiting process. These limiting processes may include additional structures —such as framings— and the end result may be a formulation in terms of some generalization of the idea of loops.

A simple example of the situation is given by the exponential of the self-linking number. Its extended form is $\exp(g_{ax\,by}X^{ax}X^{by})$. If the Xs are smooth, this is a well defined quantity in spite of the fact that $g_{ax\,by}$ is distributional. However, if one considers the Xs that correspond to a

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loop it is not, as we have discussed, and an ambiguity appears. Therefore if one wants to have the self-linking number as a well defined function in the extended loop space one has to restrict it to smooth first order multitensors, which exclude those of loops. If one defines a limiting process in which the multitensors of (framed) loops arise as a limit of smooth multitensors, the self-linking number is well defined.

11.5 Solutions of the constraints

As we have seen, the expressions for the constraints in the extended representation are very compact: they amount to the evaluation of the wavefunctions in a shifted argument. The compactness of these expressions allows us to compute in a very efficient way their action on specific states. In particular it allows us to compute very efficiently the action of the Hamiltonian constraint on the second coefficient of the Jones polynomial, which we claimed without presenting an explicit proof in chapter 10 was annihilated by the constraint. The discussion in this section serves both as proof of that fact and as an illustration of the computational economy attained by the use of the extended representation. Even if the extended representation does not in the end have intrinsic value for representing quantum gravity it is a powerful computational framework for doing calculations in the loop representation. The computation presented here will be unregularized, we will discuss the regularization of it in section 11.6.2.

The expression for the coefficient $\mathcal{A}_2(\gamma)$ in terms of the multitangent fields is

$$\mathcal{A}_2(\gamma) = h_{\mu_1\mu_2\mu_3} X^{\mu_1\mu_2\mu_3}(\gamma) + g_{\mu_1\mu_3} g_{\mu_2\mu_4} X^{\mu_1\mu_2\mu_3\mu_4}(\gamma), \tag{11.66}$$

where

$$h_{\mu_1\mu_2\mu_3} = \epsilon^{\alpha_1\alpha_2\alpha_3}\, g_{\mu_1\alpha_1}\, g_{\mu_2\alpha_2}\, g_{\mu_3\alpha_3}, \tag{11.67}$$

with

$$\epsilon^{\alpha_1\alpha_2\alpha_3} = \epsilon^{c_1c_2c_3} \int d^3t\, \delta(z_1 - t)\, \delta(z_2 - t)\, \delta(z_3 - t). \tag{11.68}$$

The generalization of this knot invariant to extended loops is straightforward

$$\mathcal{A}_2(\gamma) = \mathcal{A}_2[\mathbf{X}(\gamma)] \to \mathcal{A}_2(\mathbf{X}) = \mathcal{A}_2(\mathbf{R}), \tag{11.69}$$

where $\mathbf{X}$ is now an element of the extended group $\mathcal{D}_o$. We now analyze the application of the Hamiltonian constraint to this state in the extended representation. By (11.50) we have

$$\mathcal{H}(x)\, \mathcal{A}_2(\mathbf{R}) = 2\, h_{\mu_1\mu_2\mu_3} \left[\mathcal{F}_{ab}{}^{\mu_1}(x)\, R^{(ax,\,bx)\mu_2\mu_3} + \mathcal{F}_{ab}{}^{\mu_1\mu_2}(x)\, R^{(ax,\,bx)\mu_3}\right]$$

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$$+2\, g_{\mu_1\mu_3} g_{\mu_2\mu_4} \left[\mathcal{F}_{ab}{}^{\mu_1}(x)\, R^{(ax,\,bx)\mu_2\mu_3\mu_4} + \mathcal{F}_{ab}{}^{\mu_1\mu_2}(x)\, R^{(ax,\,bx)\mu_3\mu_4}\right]. \quad (11.70)$$

The contraction of the element of the extended algebra $\mathcal{F}_{ab}$ with the propagators leads to integrations by parts similar to those we encountered in chapter 10 while analyzing the action of the Hamiltonian constraint on the Gauss linking number. Explicitly, we have

$$\mathcal{F}_{ab}{}^{\mu_1}(x)\, g_{\mu_1\mu_3} = -\epsilon_{aba_3}\delta(x - x_3) - \partial_{a_3} g_{ax\,bx_3}, \quad (11.71)$$

$$\mathcal{F}_{ab}{}^{\mu_1\mu_2}(x)\, g_{\mu_1\mu_3} g_{\mu_2\mu_4} = g_{\mu_3[ax\; g\; bx]\,\mu_4}, \quad (11.72)$$

$$\begin{aligned}\mathcal{F}_{ab}{}^{\mu_1}(x)\, h_{\mu_1\mu_2\mu_3} &= -g_{\mu_2[\,ax} g_{\,bx]\,\mu_3} + (g_{ax\,bx_2} - g_{ax\,bx_3}) g_{\mu_2\mu_3}, \\ &\quad + \tfrac{1}{2} g_{ax\,bz}\epsilon^{def}[g_{\mu_3\,dz}\partial_{a_2}\, g_{ex_2\,fz} - g_{\mu_2\,dz}\partial_{a_3}\, g_{ex_3\,fz}],\end{aligned} \quad (11.73)$$

$$\mathcal{F}_{ab}{}^{\mu_1\mu_2}(x)\, h_{\mu_1\mu_2\mu_3} = 2\, h_{ax\,bx\,\mu_3}. \quad (11.74)$$

