Can. J. Math., Vol. XXX, No. 3, 1978, pp. 600–630

PERTURBATION OF DIRECT SUM DIFFERENTIAL OPERATORS

S. J. LEE

0. Introduction. Let I be an interval, and let $I_j \subset I$ for $1 \leqq j \leqq l < \infty$ be abutted subintervals such that $\bigcup \bar{I}_j = \bar{I}$. Let τ_j be a linear differential expression defined on I_j. In this paper we study *densely* defined operators associated with

$$(0.1) \qquad \mathscr{L}y = \tau y + By + \chi^T(t)[A(\tau y|\phi) + \iota D V(y)]$$

defined on the direct sum space $\bigoplus_1^l \mathscr{D}_1(\tau_j, p, I_j) \subset L_p(I)$. Here τ is the direct sum expression $\bigoplus_1^l \tau_j$, B is an arbitrary given norm bounded operator defined everywhere in $L_p(I)$, A and D are given $m \times r$ and $m \times N$ constant matrices, $\chi(t)$ and $\phi(t)$ are given $m \times 1$ and $r \times 1$ finite column vector functions in $(L_p(I))^m$ and $(L_q(I))^r$ respectively such that the rows of χ and ϕ are linearly independent. $V(y)$ is an arbitrary but fixed $N \times 1$ "boundary" column vector functional of y which will be explained more precisely in § 1. The expression $\mathscr{L}$ contains, as a special case, a class of linear differential expressions whose leading coefficients vanish identically on a subinterval of I. The form of (0.1) includes a wide range of interface problems (cf. the first paragraph in § 5). We shall make no special mention of those problems in what follows.

This paper is a generalization of a paper by Kemp and Lee [**17**] where a special case of (0.1) is considered when $\tau_j = \tau_1$ for all j, $A = 0$, and B is of m dimensional range. The introduction of the term $(\tau y|\phi)$ allows, among other things, the following: (i) the elements of the domain of densely defined adjoint operators generated by (0.1) may be nowhere differentiable (Theorem 3.9); (ii) the expression (0.1) contains, in some cases, a member which in turn contains terms of the form

$$\int_{I_j} y^{(k)}(t)\bar{\Phi}_{jk}(t)dt \quad (\S 4).$$

There is a large literature on operators generated by linear differential expressions plus some additional terms. For example, see the survey article by Krall [**19**]. Recently Coddington and Dijksma [**7**] investigated self-adjoint subspaces in Hilbert space generated by a single formally self-adjoint differential expression (regular or singular). Hönig ([**13**; **14**] and, in particular, [**15**]) considered

Received March 10, 1977. This research was supported in part by National Research Council of Canada grants NRC A-3053, NRC A-7673, NRC A-8130, NRC A-4823, NRC A-7197.

Volterra Stieltjes integral equations in general function space, with linear constraints. Krall [**21**] studied finite dimensional perturbations in a suitable L_p-type space, generated by a single regular differential expression. Tvrdy' [**26**; **27**] and Tvrdy' and Vejvoda [**28**] also considered integrodifferential equations on a compact interval.

We briefly summarize the contents of our paper. In § 2 we set up "reasonable boundary conditions" to study (0.1). This is done by introducing maximal and minimal operators, and adjoint expressions. These ideas are basically the same as in [**17**]. More specifically we do as follows: we define the maximal operator $T_1(\mathscr{L}, p, I)$ by (0.1) on the maximal domain, and its adjoint operator is found, but this adjoint may not have dense domain (Theorem 2.1). Our interest here is in densely defined operators. In order that the domain of $T_1^*(\mathscr{L}, p, I)$ be dense we assume through this paper that the matrix Δ_r is non-singular (Proposition 2.2). This assumption guarantees that $T_1(\mathscr{L}, p, I)$ is a closed operator (Theorem 2.3), and consequently we can define the minimal operator $\tilde{T}_0(\mathscr{L}, q, I) \equiv T_1^*(\mathscr{L}, p, I)$ in $L_q(I)$. We attempt to find densely defined closed operators $T \subset T_1(\mathscr{L}, p, I)$. This leads us to define an expression $\tilde{\mathscr{L}}$ in $L_q(I)$ ((2.11)), and a corresponding maximal operator $\tilde{T}_1(\tilde{\mathscr{L}}, q, I)$ in $L_q(I)$. We choose $\tilde{\mathscr{L}}$ so that $\mathscr{L}$ and $\tilde{\mathscr{L}}$ produce a suitable bilinear form, allowing us to define an adjoint expression (cf. Proposition 2.4), which in turn leads us to define the minimal operator $T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I) \equiv \tilde{T}_1^*(\tilde{\mathscr{L}}, q, I)$ in $L_p(I)$ (Theorem 2.5). The main object of study in § 3 is the operator T defined by (3.13.) This operator corresponds to an *arbitrary* closed linear operator between the minimal operator $T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I)$ and the maximal operator $T_1(\mathscr{L}, p, I)$ (Proposition 3.6). When the number of boundary conditions for T exceeds N, then T becomes a non-densely defined operator (Proposition 3.8 and Remark after). The adjoint of T is computed (Theorem 3.9). Each element of $\mathscr{D}(T^*)$ is the sum of two functions, one of which is smooth, and the other not. These phenomena occur also in Theorem 3.3 and Theorem 4.1 of Coddington and Dijksma [**7**] in their description of self-adjoint subspace extensions. However, the above mentioned phenomena do not appear if, in their theorems, we replace "self-adjoint subspace extensions" by "self-adjoint operator extension". The expression (0.1) can generate a symmetric operator in $L_2(I)$ only when it does not contain the term $(\tau y|\phi)$ (Theorem 3.10), and under this circumstance we can give a necessary and sufficient condition for a given perturbation to be self-adjoint (Theorem 3.11 and Corollary 3.12). Theorem 3.11 in our paper coincides with Theorem 3 in [**6**] and Theorem 4.1 (regular case) in [**7**] provided that in our theorem we take $l = 1$, B of finite dimensional range and their extensions are taken to be operator extensions, i.e., $H(0) = \{0\}$ in their notation. A necessary and sufficient condition is given for a perturbation generated by (0.1) to be symmetric without using the Cayley Transform (Theorem 3.13). It seems that such a characterization has not appeared in the literature even for the special case of $l = 1, \mathscr{L}y = \tau_1 y$. The purpose of § 4 is to see how large is the class of expressions (0.1). In some cases (for instance, regular cases)

(0.1) contains an expression which in turn contains terms of the form

$$\int_{I_j} y^{(k)}(t)\bar{\phi}_{jk}(t)dt$$

(ϕ_{jk} may be nowhere differentiable) (Theorem 4.14). In trying to convert the above integral into the form $(y|g_1)_{I_1} + (\tau_1 y|g_2)_{I_1}$, we incidently generalize the second part of Lemma XIII.2.9 of Dunford and Schwartz [**10**] (Part I of Theorem 4.15). In § 5 we briefly discuss inhomogeneous regular boundary value problems in $L_p(I)$. In § 6 we find explicitly resolvents for self-adjoint operators generated by (0.1) in singular cases. In the case when $l = 1$ and B is of finite dimensional range, resolvents of self-adjoint subspace extensions were found by a different method in [**7**].

1. Preliminaries and notation. Whenever possible we shall use the same notation as in [**17**]. If $Q_1(t)$ and $Q_2(t)$ are $m_1 \times m_2$, $m_2 \times m_3$ matrix functions in $L_p(I_j)$ and $L_q(I_j)$ respectively $(1/p + 1/q = 1,\ 1 \leqq p,\ q \leqq \infty)$, then $(Q_1|Q_2)_{I_j}$ will denote the $m_1 \times m_3$ matrix

$$\int_{I_j} Q_1(t)\bar{Q}_2(t)dt$$

(integrated componentwise). If, in particular, the interval I is used inside of which all of our analysis will take place, then we denote $(Q_1|Q_2)_I$ by $(Q_1|Q_2)$. The interior and the closure of I_j are denoted by I_j^0 and $\bar{I}_j$ respectively. The transpose and conjugate transpose of a matrix Q are denoted by Q^T and Q^* respectively. If T is a densely defined operator in $L_p(I_j)$ $(1 \leqq p < \infty)$ then the operator adjoint of T is denoted by T^*. In the case when $p = \infty$, the adjoint T^* of T is defined as in Rota [**25**]. For an operator T, $\mathscr{D}(T)$ denotes the domain of T. Suppose now that T_0 and T_1 are densely defined closed operators in $L_p(I_j)$ $(1 \leqq p \leqq \infty)$ such that $T_0 \subset T_1$. Then the T_1-topology in $\mathscr{D}(T_1)$ is the topology generated by the T_1-norm:

$$\begin{aligned} ||y||_{T_1} &= ||y||_p + ||T_1 y||_p \quad \text{if } 1 \leqq p < \infty, \\ &= \max\{||y||_\infty, ||T_1 y||_\infty\} \quad \text{if } p = \infty. \end{aligned}$$