In the last term of equation (11.73) an integral in z is assumed. The derivatives that appear in the above expressions can be integrated by parts and as a consequence act on the **R**s. Using the differential constraint we generate from them terms of lower multitensor rank. For example, from (11.71) we have

$$g_{\mu_2\mu_4}\partial_{a_3} g_{ax\,bx_3} R^{(ax,\,bx)\mu_2\mu_3\mu_4} = g_{\mu_2\mu_4}(g_{ax\,bx_2} - g_{ax\,bx_4}) R^{(ax,bx)\mu_2\mu_4}. \quad (11.75)$$

Performing these calculations, the following partial results are obtained for the four expressions quoted above

$$-\epsilon_{abc} g_{\mu_1\mu_2} R^{(ax,bx)\mu_1\,cx\,\mu_2} - (g_{ax\,bx_1} - g_{ax\,bx_2}) g_{\mu_1\mu_2} R^{(ax,\,bx)\mu_1\mu_2}, \quad (11.76)$$

$$g_{\mu_1[ax\; g\; bx]\,\mu_2} R^{(ax,\,bx)\mu_1\mu_2}, \quad (11.77)$$

$$\begin{aligned}&-g_{\mu_1[\,ax} g_{\,bx]\,\mu_2} R^{(ax,bx)\mu_1\mu_2} + (g_{ax\,bx_1} - g_{ax\,bx_2}) g_{\mu_1\mu_2} R^{(ax,\,bx)\mu_1\mu_2} \\ &-\epsilon^{def} g_{ax\,bz} g_{\mu_1\,dz} g_{ex\,fz} R^{(ax,\,bx)\mu_1},\end{aligned} \quad (11.78)$$

$$2\, h_{ax\,bx\,\mu_1} R^{(ax,\,bx)\mu_1}. \quad (11.79)$$

After some cancellations we finally obtain

$$\begin{aligned}\mathcal{H}(x)\,\mathcal{A}_2(\mathbf{R}) &= -2\,\epsilon_{abc} g_{\mu_1\mu_2} R^{(ax,bx)\mu_1\,cx\,\mu_2} \\ &\quad + 2[2h_{ax\,bx\,\mu_1} - \epsilon^{def} g_{ax\,bz} g_{\mu_1\,dz} g_{ex\,fz}] R^{(ax,\,bx)\mu_1}.\end{aligned} \quad (11.80)$$

One can check that the terms in the bracket are identical and of opposite signs, so the bracket vanishes. One can also see that the term of rank five vanishes. To see this, expand $R^{(ax,\,bx)\mu_1\,cx\,\mu_2}$ and as a result one gets,

$$R^{(ax,\,bx)\mu_1\,cx\,\mu_2} = -2\, R^{(ax\,bx\,\mu_1\,cx\,\mu_2)_c} + R^{(cx\,ax\,\mu_1\,bx\,\mu_2)_c} + R^{(bx\,cx\,\mu_1\,ax\,\mu_2)_c}. \quad (11.81)$$

which implies the contribution vanishes due to symmetry considerations when contracted with ϵ_{abc}.

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We therefore conclude that

$$\mathcal{H}(x)\,\mathcal{A}_2(\mathbf{R}) = 0. \tag{11.82}$$

We see that the explicit computation of this formal result in the extended representation involves only a few simple steps that basically amount to integrations by parts and application of the constraints satisfied by the multitensors. This should be compared with the lengthy computation in terms of loops outlined in reference [209].

An interesting point is that the computational efficiency that is obtained in the extended representation may be useful at the level of the diffeomorphism constraint. It is straightforward to show, for instance, that $\mathcal{A}_2$ is diffeomorphism invariant simply by checking that it is annihilated by the diffeomorphism constraint. This may find useful applications as a technique for searching for knot invariants.

11.6 Regularization

The extended representation provides a new scenario for analyzing the regularization problems in quantum gravity. In the loop representation regularization ambiguities appear at the level of both quantum operators *and* quantum states. Whereas the first problem is common to all the representations that one can construct for quantum gravity (and lies in the fact that the constraints involve the product of operators evaluated at the same point), the second is typical of the loop representation. In the case of quantum gravity the loop wavefunctions are knot invariants and their analytic expressions require the introduction of a regularization (framing). This difficulty does not only arise for the gravitational case. As we discussed in section 11.1 it is suggestive that even in the simple case of a free Maxwell field it is known that the quantum states in the loop representation are ill defined and a regularization is needed.

We will see that in the extended representation the problems in the definition of the wavefunctions can be solved. We are going to show that with an adequate restriction of the domain of dependence, the extended wavefunctions are well defined functionals. In the regularization of the constraints, we shall limit the analysis to the case of wavefunctions with a totally specified analytical dependence. More precisely, we shall study the action of the regularized Hamiltonian constraint over the wavefunctions that are formally annihilated by the constraint. The regularization of the constraint on the space of all wavefunctions has not yet been studied in detail.

11.6.1 The smoothness of the extended wavefunctions

Let us consider now the regularity properties of the extended wavefunctions. Generically the multitensors $X^{\underset{\sim}{\mu}}$ are distributional, as is directly inferred from the differential constraint (their derivative is a delta function). As we saw in chapter 2 any multitensor that satisfies the differential constraint can be written in the form $\mathbf{X} = \sigma[\phi] \cdot \mathbf{Y}$, where the $\mathbf{Y}$ fields satisfy the homogeneous differential constraint. For example, for the rank two component we have

$$X^{ax\,by} = Y^{ax\,by} + \phi^{\,ax}{}_{y}\, Y^{by} - \phi^{\,by}{}_{x}\, Y^{ax} - \phi^{\,ax}{}_{z}\phi^{\,by}{}_{z,c}\, Y^{cz} + \phi^{\,[by}{}_{o}\, Y^{ax]}. \tag{11.83}$$

As we discussed in chapter 2 the function ϕ fixes a prescription for the decomposition of the multitensors in transverse and longitudinal parts, $\mathbf{Y} = \delta_T \cdot \mathbf{X}$ with

$$\delta_T{}^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m} = \delta_{n,m}\ \delta_T{}^{\mu_1}{}_{\nu_1} \cdots \delta_T{}^{\mu_n}{}_{\nu_n}, \tag{11.84}$$

$$\delta_T{}^{ax}{}_{by} = \delta^{ax}{}_{by} - \phi^{\,ax}{}_{y,b}. \tag{11.85}$$

As the $\mathbf{Y}$s satisfy the homogeneous differential constraint, they can be *chosen* to be smooth functions. In that case, all the divergent behavior of the $\mathbf{X}$ is concentrated in the function ϕ. The σs control the divergent character of the group elements.

Let us define the following set of elements of the extended space: $\mathbf{X} \in \{\mathbf{X}\}_s$ if, and only if, there exists a prescription function ϕ such that $\delta_T[\phi] \cdot \mathbf{X} = \mathbf{Y}$ is a smooth function. We shall show that the wavefunctions defined on this domain are smooth in the extended variables and that this property is invariant under diffeomorphism transformations.

Given a diffeomorphism transformation $\mathbf{\Lambda}_D$ defined by $x'^a = D^a(x)$ it can be shown that $\delta_{DT} \equiv \mathbf{\Lambda}_{D^{-1}} \cdot \delta_T \cdot \mathbf{\Lambda}_D$ is a transverse projector in the prescription

$$\phi_D{}^{ax}{}_{y} = J(x)\ \frac{\partial x^a}{\partial D^b(x)} \phi^{bD(x)}{}_{D(y)}, \tag{11.86}$$

where $J(x)$ is the Jacobian of the coordinate transformation and ϕ is the function that fixes the prescription of the projector δ_T. In this prescription $\mathbf{X} = \sigma \cdot \mathbf{Y} = \mathbf{\Lambda}_{D^{-1}} \cdot \sigma_{D^{-1}} \cdot \mathbf{\Lambda}_D \cdot \mathbf{Y}$. For any diffeomorphism transformation $\mathbf{\Lambda}_D$, the transverse part of $\mathbf{\Lambda}_D \cdot \mathbf{X}$ is a smooth function with the prescription $\phi_{D^{-1}}$. In effect

$$\delta_{D^{-1}T} \cdot (\mathbf{\Lambda}_D \cdot \mathbf{X}) = \delta_{D^{-1}T} \cdot \sigma_{D^{-1}} \cdot \mathbf{\Lambda}_D \cdot \mathbf{Y} = \mathbf{\Lambda}_D \cdot \mathbf{Y}, \tag{11.87}$$

and we therefore see that there is a prescription $\phi_{D^{-1}}$ in which $\delta_T[\phi_{D^{-1}}]$ is a smooth function since $\mathbf{Y}$ is and its character is unchanged by the action of the diffeomorphism. The set $\{\mathbf{X}\}_s$ is then invariant under diffeomorphism transformations.