Note that T_1-norm can be replaced by equivalent norms $(||y||_p^p + ||T_1 y||_p^p)^{1/p}$ and $(||y||_p^2 + ||T_1 y||_p^2)^{1/2}$ when $p \neq \infty$. By a boundary value for T_1 with respect to T_0 we mean a T_1-continuous functional (i.e., continuous with respect to the T_1-topology) on $\mathscr{D}(T_1)$, annihilating $\mathscr{D}(T_0)$. If $p \neq \infty$ then any T_1-continuous functional f on $\mathscr{D}(T_1)$ can be written as

$$f(y) = (T_1 y|g_2)_{I_j} + (y|g_1)_{I_j}$$

with $g_k \in L_q(I_j)$. In addition, if f annihilates $\mathscr{D}(T_0)$, then $g_2 \in \mathscr{D}(T_0^*)$, $T_0^* g_2 = -g_1$, and thus

$$f(y) = (T_1 y|g_2)_{I_j} - (y|T_0^* g_2)_{I_j}$$

(cf. Lemma 1.1 in [**17**]). For each $j = 1, 2, \cdots, l$, let τ_j be a differential expression of order n_j:

$$\tau_j y \equiv \sum_{k=0}^{nj} p_{jk}(t) y^{(nj-k)}, \quad t \in I_j$$

where the $p_{jk}(t)$ are $(n_j - k)$ times continuously differentiable complex-valued functions defined on I_j and $p_{j0}(t) \neq 0$ for every $t \in I_j^0$. $P_{j0}(t)$ may or may not vanish at end points of I_j. We assume that the right end point of I_j is the left end point of I_{j+1}. As usual the direct sum expression $\tau \equiv \bigoplus_1^l \tau_j$ is defined as $(\tau y)(t) = (\tau_j y)(t)$ if $t \in I_j^0$. The Lagrange adjoint of τ_j is denoted by τ_j^*. Associated with each τ_j there is in $L_p(I_j)$ the maximal operator $T_1(\tau_j, p, I_j)$ and the minimal operator $T_0(\tau_j, p, I_j)$. These operators are closed operators satisfying $T_0(\tau_j, p, I_j) \subset T_1(\tau_j, p, I_j)$. For detailed properties, see Kemp [**16**] or Rota [**25**]. For a Banach space X, the dual will be denoted by X^*.

Throughout this paper we assume† *that the essential resolvent set for $T_0(\tau_j, p, I_j)$ is not empty for each $j = 1, 2, \cdots, l$.*

The above assumption implies that dim $[\mathscr{D}_1(\tau_j, p, I_j)/\mathscr{D}_0(\tau_j, p, I_j)]^*$ is finite (call it N_j) and $N_j \leqq 2n_j$ (see Rota [**25**]). Here $\mathscr{D}_1(\tau_j, p, I_j) \equiv \mathscr{D}(T_1(\tau_j, p, I_j))$, $\mathscr{D}_0(\tau_j, p, I_j) \equiv \mathscr{D}(T_0(\tau_j, p, I_j))$. The above assumption is not necessary if either $\tau_j = \tau_j^*$ and $p = q = 2$, or τ_j is regular. Because of the above assumption there exists an $N_j \times 1$ column vector function $V_j(y)$ such that the N_j rows of $V_j(y)$ form a basis for the space of boundary values for $T_1(\tau_j, p, I_j)$ with respect to $T_0(\tau_j, p, I_j)$. The dimension of the space of the boundary values for $T_1(\tau_j^*, q, I_j)$ with respect to $T_0(\tau_j^*, q, I_j)$ is also N_j and there exists an $N_j \times 1$ column vector function $\tilde{V}_j(y)$ whose N_j rows form a basis for the space of boundary values for $T_1(\tau_j^*, q, I_j)$ with respect to $T_0(\tau_j^*, q, I_j)$. These functions satisfy the following relation:

$$\begin{aligned} \langle y|z\rangle_{\tau_j, p, I_j} &\equiv (T_1(\tau_j, p, I_j)y|z)_{I_j} - (y|T_1(\tau_j^*, q, I_j))_{I_j} \\ &= \iota \tilde{V}_j^*(z) C_j(\tau_j) V_j(y) \end{aligned}$$

for every $y \in \mathscr{D}_1(\tau_j, p, I_j)$ and $z \in \mathscr{D}_1(\tau_j^*, q, I_j) \equiv \mathscr{D}(T_1(\tau_j^*, q, I_j))$. Here $C_j(\tau_j)$ is an $N_j \times N_j$ non-singular matrix depending only on τ_j.

Throughout this paper we shall assume that if either τ_j is regular, or $\tau_j = \tau_j^*$ and $p = q = 2$, then $\tilde{V}_j = V_j$.

This is not a restrictive condition. Let

$$N = N_1 + N_2 + \cdots + N_l, \quad \begin{aligned} V^T(y) &= (V_1^T(y), V_2^T(y), \cdots, V^T(y)), \\ \tilde{V}^T(z) &= (\tilde{V}_1^T(z), \tilde{V}_2^T(z), \cdots, \tilde{V}_l^T(z)). \end{aligned}$$

Thus $V(y)$ and $\tilde{V}(z)$ are $N \times 1$ column vectors and

$$\langle y|z\rangle_{\tau, p, I} \equiv (\tau y|z) - (y|\tau^* z) = i\tilde{V}^*(z) C(\tau) V(y)$$

†This assumption is not necessary if $p = \infty$. See E. A. Coddington and A. Dijksma, *Adjoint subspaces in Banach spaces with applications to ordinary differential subspaces*, Ann. Mat. Pura Appl., to appear.

for every $y \in \bigoplus_1^l \mathscr{D}_1(\tau_j, p, I_j)$ and $z \in \bigoplus_1^l \mathscr{D}_1(\tau_j^*, q, I_j)$. Here $C(\tau)$ is a $N \times N$ nonsingular matrix $\bigoplus_1^l C_j(\tau_j)$ (matrix direct sum).

We also assume throughout this paper that the rows of ϕ are linearly independent mod $\mathscr{D}_1(\tau)$.

Thus (0.1) must contain terms involving $(\tau y|\phi)$, unless A vanishes identically. We define the direct sum operators $T_0(\tau, p, I)$, $T_1(\tau, p, I)$, $T_0(\tau^*, q, I)$ and $T_1(\tau^*, q, I)$ as follows:

$$T_0(\tau, p, I) \equiv \bigoplus_1^l T_0(\tau_j, p, I_j), \quad T_1(\tau, p, I) = \bigoplus_1^l T_1(\tau_j, p, I_j),$$

$$T_0(\tau^*, q, I) \equiv \bigoplus_1^l T_0(\tau_j^*, q, I_j), \quad T_1(\tau^*, q, I) \equiv \bigoplus_1^l T_1(\tau_j^*, q, I_j).$$

Then clearly the following are satisfied:

$$\begin{gathered} T_0(\tau, p, I) \subset T_1(\tau, p, I), \quad T_0(\tau^*, q, I) \subset T_1(\tau^*, q, I), \\ T_0^*(\tau, p, I) = T_1(\tau^*, q, I), \quad T_1^*(\tau, p, I) = T_0(\tau^*, q, I), \\ T_0^*(\tau^*, q, I) = T_1(\tau, p, I), \quad T_1^*(\tau^*, q, I) = T_0(\tau, p, I), \\ \langle y|z\rangle_{\tau,p,I} = (T_1(\tau, p, I)y|z) - (y|T_1(\tau^*, q, I)z) \end{gathered}$$

for every $y \in \mathscr{D}_1(\tau, p, I) \equiv \mathscr{D}(T_1(\tau, p, I))$ and $z \in \mathscr{D}_1(\tau^*, q, I) \equiv \mathscr{D}(T_1(\tau^*, q, I))$. Using $T_1(\tau, p, I)$ we can interpret (0.1) as follows in the special case when B is of finite dimensional range and $p \neq \infty$. In this case an expression $\mathscr{L}$ has the form (0.1) if, and only if, $\mathscr{L} y = \tau y + G(t, y)$ where

(i) For a.a. $t \in I$, $G(t, y)$ is a $T_1(\tau, p, I)$-continuous functional of y on $\mathscr{D}_1(\tau, p, I)$ (not necessarily annihilating $\mathscr{D}_0(\tau, p, I) \equiv \mathscr{D}(T_0(\tau, p, I))$), and
(ii) for each fixed $y \in \mathscr{D}_1(\tau, p, I)$, $G(t, y) \in L_p(I)$.

Let $V(y) = (v_k(y))_{k=1}^N$. If $Q(t)$ is a $1 \times d$ row vector $(q_j(t))$ with entries in $\mathscr{D}_1(\tau, p, I)$, then $V(Q)$ will denote the $N \times d$ matrix with the (k, j)-entry $v_k(q_j)$.

If E is an $n \times n$ square matrix and if $E_1 = (e_{kj})$ is an $m_1 \times m_2$ submatrix, then the cofactor $\tilde{E}_1 = (\tilde{e}_{kj})$ of E_1 in E is the $m_1 \times m_2$ matrix with $\tilde{e}_{kj}$ equal to the usual cofactor of e_{kj} in E.