Let us now consider the specific wavefunctions we introduced in chapter 10. The extended loop transform of the exponential of the Chern–Simons form

$$\Psi_\Lambda(\mathbf{X}) = \int DA \, \exp(S_\Lambda[A]) \, \mathrm{Tr}(\mathbf{A} \cdot \mathbf{X}) = \sum_{n=0}^{\infty} (\mathbf{g}^{(n)} \cdot \mathbf{X}) \, \Lambda^n, \qquad (11.88)$$

where the dot indicates the contraction of indexes. We denote by $\mathbf{g}$ the products of propagators that arise in the perturbative expansion of the functional integral. As we have argued, they play the role of one of the diffeomorphism invariant metrics in the space of multitensors we were seeking in chapter 2. We recall that those metrics were, in general, objects that depended on the particular prescription one took for defining transverse and longitudinal parts.

Now, for any $\mathbf{X} \in \{\mathbf{X}\}_s$ we have

$$\Psi(\mathbf{X}) = \mathbf{g} \cdot \mathbf{X} = \mathbf{g} \cdot \sigma[\phi] \cdot \mathbf{Y} \equiv \mathbf{g}_\phi \cdot \mathbf{Y}, \qquad (11.89)$$

where one can see that $\mathbf{g}_\phi$ is a well defined distributional object that corresponds to the metric $\mathbf{g}$ in a particular prescription determined by ϕ.

This is a very important result. It implies that all the distributional character of the multitensors that is embodied in the σs is incorporated in the distributional nature of the $\mathbf{g}$s. Therefore if one chooses smooth $\mathbf{Y}$s, the wavefunctions are well defined. This fact is invariant under diffeomorphisms. One can always find a prescription in terms of which the wavefunction is written as $\mathbf{g} \cdot \mathbf{Y}$.

It is satisfying to check that by going to the extended representation and suitably restricting the domain of dependence of the wavefunctions one can remove the divergences in their definition. However, there is a price to be paid for this. As we argued before, ordinary loops are included in extended loops. The price we pay for limiting the domain of the extended wavefunctions in order to make them smooth is that we *exclude ordinary loops from the representation.* Ordinary loops *do not* correspond to smooth $\mathbf{Y}s$.

This is consistent with what we discussed before. Written purely in terms of ordinary loops the expressions for the knot invariants are divergent. Therefore they could never have arisen as a restriction of a smooth expression in terms of extended loops. The consistency goes beyond this fact. We saw that one could to a certain extent make sense of the knot invariants in terms of ordinary loops if one supplemented them with an additional structure: a framing. What this is suggesting is that in order to obtain the ordinary loop expressions from the expression of the knot invariants in terms of extended loops one has to go outside their domain of well behavedness. In order to obtain well behaved expressions, that limit should involve a choice of a prescription or regularization which translates itself in the notion of framed loops. The details of how to take this limit and derive a consistent framing from the extended representation have only been studied for particular cases and should be studied further.

11.6.2 The regularization of the constraints

As we discussed in section 11.3, the expressions for the constraints in the extended representation we have introduced are ill defined. They involve a multitensor with indices with a repeated spatial dependence. Due to the distributional character of multitensors imposed by the differential constraint (2.11) a repetition of a spatial dependence implies a divergence. Furthermore, the expression also involves an element of the algebra $\mathcal{F}_{ab}$ which may lead upon contraction to a distribution. Similar arguments apply to the diffeomorphism constraint.

To regularize the constraints we will proceed to point-split them. This is one of the simplest regularization methods one can consider. It may introduce difficulties due to its dependence on a background metric as we argued in chapter 8. It is straightforward to point-split the formal expressions for the constraints introduced in section 11.3. One takes expressions (11.36),(11.50) and point-splits the dependence on the variable x. The result is

$$\mathcal{C}^{\epsilon}_{ax}\,\Psi(\mathbf{R}) = \int d^3w \int d^3v\, f_\epsilon(w,x)\, f_\epsilon(v,x)\, \Psi(\mathcal{F}_{ab}(w) \times \mathbf{R}^{(bv)}), \quad (11.90)$$

$$\hat{\mathcal{H}}^{\epsilon}(x)\,\Psi(\mathbf{R}) =$$
$$2\int d^3w \int d^3u \int d^3v\, f_\epsilon(w,x)\, f_\epsilon(u,x)\, f_\epsilon(v,x)\, \Psi(\mathcal{F}_{ab}(w) \times \mathbf{R}^{(au,\,bv)}), \quad (11.91)$$

where f_ϵ is any appropriate symmetric smearing of the delta function. Notice that this point-splitting regularization is not uniquely determined by the formal factor ordered expression. Several sources of ambiguities arise, one of which is related to the background metric used in the smearing functions. It is also possible, but not mandatory, to preserve the gauge invariance in the regularization process. Gauge invariance is easily preserved in the extended representation by a procedure analogous to "closing the loops" in the usual representation. It has been checked that this procedure yields the same result as the non-invariant calculation we will perform here [210]. Finally additional factor ordering problems may arise due to the distributional character of the fundamental fields. We will see that distributional connections will appear naturally in the discussion.

We shall proceed as follows: we will introduce a naive point-splitting and study the action of the regularized and renormalized operators on

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the formal solutions. We will prove that there is a factor ordering that ensures consistency between the known results in the connection and the loop representation.

In section 11.5 we have shown that the invariance under diffeomorphisms of the coefficients of the expansion of the generalized transform (11.88) is ensured by construction. We also saw that with an appropriate definition of the domain of dependence, the wavefunctions can be endowed with convenient regularity properties (in particular, the smoothness dependence on the extended variables can be ensured in a diffeomorphism invariant way). All this can be explicitly confirmed by checking that the wavefunctions are annihilated by the regularized diffeomorphism constraints. Let us explicitly perform one of these calculations. This will also serve as a warm-up for the Hamiltonian case. Let us check the behavior of the regularized diffeomorphism constraint for the particular case of the extended Gauss linking number. From (11.90) we obtain

$$\mathcal{C}^{\epsilon}_{ax}\, a_1(\mathbf{R}) = \int d^3w \int d^3v\, f_\epsilon(w,x)\, f_\epsilon(v,x)\, g_{\mu_1\mu_2}\, \mathcal{F}_{ab}{}^{\mu_1}(w)\, R^{(bv)\mu_2}. \quad (11.92)$$

This result is valid for any prescription. Due to practical computational reasons we shall restrict the domain of the wavefunctions to those prescriptions connected by diffeomorphisms to the "transverse" prescription, given by

$$\phi_o{}^{ax}{}_y = \frac{1}{4\pi}\, \frac{\partial}{\partial x_a} \frac{1}{\mid x-y \mid}. \quad (11.93)$$

In the transverse prescription the free Chern–Simons propagator $g_{ax\,by}$ takes the form introduced in chapter 10. Then using (11.71) we get

$$\mathcal{C}^{\epsilon}_{ax}\, a_1(\mathbf{R}) = -\epsilon_{abc} \int d^3w \int d^3v\, f_\epsilon(w,x)\, f_\epsilon(v,x)\, R^{(bv)\,cw}, \quad (11.94)$$

where

$$R^{(bv)\,cw} = Y^{bv\,cw} + Y^{cw\,bv}, \quad (11.95)$$

is a smooth function symmetric under the interchange of the indices b and c (using the fact that the integration points are indistinguishable) contracted with an antisymmetric tensor. The last expression is well defined and we therefore have

$$\mathcal{C}^{\epsilon}_{ax}\, a_1(\mathbf{R}) = 0. \quad (11.96)$$

Notice that no divergences occur in (11.94) and we do not need to take the limit when ϵ goes to zero. The diffeomorphism constraint is perfectly well defined and no renormalization is needed. A similar result holds for $\mathcal{A}_2$ in the sense that no renormalization is needed, although the expression only vanishes when the regulator is removed. This situation is likely to be repeated for all other invariants constructed from Chern–Simons theory.

Let us analyze now the action of the regularized Hamiltonian constraint. This will allow us to put on a rigorous footing the formal results introduced in chapter 10 concerning the transform of the Chern–Simons state. We will not present a complete account here, but we will concentrate on the most elaborate calculation, the action of the Hamiltonian constraint on the second coefficient of the infinite expansion of the Jones polynomial, $\mathcal{A}_2(R)$. This result is of interest in itself since $\mathcal{A}_2(R)$ is the first non-trivial non-degenerate solution to the Wheeler–DeWitt equation with vanishing cosmological constant. We will end this section with some discussion of the rest of the calculation of the action of the Hamiltonian with cosmological constant on the extended Kauffman bracket.

The action of the regularized Hamiltonian constraint on the second coefficient $\mathcal{A}_2(R)$ is

$$\begin{aligned}\hat{\mathcal{H}}^{\epsilon}(x)\,\mathcal{A}_2(\mathbf{R}) = \int d^3w \int d^3u \int d^3v\, f_\epsilon(w,x)\, f_\epsilon(u,x)\, f_\epsilon(v,x) \\ \times\{-\,\epsilon_{abc} g_{\mu_1\mu_2} R^{(au,\,bv)\mu_1\;cw\,\mu_2} \\ +[2\,h_{aw\,bw\,\mu_1} - \epsilon^{def} g_{aw\,bz} g_{\mu_1\,dz} g_{eu\,fz}] R^{(au,\,bv)\mu_1} \\ +(g_{aw\,bu} - g_{aw\,bv})\, g_{\mu_1\mu_2} R^{(au\,\mu_1\;bv\,\mu_2)_c}\}. \qquad (11.97)\end{aligned}$$