2. Minimal and maximal operators, adjoint pairs. Throughout this section let $\mathscr{L}$ be as in (0.1). First let us define the maximal operator $T_1(\mathscr{L}, p, I)$ in $L_p(I)$ associated with $\mathscr{L}$ as follows:

$$T_1(\mathscr{L}, p, I)y = \mathscr{L} y, \quad \mathscr{D}(T_1(\mathscr{L}, p, I)) \equiv \mathscr{D}_1(\mathscr{L}, p, I) = \mathscr{D}_1(\tau, p, I).$$

Clearly $\mathscr{D}_1(\mathscr{L}, p, I)$ is dense in $L_p(I)$ so that $T^*(\mathscr{L}, p, I)$ exists. To determine the domain of, and formula for, $T_1^*(\mathscr{L}, p, I)$, we first define an operator F on $L_q(I)$ by

$$F(z) = z + (z|\chi^T A)\phi.$$

https://doi.org/10.4153/CJM-1978-053-7 Published online by Cambridge University Press

THEOREM 2.1.

$$\mathscr{D}(T_1^*(\mathscr{L}, p, I)) = \{z \in L_q(I) \colon F(z) \in \mathscr{D}_1(\tau^*, q, I) \text{ and } \tilde{V}(F(z)) + c^{*-1}(\tau)(z|D^T\chi) = 0\}$$

$$T^*(\mathscr{L}, p, I)z = \tau^*(F(z)) + B^*z.$$

Proof. Take $z \in \mathscr{D}(T_1^*(\mathscr{L}, p, I))$ and put $T_1^*(\mathscr{L}, p, I)z = w$. Then for any $y \in \mathscr{D}_1(\tau, p, I)$, $(\tau y + By + \chi^T A(\tau y|\phi) + i\chi D V(y)|z) = (y|w)$. Thus, using (0.1) and the above, for $y \in \mathscr{D}_0(\tau, p, I)$,

$$(\tau y|F(z)) = (y|w - B^*z).$$

Hence, $F(z) \in \mathscr{D}_1(\tau^*, q, I)$ and $\tau^*(F(z)) = w - B^*z$. Now, for any $y \in \mathscr{D}_1(\tau, p, I)$,

$$(\tau y|F(z)) + \iota(\chi^T D V(y)|z) = (y|\tau^*(F(z))).$$

Since $F(z) \in \mathscr{D}_1(\tau^*, q, I)$,

$$[\tilde{V}(F(z)) + C^{-1*}(\tau)(z|D^T\chi)]^* V(y) = 0$$

for any $y \in \mathscr{D}_1(\tau, p, I)$. This implies the second part of the theorem because $V(y)$ can be an arbitrary $N \times 1$ constant vector. This completes the proof.

The above theorem tells us that elements in $\mathscr{D}(T_1^*(\mathscr{L}, p, I))$ need not be differentiable, and the domain may not be dense in $L_q(I)$. Since we are interested in operators with dense domains we shall find a condition under which the domain is dense. First let us define an $r \times r$ constant complex matrix:

$$\Delta_r = I_r + (\phi|\chi^T A)$$

with I_r denoting the $r \times r$ identity matrix.

PROPOSITION 2.2. *Suppose Δ_r is non-singular. Then*
(i) *F is a homeomorphism from $L_q(I)$ onto $L_q(I)$, and*

$$F^{-1}(z) = z - (z|\chi^T A)\Delta_r^{-1}\phi.$$

(ii) *$\mathscr{D}(T_1^*(\mathscr{L}, p, I))$ is dense in $L_q(I)$.*

Proof. Part (i) is clear. To prove (ii), let us define a manifold

$$\mathscr{Y} = \{z \in \mathscr{D}_1(\tau^*, q, I) \colon \tilde{V}(z) + C^{*-1}(\tau)(F^{-1}(z)|D^T\chi) = 0\}.$$

Then $\mathscr{Y}$ is dense in $L_q(I)$ by Lemma 2.2 in [**17**]. Thus $\mathscr{D}(T_1^*(\mathscr{L}, p, I)) = F^{-1}(\mathscr{Y})$ is dense in $L_q(I)$. This completes the proof.

In the remainder of this paper we shall assume that Δ_r is non-singular.

By the above assumption $T_1^{**}(\mathscr{L}, p, I)$ exists, and we have

THEOREM 2.3. *$T_1^{**}(\mathscr{L}, p, I) = T_1(\mathscr{L}, p, I)$, and, in particular, $T_1(\mathscr{L}, p, I)$ is closed.*

Proof. Since $T_1(\mathscr{L}, p, I) \subset T_1^{**}(\mathscr{L}, p, I)$, it is sufficient to show that $T_1^{**}(\mathscr{L}, p, I) \subset T_1(\mathscr{L}, p, I)$. Take any $y \in \mathscr{D}(T_1^{**}(\mathscr{L}, p, I))$ and put

https://doi.org/10.4153/CJM-1978-053-7 Published online by Cambridge University Press

$T_1^{**}(\mathscr{L}, p, I)y = w$. Then for any $z \in \mathscr{D}(T_1^*(\mathscr{L}, p, I))$, $(\tau^*(F(z)) + B^*z|y) = (z|w)$. Therefore

$$(2.2) \qquad (\tau^* g|y) = (F^{-1}(g)|w - By)$$

for any $g \in \mathscr{Y}$. Let $[c, d]$ be a compact subinterval contained in I_1^0. Then (2.2) holds, in particular, for any $g \in \mathscr{Y}$ vanishing outside $[c, d]$. Let us put $g^{(n_1)}(t) = h(t)$. Then

$$(2.3) \qquad g^{(n_1-k)}(t) = \int_c^t \frac{(t-s)^{k-1}}{(k-1)!} h(s)ds, \quad (1 \leqq k \leqq n_1),$$

and h is orthogonal to $1, s, \cdots, s_{n_1-1}$ in $L_2[c, d]$. We also note that if $h \in L_2(c, d)$ and is perpendicular to $1, s, \cdots, s_{n_1-1}$ in $L_2(c, d)$ then g defined by $g^{(n_1)}(t) = h(t)$ if $t \in [c, d]$, $g = 0$ for $t \notin [c, d]$ belongs to $\mathscr{Y} \cap C_0(I_1^0)$. Since $g \in \mathscr{Y} \cap C_0(I^0)$, we must have

$$(g|\chi^T D - \chi^T A \overline{(\Delta_r^{-1}\phi|\chi^T D)}) = 0.$$

Thus if we put

$$(2.4) \qquad u(t) = \chi^T(t)[D - A\overline{(\Delta_r^{-1}\phi|\chi^T D)}]$$

then

$$0 = (g|u) = \int_c^d h(s)\left[\int_s^d \frac{(t-s)^{n_1-1}}{(n_1-1)!}\overline{u(t)}dt\right]ds.$$

We now express (2.2) in terms of h. Let us put

$$\tau_1^* g = \sum_{k=0}^{n_1} \hat{p}_{1k}(t) g^{(n_1-k)}$$

and

$$(2.5) \qquad r(t) = w - By - \chi^T A\overline{(\Delta_r^{-1}\phi|w - By)}, \quad t \in [c, d].$$

Then

$$\begin{aligned} 0 &= (\tau^* g|y) - (F^{-1}(g)|w - By) \\ &= \int_c^d \bar{y}(t)\left\{\hat{p}_{10}(t)g^{(n_1)}(t) + \sum_{k=0}^{n_1} \hat{p}_{1k}(t)g^{(n_1-k)}(t)\right\}dt \\ &\qquad - \int_c^d \bar{r}(t)\left\{\int_c^t \frac{(t-s)^{n_1-1}}{(n_1-1)!}h(s)ds\right\}dt. \end{aligned}$$

Thus, using (2.3) and interchanging the order of integration which can be justified by Fubini's theorem, the above can be rewritten as

$$\int_c^d h(s)\tilde{\psi}(s)ds = 0,$$

https://doi.org/10.4153/CJM-1978-053-7 Published online by Cambridge University Press

where

$$\psi(s) = \bar{\hat{p}}_{10}(s)y(s) + \sum_{k=1}^{n_1} \int_s^d \frac{(t-s)^{k-1}}{(k-1)!} \bar{\hat{p}}_{1k}(t)y(t)dt - \int_s^d \frac{(t-s)^{n_1-1}}{(n_1-1)!} r(t)dt. \tag{2.6}$$

Therefore ψ is a linear combination of $1, s, \cdots, s^{n_1-1}$ and the N entries of

$$\int_s^d \frac{(t-s)^{n_1-1}}{(n_1-1)!} u(t)dt$$

in $L_2(c, d)$. Since $\hat{p}_{10}(t) = (-1)^{n_1}\bar{p}_{10}(t) \neq 0$ for $t \in [c, d]$ and so it is easy to see that y is $(n_1 - 1)$ times differentiable on $[c, d]$ and

$$\sum_{k=0}^{n_1-1} (\bar{\hat{p}}_{1k}y)^{(n_1-1-k)}(-1)^k + (-1)^{n_1-1} \int_s^d \bar{\hat{p}}_{1n_1}(t)y(t)dt - (-1)^{n_1-1}\int_s^d r(t)dt = \alpha + (-1)^{n_1-1}\int_s^d u(t)dt\beta_1$$

for a.a. $s \in [c, d]$ for some constant α and an $N \times 1$ constant column vector β_1. From this we conclude that $y^{(n_1-1)}$ is absolutely continuous on $[c, d]$, and differentiating again we have

$$\tau_1 y - u(t)\beta_1 = r(t) \tag{2.7}$$

for a.a. $t \in [c, d]$, and hence in I_1. The above shows that $y^{(n_1-1)}$ is absolutely continuous on every compact subinterval of $I_1{}^0$ and $\tau_1 y \in L_p(I_1)$. Since I_1 was arbitrary we see that $y \in \mathscr{D}_1(\tau, p, I)$ and

$$\tau_j y - u(t)\beta_j = r(t), \quad t \in I_j \tag{2.8}$$

where β_j is a constant $N \times 1$ column vector depending only on I_j. If we define $\Omega(t)$ in $L_p(I)$ by $\Omega(t) = u(t)\beta_j$ if $t \in I_j$, then, in view of (2.4) and (2.5), formula (2.8) is rewritten

$$w - By = \tau y - \Omega(t) + (w - By|\phi^T\Delta_r^{-1T})A^T\chi \tag{2.9}$$

for a.a. $t \in I$. Thus

$$\begin{aligned}(w - By|\phi^T\Delta_r^{-1T}) &= (\tau y - \Omega|\phi^T\Delta_r^{-1T})\Delta_r^* \\ &= (\tau y - \Omega|\phi^T).\end{aligned}$$

Hence (2.9) can be rewritten

$$w - By = \tau y - \Omega(t) + (\tau y - \Omega|\phi^T)A^T\chi. \tag{2.10}$$

We shall determine Ω. Now for any $z \in \mathscr{D}(T_1^*(\mathscr{L}, p, I))$,

$$(\tau^*(F(z))|y) = (F(z)|\tau y) - (z|\Omega + (\Omega|\phi^T)A^T\chi).$$

https://doi.org/10.4153/CJM-1978-053-7 Published online by Cambridge University Press

Hence by the definition of $\langle y|z\rangle_{\tau,p,I}$ we can obtain

$$\iota V^*(y)C(\tau^*)\tilde{V}(F(z)) = -(z|\Omega + (\Omega|\phi^T)A^T\chi).$$

Since $z \in \mathscr{D}(T_1^*(\mathscr{L}, p, I))$ and $C^*(\tau^*) = C(\tau)$,

$$\iota V^*(y)(z|D^T\chi) = (z|\Omega + (\Omega|\phi^T)A^T\chi).$$

Thus the denseness of $\mathscr{D}(T_1^*(\mathscr{L}, p, I))$ in $L_q(I)$ implies that

$$V^T(y)D^T\chi = \iota[\Omega + (\Omega|\phi^T)A^T\chi).$$

Substituting the above into (2.10), we see that $w = \mathscr{L}y$. This completes the proof.

By Theorem 2.3 we have the natural minimal operator $\tilde{T}_0(\mathscr{L}, q, I)$ in $L_q(I)$ defined by

$$\tilde{T}_0(\mathscr{L}, q, I) \equiv T_1^*(\mathscr{L}, p, I).$$

Thus

$$\tilde{T}_0(\mathscr{L}, q, I)z = T_1^*(\mathscr{L}, p, I)z = \tau^*(F(z)) + B^*z.$$

Remark. In [**17**] a different notation for the above $\tilde{T}_0(\mathscr{L}, q, I)$ is used. Note that $\tilde{T}_0(\mathscr{L}, q, I)$ does *not* depend on $\tilde{\mathscr{L}}$ which will be defined later.

Suppose now that T is a closed operator with dense domain such that

$$T \subset T_1(\mathscr{L}, p, I).$$

Then

$$\tilde{T}_0(\mathscr{L}, q, I) \subset T^*.$$

In particular,

$$T^*z = \tau^*(F(z)) + B^*z, \quad z \in \mathscr{D}(\tilde{T}_0(\mathscr{L}, q, I)).$$

We also note that in this formula for $T_1^*(\mathscr{L}, p, I)$ no terms involving $(\tau^*z|\tilde{\phi})$ do appear. Therefore it is natural to expect that T^*z can be associated with an expression of the form

$$(2.11) \quad \tilde{\mathscr{L}}z = \tau^*(F(z)) + \tilde{B}z + \iota\tilde{\chi}^T\tilde{D}\tilde{V}(F(z)), \quad z \in F^{-1}(\mathscr{D}_1(\tau^*, q, I))$$

where $\tilde{B}$ is a norm bounded operator defined everywhere in $L_q(I)$, $\tilde{\chi}$ is an $\tilde{m} \times 1$ column vector function whose $\tilde{m}$ rows are linearly independent in $L_q(I)$ and $\tilde{D}$ is a $\tilde{m} \times N$ constant matrix. The expression $\tilde{\mathscr{L}}$ acts on $F^{-1}(\mathscr{D}_1(\tau^*, q, I))$. Associated with $\tilde{\mathscr{L}}$ there is the maximal operator $\tilde{T}_1(\tilde{\mathscr{L}}, q, I)$ in $L_q(I)$ defined by

$$\tilde{T}_1(\tilde{\mathscr{L}}, q, I)z = \tilde{\mathscr{L}}z, \quad z \in F^{-1}(\mathscr{D}_1(\tau^*, q, I)).$$

We note here that the operator $\tilde{T}_1(\tilde{\mathscr{L}}, q, I)$ depends on $\mathscr{L}$ because $\tilde{\mathscr{L}}$ depends on F. However, if F is the identity operator, $\tilde{\mathscr{L}}$ is independent of $\mathscr{L}$.

First we have the following.

Definition. We say that the expression $\tilde{\mathscr{L}}$ in (2.11) is *adjoint to* $\mathscr{L}$ in (0.1) if

$$\tilde{B}z = B^*z + \iota\tilde{\chi}^T\tilde{D}C^{*-1}(\tau)(z|D^T\chi)$$

for every $z \in L_q(I)$. Similarly we say that $\mathscr{L}$ is *adjoint to* $\tilde{\mathscr{L}}$ if

$$By = \tilde{B}^*y + \iota\chi^T DC^{*-1}(\tau^*)(y|\tilde{D}^T\tilde{\chi}).$$

We note from the above definition that $\mathscr{L}$ is an adjoint to $\tilde{\mathscr{L}}$ if and only if $\tilde{\mathscr{L}}$ is adjoint to $\mathscr{L}$. In particular, if B is of finite dimensional range, then by altering D, $\tilde{D}$, χ and $\tilde{\chi}$, we see that the concept of "adjoint" discussed here coincides with that in [**17**]. We have the following properties of $\tilde{\mathscr{L}}$.

PROPOSITION 2.4. *Suppose* $\tilde{\mathscr{L}}$ *is any adjoint expression to* $\mathscr{L}$ *defined by* (2-11) *for some* $\tilde{\chi}, \tilde{D}, \tilde{B}$. *Then*

(i) $(\mathscr{L}y|z) - (y|\tilde{\mathscr{L}}z) = \iota[\tilde{V}(F(z)) + C^{*-1}(\tau)D^*(z|\chi)]^*C(\tau)[V(y) + C^{-1}(\tau)\tilde{D}^*(y|\tilde{\chi})]$

for $y \in \mathscr{D}_1(\tau, p, I)$, $z \in F^{-1}(\mathscr{D}_1(\tau^*, q, I))$.

(ii) $\tilde{T}_0(\tilde{\mathscr{L}}, q, I) \subset \tilde{T}_1(\tilde{\mathscr{L}}, q, I)$.

(iii) $z \in \mathscr{D}(\tilde{T}_0(\tilde{\mathscr{L}}, q, I))$ *if, and only if* $z \in F^{-1}(\mathscr{D}_1(\tau^*, q, I))$ *and* $(\mathscr{L}y|z) = (y|\tilde{\mathscr{L}}z)$ *for all* $y \in \mathscr{D}_1(\tau, p, I)$.

Proof. Since $\tilde{T}_0(\tilde{\mathscr{L}}, q, I) = T_1^*(\mathscr{L}, p, I)$, (i) and (ii) follow immediately. The "only if" part in (iii) is obvious from the definition of $\mathscr{D}(T_1^*(\mathscr{L}, p, I))$. The "if" part of (iii) follows from (ii) and Theorem 2.3.

The proof of the following theorem can be carried out by an argument similar to that of Theorem 2.3. Thus we merely state it without proof.

THEOREM 2.5. *Suppose the* $\tilde{\mathscr{L}}$ *defined by* (2.11) *is adjoint to the* $\mathscr{L}$ *defined by* (0.1). *Then*

(i) $\mathscr{D}(T_1^*(\tilde{\mathscr{L}}, q, I)) = \{y \in \mathscr{D}_1(\tau, p, I): V(y) + C^{-1}(\tau)\tilde{D}^*(y|\tilde{\chi})] = 0\}$, $T_1^*(\tilde{\mathscr{L}}, q, I)y = \mathscr{L}y$.

(ii) $\tilde{T}_1(\tilde{\mathscr{L}}, q, I) = \tilde{T}_1^{**}(\tilde{\mathscr{L}}, q, I)$.

By the above theorem if the $\mathscr{L}$ and $\tilde{\mathscr{L}}$ defined by (0.1) and (2.11) are an adjoint pair, then we have the natural minimal operator $T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I)$ in $L_p(I)$ defined by

$$T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I) \equiv \tilde{T}_1^*(\tilde{\mathscr{L}}, q, I).$$

Thus, the definition of $T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I)$ together with Theorem 2.5 yield

$$T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I) \subset T_1(\mathscr{L}, p, I),$$
$$T_0^*(\mathscr{L}, \tilde{\mathscr{L}}, p, I) = \tilde{T}_1(\tilde{\mathscr{L}}, q, I).$$

We also note that if the $\tilde{\mathscr{L}}$ defined by (2.11) is adjoint to the $\mathscr{L}$ defined by (0.1), then

$$(2.12)\quad (T_1(\mathscr{L}, p, I)y|z) - (y(\tilde{T}_1(\tilde{\mathscr{L}}, q, I)z) = \iota[\tilde{V}(F(z)) + C^{*-1}(\tau)D^*(z|\chi)]^*C(\tau)[V(y) + C^{-1}(\tau)\tilde{D}^*(y|\tilde{\chi})]$$

https://doi.org/10.4153/CJM-1978-053-7 Published online by Cambridge University Press

The relations between maximal operators $T_1(\mathscr{L}, p, I)$, $\tilde{T}_1(\tilde{\mathscr{L}}, q, I)$ and minimal operators $\tilde{T}_0(\mathscr{L}, q, I)$, $T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I)$ can be illustrated by the following diagram:

$$\begin{tikzcd}
T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I) \subset T_1(\mathscr{L}, p, I) \arrow[d, phantom] \\
\tilde{T}_0(\mathscr{L}, q, I) \subset \tilde{T}_1(\tilde{\mathscr{L}}, q, I)
\end{tikzcd}$$

$$\begin{tikzcd}
T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I) \arrow[dr, leftrightarrow] & \subset & T_1(\mathscr{L}, p, I) \arrow[dl, leftrightarrow] \\
\tilde{T}_0(\mathscr{L}, q, I) & \subset & \tilde{T}_1(\tilde{\mathscr{L}}, q, I)
\end{tikzcd}$$

where "$\leftrightarrow$" means "adjoint to each other". We also note that $\tilde{T}_1(\tilde{\mathscr{L}}, q, I)$ and $T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I)$ depend on $\tilde{\mathscr{L}}$, but $\tilde{T}_0(\mathscr{L}, q, I)$ does not depend on $\tilde{\mathscr{L}}$.