If we now compare this with the unregulated result that we obtained in section 11.5, equation (11.80), we notice that there is an extra term, the last one in (11.97). We call this the "anomalous term". In the unregulated calculation, the variable $\mathbf{R}^{(au,\,bv)}$ appeared as $\mathbf{R}^{(ax,\,bx)}$ and satisfied the differential constraint based at the point x. In the regulated case, the variable $\mathbf{R}^{(au,\,bv)}$ satisfies a similar equation,

$$\begin{aligned}\partial_{\mu_i} R^{(au,\,bv)\mu_1\ldots\mu_i\ldots\mu_n} = [\delta(x_i - x_{i-1}) - \delta(x_i - x_{i+1})]\, R^{(au,\,bv)\mu_1\ldots\not{\mu}_i\ldots\mu_n} \\ +\,[\delta(x_i - u) - \delta(x_i - v)]\,(-1)^{n-i} R^{(au\,\mu_1\ldots\mu_{i-1}\;bv\,\mu_n\ldots\mu_{i+1})_c}, \qquad (11.98)\end{aligned}$$

instead of the usual differential constraint. In the above expression one should identify $x_0 = u$ and $x_{n+1} = v$.

To consider the limit of (11.97) when one removes the regulators, one needs to take into account the divergences that come from the group elements (through the matrix σ) and from the **g**s. The first observation is that both types of contributions are of the same order.

In order to see this we compare the first term in (11.97), which has divergences due to σ (the repeated indices in **R**) and the anomalous term which has divergences due to **g** which in the limit means both indices are evaluated at the same point.

We start with the rank five group elements $R^{(au,\,bv)\mu_1\;cw\,\mu_2}$. If one recalls

https://doi.org/10.1017/9781009290203.013 Published online by Cambridge University Press

the definition of σ from chapter 2 and expands $\mathbf{X} = \sigma \cdot \mathbf{Y}$ one finds a large number of terms. One can see that all these terms have a structure of divergences that is characterized by

$$\phi_o{}^{au}{}_v \, Y^{(bv\,\mu_1\,cw\,\mu_2)_c}, \tag{11.99}$$

with $Y^{(bv\,\mu_1\,cw\,\mu_2)_c}$ a regular function in the limit $\epsilon \to 0$. This expression gives the leading divergence of the rank five term in (11.97).

These leading divergences are exactly the same as those that arise from the anomalous term. In order to see this first notice that

$$\epsilon_{bca}\phi_o{}^{au}{}_v \, Y^{(bv\,\mu_1\,cw\,\mu_2)_c} = g_{bu\,cv} \, Y^{(bv\,\mu_1\,cw\,\mu_2)_c}, \tag{11.100}$$

whereas in the anomalous term one has a contribution $g_{bu\,cv} \, Y^{(bv\,\mu_1\,cw\,\mu_2)_c}$. This last expression apparently has a different divergence structure since it involves an $\mathbf{R}$ instead of a $\mathbf{Y}$ but it turns out that the contraction with $g_{\mu_1\,\mu_2}$ "erases" the extra divergences introduced by the $\mathbf{R}$ and the order is the same. Therefore in the limit $u \to v$ both the anomalous term and the first term of (11.97) only have divergences due to the presence of $g_{bu\,cv}$.

The result (11.100) ensures, due to the same symmetry properties used in the formal calculation, that the contribution of the first term in (11.97) vanishes. Indeed, one gets from (11.81)

$$-\,2\,R^{(ax\,bx\,\mu_1\,cx\,\mu_2)_c} + R^{(cx\,ax\,\mu_1\,bx\,\mu_2)_c} + R^{(bx\,cx\,\mu_1\,ax\,\mu_2)_c}, \tag{11.101}$$

contracted with ϵ_{abc} and integrated in u, v, w. One can relabel the dummy indices a, b, c and the integration variables u, v, w in such a way that the three terms in the above expression are equal. The contribution from the first term in (11.97) therefore cancels before removing the regulator.

One can see that the second term in (11.97) also vanishes when one removes the regulator for exactly the same reasons mentioned in the formal calculation since no singularities are involved in the canceling terms.

In order to consider the anomalous term we rearrange slightly the form for it that appears in (11.97). First of all we notice that the contributions to the anomalous term of the two **g**s in the parenthesis actually are the same and add up, giving a single **g** and a factor of 2. The way to see this is to write the **g**s explicitly. Each includes an ϵ_{abc}, which contracted with the $\mathbf{R}$ yields an expression antisymmetric in u, v and therefore the terms add up. Moreover, we notice that the contraction of $g_{\mu_1\,\mu_2}$ with $\mathbf{R}$ is equivalent to the contraction with $\mathbf{Y}$ as we argued in section 11.6.1. We then have

$$2\int d^3w \int d^3u \int d^3v \, f_\epsilon(w,x)\, f_\epsilon(u,x)\, f_\epsilon(v,x)\, g_{aw\,bv}\, g_{\mu_1\mu_2} R^{(au\,\mu_1\,bv\,\mu_2)_c} =$$
$$\frac{2}{\sqrt{2\pi}\,\epsilon}\,\epsilon_{abc}\, g_{\mu_1\mu_2}\; \partial^{cy}\, R^{(ax\,\mu_1\,by\,\mu_2)_c}\big|_{y=x} + O(\epsilon), \tag{11.102}$$