3. Adjoint operators, symmetric perturbations. We shall show how to find adjoint operators and closed symmetric operators using the adjoint expression defined in the previous section. Throughout this section, unless otherwise mentioned, T will be the operator

$$(3.13)\quad Ty = \mathscr{L}y \equiv \tau y + By + \chi^T(t)[A(\tau y|\phi) + \iota DV(y)],$$

$$(3.14)\quad \mathscr{D}(T) = \{y \in \mathscr{D}_1(\tau, p, I): P[V(y) + C^{-1}(\tau)\tilde{D}^*(y|\tilde{\chi})] = 0.$$

Here

(i) $\tilde{D}$ is a given $\tilde{m} \times N$ constant matrix, P is an $M \times N$ constant matrix of rank $M \leqq N$ $(\tilde{m} < \infty)$.

(ii) $\tilde{\chi}(t)$ is a given $\tilde{m} \times 1$ column vector function whose rows are linearly independent in $L_q(I)$.

First we have

PROPOSITION 3.6. (i) *The operator T defined by* (3.13) *is closed.*

(ii) *If the $\tilde{\mathscr{L}}$ defined by* (2.11) *is adjoint to the $\mathscr{L}$ defined by* (0-1) *then any closed operator between $T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I)$ and $T_1(\mathscr{L}, p, I)$ has domain as in* (3.14).

Proof. To prove this proposition we shall make use of the following theorem: If T_0 and T_1 are densely defined closed linear operators in $L_p(I)$ with $T_0 \subset T_1$, then an operator T' between T_0 and T_1 is a closed operator if and only if $\mathscr{D}(T')$ is a T_1-closed subspace of $\mathscr{D}(T_1)$ containing $\mathscr{D}(T_0)$, (cf, for example, $X11.45$ in [**10**]). We now prove (i). First we note that $T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I) \subset T \subset T_1(\mathscr{L}, p, I)$. It is sufficient to show that $\mathscr{D}(T)$ is $T_1(\mathscr{L}, p, I)$-closed. But $\mathscr{D}(T)$ is the kernel of the M $T_1(\tau, p, I)$-continuous functionals, and thus is $T_1(\tau, p, I)$-closed. However, from the definition of (0.1), there exists a constant K such that

$$||y||_{T_1(\tau, p, I)} \leqq K||y||_{T_1(\mathscr{L}, p, I)}$$

for all $y \in \mathscr{D}_1(\tau, p, I)$. Therefore using the closed graph theorem, the $T_1(\tau, p, I)$-topology and $T_1(\mathscr{L}, p, I)$-topology in $\mathscr{D}_1(\tau, p, I)$ coincide with each other. Therefore $\mathscr{D}(T)$ is $T_1(\mathscr{L}, p, I)$-closed. We now prove (ii). Suppose T is any

https://doi.org/10.4153/CJM-1978-053-7 Published online by Cambridge University Press

closed operator between $T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I)$ and $T_1(\mathscr{L}, p, I)$. For each $z \in F^{-1}(\mathscr{D}_1(\tau^*, q, I)$ and $y \in \mathscr{D}_1(\tau, p, I)$, we define

$$\langle\langle y|z\rangle\rangle_{\mathscr{L},\tilde{\mathscr{L}}} = (\mathscr{L}y|z) - (y|\tilde{\mathscr{L}}z).$$

Then for each fixed $z \in F^{-1}(\mathscr{D}_1(\tau^*, q, I)$, the map $y \mapsto \langle\langle y|z\rangle\rangle_{\mathscr{L},\tilde{\mathscr{L}}}$ is $T_1(\tau, p, I)$-continuous on $\mathscr{D}_1(\tau, p, I)$. Thus, if we put

$$H_z(y) = \langle\langle y|z\rangle\rangle_{\mathscr{L},\tilde{\mathscr{L}}}, \quad y \in \mathscr{D}_1(\tau, p, I),$$

then, since T is closed,

$$\mathscr{D}(T) = \bigcap_{z \in \mathscr{D}(T^*)} \{y \in \mathscr{D}_1(\tau, p, I) : H_z(y) = 0\}.$$

By the form in (2.12),

$$H_z(y) = g_z{}^*[V(y) + C^{-1}(\tau)\tilde{D}^*(y|\tilde{\chi})], \quad z \in \mathscr{D}(T^*),$$

for some $N \times 1$ constant vector g_z depending on z. Thus $\mathscr{D}(T)$ has the form as in (3.14). This completes the proof.

In the course of the proof of above proposition, we have also proved:

PROPOSITION 3.7. *Suppose the* $\tilde{\mathscr{L}}$ *defined by* (2.11) *is adjoint to* $\mathscr{L}$ *defined by* (0.1). *Then the dimension of the space of boundary values for* $T_1(\mathscr{L}, p, I)$ *with respect to* $T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I)$ *is* N.

Remark. A different method was used to prove Theorem 3.1 in Kemp and Lee [**17**]. This method is not satisfactory in our case because the terms such as $(\tau y|\phi)$ are involved in the definition of (0.1).

Proposition 3.6 does not give us any information on the denseness of $\mathscr{D}(T)$ in set (3.14) in the case when the number of boundary conditions exceeds the dimension N. As we shall see later, such cases will lead us to non-dense domains.

Suppose now that P_1 and Q_1 are $M_1 \times N$ and $M_1 \times \tilde{m}$ constant matrices, and put

$$(3.15) \quad \mathscr{D}' = \{y \in \mathscr{D}_1(\tau, p, I) : P_1 V(y) + Q_1(y|\tilde{\chi}) = 0\}.$$

Note first that if $M_1 \leqq N$, then $\mathscr{D}'$ can be rewritten as in the form of (3.14) (see the proof of Theorem 3.5 in [**17**]).

PROPOSITION 3.8. *Suppose*

(i) $M_1 > N$, *and*

(ii) P_1 *is the compound matrix* $\begin{bmatrix} P_2 \\ E\,P_2 \end{bmatrix}$ *for an* $M_2 \times N$ *matrix* P_2 *of rank* $M_2 \leqq N$, *and an* $(M_1 - M_2) \times M_2$ *matrix* E.

If we write $Q_1 \equiv \begin{bmatrix} Q_2 \\ Q_3 \end{bmatrix}$ *where* Q_2 *is an* $M_2 \times \tilde{m}$ *matrix and* Q_3 *is an* $(M_1 - M_2)$

$\times\ \tilde{m}$ *matrix, then* $\mathscr{D}'$ *defined by* (3.15) *is dense in* $L_p(I)$ *if, and only if* $EQ_2 = Q_3$. *In this case*

$$\mathscr{D}' = \{y \in \mathscr{D}_1(\tau, p, I): P_2V(y) + Q_1(y|\tilde{\chi}) = 0\}.$$

Proof. Note first that $y \in \mathscr{D}'$ if, and only if $P_2V(y) + Q_2(y|\tilde{\chi}) = 0$, $(Q_3 - EQ_2)(y|\tilde{\chi}) = 0$. The result is now immediate because the rows of $\tilde{\chi}(t)$ are linearly independent in $L_q(I)$ and the set of y such that $P_2V(y) + Q_2(y|\tilde{\chi}) = 0$ is dense in $L_p(I)$.

Remark. In the course of the proof of the above proposition, we also proved the following: If $\mathscr{D}''$ denotes the linear space of y such that $PV(y) + Q(y|\tilde{\chi}_1) = 0$ and $(y|\tilde{\chi}_2) = 0$ where P and Q are constant matrices and P is of rank $M \leqq N$, and if the rows of $\tilde{\chi}_2$ are linearly independent in $L_q(I)$, then $\mathscr{D}''$ cannot be dense in $L_p(I)$.

However we still can define an operator $\mathscr{D}''$ using (0.1). This will give rise to non-densely defined operators. Such operators have been investigated, for example, by Krall [**20**; **21**] for regular cases, and Coddington and Dijksma [**7**] for the case $p = q = 2$, the number l of intervals I_j is 1 and $\tau_1^* = \tau_1$. The method used by Coddington and Dijksma are radically different from that of Krall [**20**; **21**].

We now prove

THEOREM 3.9. *Suppose*

(i) $\mathscr{L}$ *and* $\tilde{\mathscr{L}}$ *are an adjoint pair as defined in* (0.1) *and* (2.11), *respectively, and*

(ii) $\tilde{T}$ *is the operator in* $L_q(I)$ *defined by:*

$$\tilde{T}z = \tilde{\mathscr{L}}z, \quad z \in \mathscr{D}(\tilde{T}),$$
$$\mathscr{D}(\tilde{T}) = \{g - (g|\chi^TA)\Delta_r^{-1}\phi: g \in \mathscr{D}_1(\tau^*, q, I) \text{ and}$$
$$\tilde{P}[\tilde{V}(g) + C^{*-1}(\tau)D^*(g|\chi) - C^{*-1}(\tau)D^*(g|\chi^TA)\Delta_r^{-1}(\phi|\chi)] = 0\}.$$

Here $\tilde{P}$ *is an* $\tilde{M} \times N$ *constant matrix of rank* $\tilde{M} \leqq N$. *Then the operator* T *defined by* (3.13) *and the above* $\tilde{T}$ *are adjoint each other if and only if* $M + \tilde{M} = N$ *and* $PC^{-1}(\tau)\tilde{P}^* = 0_{M\times\tilde{M}}$.

Proof. The proof is similar to that of Theorem 3.5 of Kemp and Lee [**17**]. However, for completeness, we shall outline it. First, put $g = F(z)$ for $z \in \mathscr{D}(\tilde{T})$. Then

$$\mathscr{D}(\tilde{T}) = \{z \in L_q(I): z \in F^{-1}(\mathscr{D}_1(\tau^*, q, I)), \tilde{P}[\tilde{V}(F(z)) + C^{*-1}(\tau)D^*(z|\chi)] = 0\}.$$

Since P is of rank $M \leqq N$ there exists a non-singular $M \times M$ matrix E such that $(EP)(EP)^* = I_M$, and an $(N - M) \times M$ matrix R such that the compound matrix $\begin{bmatrix} EP \\ R \end{bmatrix}$ is unitary. Thus $y \in \mathscr{D}(T)$ if, and only if there exists a constant $(N - M) \times 1$ column vector η such that

$$V(y) + C^{-1}(\tau)\tilde{D}^*(y|\tilde{\chi}) = R^*\eta.$$

Note also that for *any* $(N - M) \times 1$ constant column vector η there exists a $y \in \mathscr{D}_1(\tau, p, I)$ satisfying the above equation. This is because the N rows of $V(y) + C^{-1}(\tau)\tilde{D}^*(y|\tilde{\chi})$ are linearly independent functionals in y on $\mathscr{D}_1(\tau, p, I)$, which is dense in $L_p(I)$. We shall characterize $\mathscr{D}(T^*)$ in terms of the above η. First we note that

$$T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I) \subset T \subset T_1(\mathscr{L}, p, I).$$

Thus $\tilde{T}_0(\mathscr{L}, q, I) \subset T^* \subset \tilde{T}_1(\tilde{\mathscr{L}}, q, I)$. Hence, using (i) of Proposition 2.4 we see that $z \in \mathscr{D}(T^*)$ if, and only if

$$0 = [\tilde{V}(F(z)) + C^{*-1}(\tau)D^*(z|\chi)]^*C(\tau)[V(y) + C^{-1}(\tau)\tilde{D}^*(y|\tilde{\chi})]$$

for every $y \in \mathscr{D}(T)$. Therefore $z \in \mathscr{D}(T^*)$ if, and only if

$$[\tilde{V}(F(z)) + C^{*-1}(\tau)D^*(z|\chi)]^*C(\tau)R^*\eta = 0$$

for any $(N - M) \times 1$ constant column vector η. Thus, letting $\tilde{P}_1 = RC^*(\tau)$, we see that $z \in \mathscr{D}(T^*)$ if, and only if

$$\tilde{P}_1[\tilde{V}(F(z)) + C^{*-1}(\tau)D^*(z|\chi)] = 0.$$

Moreover $PC^{-1}(\tau)(\tilde{P}_1)^* = PC^{-1}(\tau)C(\tau)R = 0_{M\times(N-M)}$, and $\tilde{P}_1$ is of rank $N - M$.

We can now prove the "only if" part. Suppose $\tilde{T} = T^*$. Then, in particular,

$$\begin{aligned}\mathscr{D}(\tilde{T}) &= \{z \in F^{-1}(\mathscr{D}_1(\tau^*, q, I): \tilde{P}[\tilde{V}(F(z)) + C^{*-1}(\tau)D^*(z|\chi)] = 0\}\\ &= \mathscr{D}(T^*) = \{z \in F^{-1}(\mathscr{D}_1(\tau^*, q, I): \tilde{P}_1[\tilde{V}(F(z))\\ &\qquad + C^{*-1}(\tau)D^*(z|\chi)] = 0\}.\end{aligned}$$

Thus $\tilde{M} = N - M$ and there exists a non-singular $\tilde{M} \times \tilde{M}$ matrix E such that $\tilde{P} = E\tilde{P}_1$. Thus $PC^{-1}(\tau)\tilde{P}^* = 0$. The "if" part is obvious. This completes the proof.

The following theorem gives us a necessary condition for the T defined by (3.13) to be symmetric.

THEOREM 3.10. *Suppose* $p = q = 2$ *and the operator* T *defined by* (3.13) *is symmetric. Then*

(i) $A \equiv 0$ *and* F *is the identity operator on* $L_2(I)$.

(ii) $\tau = \tau^*$ *if, and only if* $B^* - B$ *is of finite dimensional range.*

Proof. Let us define an expression $\tilde{\mathscr{L}}$ by

$$\mathscr{L}y \equiv \tau^*(F(y)) + B^*y + \iota\tilde{\chi}^T\tilde{D}[C^{*-1}(\tau)(y|D^T\chi) + \tilde{V}(F(y))]$$

for $y \in F^{-1}(\mathscr{D}_1(\tau^*, 2, I))$. Then $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are an adjoint pair and $T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I) \subset T^* \subset \tilde{T}_1(\tilde{\mathscr{L}}, 2, I)$. Hence, for any $y \in \mathscr{D}(T_0(\mathscr{L}, \tilde{\mathscr{L}}, 2, I))$

we must have $y \in F^{-1}(\mathscr{D}_1(\tau^*, 2, I))$ and $\mathscr{L}y = \tilde{\mathscr{L}}y$. In particular, $(y|\chi^T A)\phi \in \mathscr{D}_1(\tau^*, 2, I)$ for any $y \in \mathscr{D}(T_0(\mathscr{L}, \tilde{\mathscr{L}}, 2, I))$. Since the m rows of χ are linearly independent in $L_2(I)$ and $\mathscr{D}(T_0(\mathscr{L}, \tilde{\mathscr{L}}, 2, I))$ is dense in $L_2(I)$, the map $y \mapsto (y|\chi^T)$ from $\mathscr{D}(T_0(\mathscr{L}, \tilde{\mathscr{L}}, 2, I))$ into $\mathbf{C}^m$ is surjective. Therefore, for any $1 \times m$ constant vector c, $c\bar{A}\phi \in \mathscr{D}_1(\tau^*, 2, I)$. This is possible only if $A \equiv 0$ because by assumption the rows of ϕ are linearly independent mod $\mathscr{D}_1(\tau^*)$. Therefore $A \equiv 0$ and thus F is an identity operator on $L_p(I)$. Since F is the identity operator on $L_p(I)$ and $\mathscr{L} = \tilde{\mathscr{L}}$ on $\mathscr{D}(T_0(\mathscr{L}, \tilde{\mathscr{L}}, 2, I))$, we see that

$$(3.16)\quad (\tau - \tau^*)y = (B^* - B)y + \iota\chi^T(t)DC^{-1}(\tau)\tilde{D}^*(y|\tilde{\chi}) + \iota\tilde{\chi}^T\tilde{D}[\tilde{V}(y) + C^{*-1}(\tau)(y|D^T\chi)]$$

for every $y \in \mathscr{D}(T_0(\mathscr{L}, \tilde{\mathscr{L}}, 2, I))$. Part (ii) follows immediately from (3.16) since $\mathscr{D}(T_0(\mathscr{L}, \mathscr{L}, 2, I))$ is dense in $L_2(I)$. This completes the proof.

Remark. When $B^* - B$ is of *infinite* dimensional range it is not clear whether or not the relation (3.16) implies $\tau = \tau^*$. This suggests that (0.1) possibly generates a symmetric operator in the case when $\tau \not\equiv \tau^*$, $p = q = 2$. Of course, if this happens, then the range of $B^* - B$ must be infinite dimensional.

THEOREM 3.11. *Suppose $p = q = 2$, $\tau = \tau^*$ and $\hat{T}$ is the operator*

$$\hat{T}y = \tau y + By + \iota\chi^T DV(y),$$
$$\mathscr{D}(\hat{T}) = \{y \in \mathscr{D}_1(\tau, p, I): P[V(y) + C^{-1}(\tau)(y|\tilde{D}^T\tilde{\chi})] = 0\}$$

for some $M \times N$ constant matrix P of rank $M \leqq N$. Then $\hat{T}$ is self-adjoint if and only if

(i) $B^*y = By + \iota\chi^T DC^{-1}(\tau)(y|D^T\chi), \quad y \in L_2(I)$.
(ii) $N = 2M, \quad PC^{-1}(\tau)P^* = 0$.
(iii) $\mathscr{D}(\hat{T}) = \{y \in \mathscr{D}_1(\tau, 2, I): P[V(y) + C^{-1}(\tau)(y|D\chi)] = 0\}$.

Proof. The "if" part is trivial. We shall prove the "only if" part. Suppose $\hat{T} = \hat{T}^*$. We note first that $V(y) = \tilde{V}(y)$. The expressions $\mathscr{L}$ and $\tilde{\mathscr{L}}$ defined by:

$$\mathscr{L}y = \tau y + By + \iota\chi^T DV(y),$$
$$\tilde{\mathscr{L}}y = \tau y + B^*y + \iota\tilde{\chi}^T\tilde{D}[C^{-1}(\tau)(y|D^T\chi) + V(y)],$$

are an adjoint pair, and

$$T_0(\mathscr{L}, \mathscr{L}, 2, I) \subset \hat{T} \subset T_1(\mathscr{L}, 2, I).$$

This relation implies (i). By Theorem 3.9 there exists an $(N - M) \times N$ matrix $\tilde{P}$ of rank $N - M$ such that

$$PC^{-1}(\tau)\tilde{P}^* = 0 \quad \text{and}$$
$$\mathscr{D}(\hat{T}) = \mathscr{D}(\hat{T}^*) = \{y \in \mathscr{D}_1(\tau, 2, I): \tilde{P}[V(y) + C^{-1}(\tau)(y|D^T\chi)] = 0\}.$$

By (i), $\mathscr{L}$ is adjoint to itself. Thus applying Theorem 3.9 again, we can find

https://doi.org/10.4153/CJM-1978-053-7 Published online by Cambridge University Press

an $M \times N$ constant matrix P_1 of rank M such that $P_1 C^{-1}(\tau)\tilde{P}^* = 0$ and

$$\mathscr{D}(\hat{T}) = \{y \in \mathscr{D}_1(\tau, 2, I): P_1[V(y) + C^{-1}(\tau)(y|D^T\chi)] = 0\}.$$

Therefore $M = N - M$, and there exists a non-singular $M \times M$ matrix E such that $P_1 = EP$. This implies (iii). This completes the proof.