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where we have used a Gaussian regulator $f_\epsilon(\vec{z}) = (\sqrt{\pi}\epsilon)^{-3}\exp(-z^2\epsilon^{-2})$. This result is obtained by writing $g_{aw\,bv}$ as $\epsilon_{abc}\partial^c(1/|w-v|)$, expanding $R^{(au\,\mu_1\,bv\,\mu_2)\,c}$ in the limit $v \to u$, and explicitly performing the Gaussian integrals.

As we have already discussed, the contraction of $\mathbf{g}$ with $\mathbf{R}$ in non-contiguous indices is a regular expression and therefore the result is well defined without singularities.

We therefore see that in order to have a finite expression for the Hamiltonian we need to renormalize the point-split version by a factor ϵ. The end result for the regularized and renormalized Hamiltonian is

$$\begin{aligned}\mathcal{H}_r(x)\,\mathcal{A}_2(\mathbf{R}) &= \lim_{\epsilon\to 0}\ \epsilon\,\mathcal{H}^\epsilon(x)\,\mathcal{A}_2(\mathbf{R})\\ &= \sqrt{\frac{2}{\pi}}\,\epsilon_{abc}\,g_{\mu_1\mu_2}\,\partial^{cy}\,R^{(ax\,\mu_1\,by\,\mu_2)_c}\big|_{y=x}. \qquad (11.103)\end{aligned}$$

We conclude that the renormalized Hamiltonian constraint does *not* annihilate the generalized diffeomorphism invariant corresponding to the second coefficient of the Alexander–Conway knot polynomial in the point-splitting regularization procedure we have followed.

This leads immediately to an apparent contradiction. We argued in section 11.5 that as a consequence of the Kauffman polynomial being a state with cosmological constant, the vacuum Hamiltonian with $\Lambda = 0$ had to annihilate $\mathcal{A}_2(\gamma)$. We now see that in a regularized calculation it does not. But the Kauffman bracket arose as the transform of an exact state in the connection representation, independent of regularization problems, the exponential of the Chern–Simons form. How can all these apparently contradicting facts be compatible?

The answer lies in the hypotheses made in order to claim that the exponential of the Chern–Simons form was a solution of the Hamiltonian constraint of quantum gravity in the connection representation. As we argued in chapter 7 this result is quite robust, depending only on choosing a factor ordering with functional derivatives to the right. Because the cancellation between the vacuum Hamiltonian constraint and the cosmological constant term arose with the computation of only one functional derivative one expected the result to be quite robust under changes in regularization procedures. This is true. However, implicit assumptions are made in the domain of dependence of the wavefunctions. For instance, one typically assumes the connections to be smooth. If the connections are not smooth the definition of even apparently trivial multiplicative operators like the field tensor F^i_{ab} becomes problematic and has to be regularized.

Why should one consider distributional connections at all? The problem arises in the functional integrals used to define the loop transform.

Functional integrals have contributions from non-smooth fields. This can be seen even in simple examples of finite-dimensional quantum mechanics. If one considers the path integral formulation of a free particle, the integral has contributions from discontinuous paths when performing the partition to compute it. It is therefore natural to consider distributional connections if one is to perform the transform with usual functional integrals, such as the ones we explicitly used when performing the perturbative expansion.

It turns out that the anomaly we find when regulating the calculation of the action of the Hamiltonian on the $\mathcal{A}_2(\gamma)$ coefficient can be corrected with the introduction of a counterterm. A counterterm is a regularized term which vanishes when acting on an extended Wilson functional constructed with non-distributional, smooth connections. Consider, for example, the following expression, symmetric under the interchange of the internal indices,

$$(A^i_{aw}A^j_{bu} - A^i_{aw}A^j_{bv})\frac{\delta}{\delta A^{(j}_{bv}}\frac{\delta}{\delta A^{i)}_{au}}W_A(\mathbf{R}) =$$
$$\{\,\mathrm{Tr}(A_{(aw\,|\,\underset{\sim}{\mu}]\,bu)\,\underset{\sim}{\nu}}) - \mathrm{Tr}(A_{(aw\,|\,\underset{\sim}{\mu}]\,bv)\,\underset{\sim}{\nu}})\,\}\,R^{(au\,\underset{\sim}{\mu}\,bv\,\underset{\sim}{\nu})_c}. \qquad (11.104)$$

It is clear that this term vanishes in the limit $\epsilon \to 0$ if the connections are regular functions, but it may have a non-trivial contribution if the connections are distributions. The corresponding regularized expression in the extended space is

$$\mathcal{C}^{\,\epsilon} = R^{(au\,\underset{\sim}{\mu}\,bv\,\underset{\sim}{\nu})_c}\Big(\frac{\delta}{\delta R^{(aw\,|\,\underset{\sim}{\mu}]\,bu)\,\underset{\sim}{\nu}}} - \frac{\delta}{\delta R^{(aw\,|\,\underset{\sim}{\mu}]\,bv)\,\underset{\sim}{\nu}}}\Big). \qquad (11.105)$$

This expression generates anomalous type contributions. For example,

$$\mathcal{C}^{\,\epsilon}(g_{\mu_1\mu_2}\,R^{\mu_1\mu_2}) = 2(g_{aw\,bu} - g_{aw\,bv})\,R^{(au\,bv)_c}. \qquad (11.106)$$

Could it be that by adding expressions like the above one to the Hamiltonian one can cancel the anomalous terms? The answer is in the affirmative. The precise counterterm is given by the difference of two terms, $\mathcal{C}_2 - \mathcal{C}_1$,

$$\mathcal{C}_1 = R^{(au\,\underset{\sim}{\mu}\,bv\,\underset{\sim}{\nu})_c}\left(\frac{\delta}{\delta R^{aw\,\underset{\sim}{\mu}\,bu\,\underset{\sim}{\nu}}} - \frac{\delta}{\delta R^{aw\,\underset{\sim}{\mu}\,bv\,\underset{\sim}{\nu}}}\right), \qquad (11.107)$$

$$\mathcal{C}_2 = (\,R^{(au\,bv)\,\underset{\sim}{\alpha}} + \tfrac{1}{2}R^{[au\,bv]\,\underset{\sim}{\alpha}})\left(\frac{\delta}{\delta R^{(aw\,bu)_c\,\underset{\sim}{\alpha}}} - \frac{\delta}{\delta R^{(aw\,bv)_c\,\underset{\sim}{\alpha}}}\right), \qquad (11.108)$$

where $R^{[au\,bv]\,\underset{\sim}{\alpha}}$ is given by expression (11.56) without the $(-1)^{n(\underset{\sim}{\mu})}$ factor and without the "rerouting" action (the index $\underset{\sim}{\mu}^{-1}$ is replaced by $\underset{\sim}{\mu}$.) Remarkably, these expressions also have a simple form in the connection representation,

$$\mathcal{C}_1 := (A^i_{aw}A^k_{bu} - A^i_{aw}A^k_{bv})\frac{\delta}{\delta A^i_{au}}\frac{\delta}{\delta A^k_{bv}}, \tag{11.109}$$

$$\mathcal{C}_2 := (A^i_{aw}A^i_{bu} - A^i_{aw}A^i_{bv})\frac{\delta}{\delta A^k_{au}}\frac{\delta}{\delta A^k_{bv}}. \tag{11.110}$$

With this *single* counterterm all the anomalous contributions to the action of the Hamiltonian constraint on the a_2, the Kauffman bracket and the exponential of the self-linking number cancel. The fact that a single counterterm is responsible for all the cancellations is remarkable and shows that the construction is not just a gimmick to fix the anomaly problem, but might well be a genuine counterterm arising from quantum gravity. The fact that the counterterm has a simple and precise expression in the connection representation raises the hope that a better intuitive explanation of it could be gained by viewing it in this context. At present this issue is not settled: could it be that $\mathcal{C}_2 - \mathcal{C}_1$ is what one needs to add to the Hamiltonian in the connection representation in order to annihilate the exponential of the Chern–Simons form when distributional connections are allowed? Could it reflect the fact that in that case a non-trivial contribution to the measure arises? These issues are currently being studied.

11.7 Conclusions

We constructed a representation for quantum gravity based on extended loops. We studied the space of wavefunctions and promoted the constraints to wave equations. The wavefunctions are linear functionals of the multitensors and the constraints are first order functional differential operators. This introduces computational simplifications that allow to operate very efficiently with the constraints. The price paid for this is that one loses the simple geometric characterization of the solutions of the diffeomorphism constraint in terms of knot classes. One has to deal with that constraint as another functional equation. In spite of this, the knot invariants derived from Chern–Simons theory that were formal solutions of the constraints in terms of loops admit a straightforward extension to the space of multitensors. We checked formally that they solved the constraints. We then studied a regularization and showed that the solutions found also solved the constraints in a rigorous regularized way through the introduction of appropriate counterterms. The situation regarding the regularization of the constraints is still unsatisfactory, since although we can recover in a regularized fashion all of the formal results, we do not have a physical argument for the introduction of the counterterms. The fact that they have a simple expression in the connection representation raises the hope that some physical insight might be gained into their origin. The results obtained are just a first step in the regularization process, the next step being the computation of the algebra of constraints.