Theorem 3.11 does not tell us conditions on D, χ for which T is self-adjoint. However, in the course of the proof of the above theorem we also have proved

COROLLARY 3.12. *Suppose that* $p = q = 2$, $\tau = \tau^*$. *Then the operator* $\hat{T}$ *defined in Theorem* 3.11 *is self-adjoint if, and only if*

(i) $B^*y = By + \iota\chi^T DC^{-1}(\tau)(y|D^T\chi)$ for $y \in L_2(I)$.

(ii) $N = 2M$, $PC^{-1}(\tau)P^* = 0$.

(iii) $\chi^T(t)DC^{-1}(\tau)P^* = \tilde{\chi}^T(t)\tilde{D}C^{-1}(\tau)P^*$.

Remark. The above condition (i) implies that in addition if $B = B^*$, then $DC^{-1}(\tau)D^* = 0$.

Remark. Theorem 3.11 and Corollary 3.12 coincide with Theorem 3 of Coddington [**6**], and Theorem 4.1 (regular case) of Coddington and Dijksma [**7**] provided that $l = 1$ in our case and their extensions are *operator extensions*, i.e., $H(0) = \{0\}$ in their notation.

Remark. Theorem 3.11 also proves the following: Let $\{\gamma^-, \gamma^+\}$ denote the deficiency indices of $\tau_1 = \tau_1^*$ in $L_2(I_1)$. Then $\gamma^- = \gamma^+$ if $\gamma^- + \gamma^+$ is an even number, and there exists a $((\gamma^- + \gamma^+)/2) \times (\gamma^- + \gamma^+)$ constant matrix P such that $PC^{-1}(\tau_1)P^* = 0$.

Next we shall find a necessary and sufficient condition for a given closed perturbed operator to be symmetric. As we will see later, any closed symmetric perturbation can be obtained by examining the $N \times N$ non-singular matrix $C(\tau)$. In the proof of the following theorem, we do not make use of Cayley transform.

THEOREM 3.13. *Suppose* $p = q = 2$, $\tau = \tau^*$. *Let* T *be the operator*

$$Ty = \tau y + By + \iota\chi^T DV(y),$$
$$\mathscr{D}(T) = \{y \in \mathscr{D}_1(\tau, 2, I): P[V(y) + C^{-1}(\tau)D^*(y|\chi)] = 0\}$$

where P *is an* $M \times N$ *constant matrix of rank* $M \leqq N$. *Then* T *is symmetric if, and only if*

(i) $(B^* - B)y = \iota\chi^T(t)DC^{-1}(\tau)(y|D^T\chi)$, $y \in L_2(I)$, *and*

(ii) *there exists an* $(N - M) \times M$ *constant matrix* E *of rank* $N - M \leqq M$ *such that* $PC^{-1}(\tau)(EP)^* = 0$.

Proof. Define an expression $\mathscr{L}$ on $\mathscr{D}_1(\tau, 2, I)$ by

$$\mathscr{L}y = \tau y + By + \iota\chi^T(t)DV(y).$$

First we prove the "if" part. Condition (i) implies that $\mathscr{L}$ is adjoint to itself, so we take $\tilde{\mathscr{L}} = \mathscr{L}$. By Sylvester's inequality the rank of EP is $N - M$. Set $\tilde{P} = EP$. Thus $PC^{-1}(\tau)\tilde{P}^* = 0$. Let $\mathscr{D}'$ denote the set of y in $\mathscr{D}_1(\tau, p, I)$ such that $\tilde{P}[V(y) + C^{-1}(\tau)D^*(y|\chi)] = 0$. Let T' denote the operator defined on $\mathscr{D}'$ by $T'y = \mathscr{L}y$. Then, by Theorem 3.9, $T' = T^*$. Clearly $\mathscr{D}(T) \subset \mathscr{D}'$. Therefore $T \subset T^*$, so that T is symmetric.

We now prove the "only if" part. First we note that Theorem 3.10 implies (i) of our assertion. Thus $\mathscr{L}$ defined above is adjoint to itself and so

$$T_0(\mathscr{L}, \mathscr{L}, 2, I) \subset T \subset T_1(\mathscr{L}, 2, I)$$

where as before we take $\tilde{\mathscr{L}} = \mathscr{L}$. Note that $T_0(\mathscr{L}, \mathscr{L}, 2, I)$ is a closed symmetric operator in $L_2(I)$. By Theorem 3.9 there exists an $(N - M) \times N$ constant matrix $\tilde{P}$ of rank $N - M$ such that

$$\mathscr{D}(T^*) = \{y: y \in \mathscr{D}_1(\tau, 2, I), \tilde{P}[V(y) + C^{-1}(\tau)D^*(y|\chi)] = 0\}$$

and $PC^{-1}(\tau)\tilde{P}^* = 0$. We note that $\mathscr{D}(T_1(\mathscr{L}, 2, I))$ and

$$\mathscr{D}_0(\mathscr{L}) \equiv \mathscr{D}(T_0(\mathscr{L}, \mathscr{L}, 2, I))$$

are Hilbert spaces with the inner product

$$(f|g)_{T_1} = (f|g) + (\mathscr{L}f|\mathscr{L}g).$$

We can regard $\mathscr{D}(T)/\mathscr{D}_0(\mathscr{L})$ and $\mathscr{D}(T^*)/\mathscr{D}_0(\mathscr{L})$ as subspaces of the Banach space $\mathscr{D}_1(\tau, 2, I)/\mathscr{D}_0(\mathscr{L})$. Thus, since $\mathscr{D}(T) \subset \mathscr{D}(T^*)$ by assumption, we see that $\mathscr{D}(T)/\mathscr{D}_0(\mathscr{L}) \subset \mathscr{D}(T^*)/\mathscr{D}_0(\mathscr{L})$. However $[\mathscr{D}(T)/\mathscr{D}_0(\mathscr{L})]^*$ is the space of functionals generated by the M rows of $P[V(y) + C^{-1}(\tau)D^*(y|\chi)]$, and $[\mathscr{D}(T^*)/\mathscr{D}_0(\mathscr{L})]^*$ is the space of functionals generated by the $N - M$ rows of $\tilde{P}[V(y) + C^{-1}(\tau)D^*(y|\chi)]$. Thus there exists an $(N - M) \times M$ constant matrix E such that

$$\tilde{P}[V(y) + C^{-1}(\tau)D^*(y|\chi)] = EP[V(y) + C^{-1}(\tau)D^*(y|\chi)]$$

for every $y \in \mathscr{D}_1(\tau, 2, I)$. This implies that $\tilde{P} = EP$ and $PC^{-1}(\tau)P^*E^* = 0$. It is clear that E is of rank $N - M \leqq M$. This completes the proof.

4. Extensions to other forms of expressions. In this section we shall see how large is the class of expressions (0.1).

THEOREM 4.14. *Let τ_1 be as in* § 1. *Suppose that*

(i) $1 \leqq p < \infty$;

(ii) *the coefficients $p_{1k}(t)$ $(0 \leqq k \leqq n_1)$ of τ_1 (n_1 is the order of τ_1) are uniformly bounded by a constant ζ on I_1; and*

(iii) *there exists a constant ϵ such that*

$$|p_{10}(t)| \geqq \epsilon > 0 \quad \text{for all } t \in I_1.$$

Then for each $k = 0, 1, 2, \cdots, n_1$ there exists a constant $K_k < \infty$ such that

$$||y^{(k)}||_p \leqq K_k[||y||_p + ||\tau_1 y||_p]$$

for all $y \in \mathscr{D}_1(\tau_1, p, I_1)$. *In particular, for each fixed* $\phi_k \in L_q(I_1)$, *the map* $y \mapsto (y^{(k)}|\phi_k)_{I_1}$ *defines a* $T_1(\tau_1, p, I_1)$*-continuous functional on* $\mathscr{D}_1(\tau_1, p, I_1)$.

Proof. If $0 \leqq k \leqq n_1 - 1$, then the result follows directly from the definition of $\mathscr{D}_1(\tau_1, p, I_1)$ and Theorem 1 of Halperin and Pitt [**12**]. We now prove the result for $k = n_1$. For $y \in \mathscr{D}_1(\tau_1, p, I_1)$.

$$|y^{(n_1)}(t)| = \left|\left(\tau_1 y - \sum_{\sigma=1}^{n_1} p_{1k} y^{(n_1-\sigma)}\right)/p_{10}\right|(t)$$
$$\leqq \frac{1}{\epsilon}\left(|(\tau_1 y)(t)|^p + \sum_{\sigma=1}^{n_1} |y^{(n_1-\sigma)}(t)|^p\right)^{1/p}(1 + n_1\zeta^q)^{1/q}$$

where $1/p + 1/q = 1$. Thus

$$||y^{(n_1)}||_p^p \leqq K\left(||\tau_1 y||_p^p + \sum_{\sigma=1}^{n_1} ||y^{(n_1-\sigma)}||_p^p\right) \leqq K_{n_1}||y||^p_{T_1(\tau_1, p, I_1)}$$

using the first part, where K_1, K_{n_1} are some constants. This proves the first part of the result. The last part of our assertion is obvious because $||y||_p \leqq ||y||_{T_1(\tau, p, I)}$.

The above theorem implies that if $1 \leqq p < \infty$ and if all the coefficients $p_{jk}(t)$ $(0 \leqq k \leqq n_j)$ of τ_j $(1 \leqq j \leqq l)$ satisfy the conditions of Theorem 4.14 (this will be the case, for instance, if τ_j is regular on I_j for each $j = 1, 2, \cdots, l$), then the expression $\mathscr{L}$ defined by (0.1) contains a term which in turn has the following form:

$$\mathscr{L}_1 y = \tau y + By + \chi^T(t)[A_1 G(y) + \iota DV(y)]$$

where (i) A_1 is a $m \times l$ constant matrix, (ii) $G(y)$ is the $l \times 1$ *column* vector (d_j) where

$$d_j = \sum_{k=1}^{nj} (y^{(nj-k)}|\phi_{jk})_{Ij} \quad (\phi_{jk} \in L_q(I_j))$$

may not be differentiable on I_j. However a direct attempt to apply our theory developed so far to the expression $\mathscr{L}_1$ is not satisfactory because formally $\mathscr{L}_1$ and (0.1) are different from each other. Therefore, it is desirable that we convert an integral of the form

$$\int_{I_j} y^{(nj-k)}(t)\bar{\phi}_{jk}(t)dt$$

into the form $(\tau_j y|\psi_{1j})_{I_j} + (y|\psi_{2j})_{I_j}$. We shall show how this can be done. The following notation will be needed later: If S is a formal differential expression, $Sy \equiv \sum_{\sigma=0}^n p_\sigma(t)y^{(n-\sigma)}$, then for each fixed $k = 0, 1, \cdots, n$, $S^{(k)}$ and $S^{*(k)}$ will denote the differential expressions of order k defined by:

$$S^{(k)}y = \sum_{\sigma=0}^{k} p_\sigma(t)y^{(n-\sigma)}, \quad S^{*(k)}y = \sum_{\sigma=0}^{k} (-1)^{n-k}(\bar{p}_\sigma(t)y)^{(n-\sigma)}.$$

Note that if $k = 0$ then $S^{*(0)}y$ is the Lagrange adjoint of S.

We shall define certain functionals which will be used later. Suppose $[c, d]$ is a fixed compact subinterval of $I_1{}^0$ and $1 \leqq p \leqq \infty$. For each $k = 1, 2, \cdots, n_1$, let $P_{k-1}(s)$ be a polynomial in s of degree $\leqq k - 1$, and let g_{k1} and g_{k2} be functions in $L_q(I_1)$ satisfying the following additional conditions: For the case when $1 \leqq k \leqq n_1 - 1$, $g_{k2}(t)$ is $(n_1 - k - 1)$ times differentiable on I_1 and $g_{k2}{}^{(n_1-k-1)}(t)$ is absolutely continuous on every compact subinterval of $I_1{}^0$; For the case when $k = n_1$, the g_{kn_1} is locally integrable on I_1. With g_{ki} and P_{k-1} defined as above we define functionals $\tilde{\alpha}_k \equiv \tilde{\alpha}_k(c, d, y, g_{k2})$, $\tilde{\beta}_k \equiv \tilde{\beta}_k(c, d, y, g_{k2})$, $\tilde{\gamma}_k \equiv \tilde{\gamma}_k(c, d, y, g_{k1})$ and $\tilde{\delta}_k \equiv \tilde{\delta}_k(c, d, P_{k-1})$ as follows:

$$\begin{aligned}
\tilde{\alpha}_k &= \int_c^d \bar{g}_{k2}(t)\Bigg\{\sum_{\sigma=0}^{k-1} p_{1,n_1-\sigma}(t)y^{(\sigma)}(c) + (t-c)y'(c)p_{1,n_1}(t) \\
&\qquad + \sum_{\substack{1\leqq i\leqq k-2\\ n_1-k+i+1\leqq\sigma\leqq n_1}} p_{1\sigma}(t)y^{(k-i)}(c) \\
&\qquad \times (t-c)^{\sigma-n_1+k-i}/(\sigma - n_1 + k - i)!\Bigg\}dt \quad \text{if } 3 \leqq k \leqq n_1 \\
&= -\int_c^d \bar{g}_{k2}(t)\Bigg\{\sum_0^1 p_{1\sigma}(t)y^{(\sigma)}(c) + y'(c)(t-c)p_{1,n_1}(t)\Bigg\}dt \quad \text{if } k = 2 \\
&= -\int_c^d p_{1,n_1}(t)\bar{g}_{k2}(t)y(c)dt \quad \text{if } k = 1.
\end{aligned}$$

$$\begin{aligned}
\tilde{\beta}_k &= -\sum_{\substack{1\leqq\sigma\leqq n_1-k-i\\ 1\leqq i\leqq n_1-k-1}} (-1)^{n_1-k-\sigma+i}[y^{(n_1-k-\sigma+i)}(t)(p_{1\sigma}\bar{g}_{k2})^{(n_1-k-\sigma-i)}(t)]_c^d \\
&\qquad \text{if } 1 \leqq k \leqq n_1 - 2, \\
&= -[y^{(n-1)}(t)p_{10}(t)\bar{g}_{k2}(t)]_c^d \quad \text{if } k = n_1 - 1, \\
&= 0 \quad \text{if } k = n_1.
\end{aligned}$$

$$\begin{aligned}
\tilde{\gamma}_k &= -\int_c^d \bar{g}_{k1}(t)\Bigg\{y(c) + \frac{(t-c)^{k-1}y^{(k-1)}(c)}{(k-1)!} \\
&\qquad + \sum_{1\leqq i\leqq k-2} \frac{y^{(k-i-1)}(c)(t-c)^{k-i-1}}{(k-i-1)!}\Bigg\}dt \quad \text{if } 3 \leqq k \leqq n_1, \\
&= -\int_c^d (1 + t - c)\bar{g}_{k1}(t)y(c)dt \quad \text{if } k = 2, \\
&= -\int_c^d \bar{g}_{k1}(t)y(c)dt \quad \text{if } k = 1.
\end{aligned}$$

$$\tilde{\delta}_k = -\sum_{0\leqq\sigma\leqq k-1} (-1)^{k-1}[y^{(k-\sigma-1)}(t)P_{k-1}{}^{(\sigma)}(t)]_c^d.$$

Finally we define

$$\tilde{\Omega} \equiv \tilde{\Omega}(c, d, y, P_{k-1}, g_{k1}, g_{k2}) = \tilde{\alpha}_k + \tilde{\beta}_k + \tilde{\gamma}_k + \tilde{\delta}_k.$$

Then this has the property that if $y \in \mathscr{D}_1(\tau_1, p, I_1)$ vanishes at c and d, then $\tilde{\Omega}(c, d, y, P_{k-1}, g_{k1}, g_{k2}) = 0$.

THEOREM 4.15. I. *Suppose* $1 \leqq p \leqq \infty$ *and* k *be an arbitrary, but fixed integer such that* $1 \leqq k \leqq n_1$ (n_1 = *order of* τ_1). *Suppose* ϕ_k, g_{k1} *and* g_{k2} *are given functions in* $L_q(I_1)$. *If*

$$(y^{(k)}|\phi_k)_{I_1} = (y|g_{k1})_{I_1} + (\tau_1 y|g_{k2})_{I_1} \tag{4.17}$$

for every $y \in \mathscr{D}_1(\tau_1, p, I_1)$, *then we have the following:*

(i) *If* $k \neq n_1$, *then* $g_{k2}(t)$ *is* $(n_1 - k - 1)$ *times differentiable on* I_1 *and* $g_{k2}^{(n_1-k-1)}(t)$ *is absolutely continuous on every compact subinterval of* I_1^0.

(ii) *For every compact subinterval* $[c, d]$ *of* I_1^0, *there exists a polynomial* $P_{k-1}(s)$ *of degree* $\leqq k - 1$ *such that for* a.a. $s \in [c, d]$,

$$\phi_k(s) = (\tau_1^{*(n_1-k)} g_{k2})(s) + P_{k-1}(s) + \int_s^d \left\{ \sum_{\sigma=n_1-k+1}^{n_1} \frac{P_{1\sigma}(t)(t-s)^{\sigma-n_1+k-1}}{(\sigma - n_1 + k - 1)!} g_{k2}(t) + \frac{(t-s)^{k-1}}{(k-1)!} g_{k1}(t) \right\} dt.$$

(iii) *For every* $y \in \mathscr{D}_1(\tau_1, p, I_1)$ *and* $[c, d] \subset I_1^0$,

$$\int_c^d y^{(k)}(t)\bar{\phi}_k(t)dt = \int_c^d \{y\bar{g}_{k1} + (\tau_1 y)\bar{g}_{k2}\}dt + \tilde{\Omega}(c, d, y, P_{k-1}, g_{k1}, g_{k2}), \tag{4.18}$$

and

$$\lim_{(c,d)\to I_1} \tilde{\Omega} = 0.$$

II. *Conversely, suppose* $1 \leqq p \leqq \infty$ *and let* k *be a fixed integer such that* $1 \leqq k \leqq n_1$. *Suppose further that* g_{k1} *and* g_{k2} *are given functions in* $L_q(I_1)$ *satisfying the following condition:*

(i) *If* $k \neq n_1$, *then* $g_{k2}(t)$ *is* $(n_1 - k - 1)$ *times differentiable on* I_1 *and* $g_{k2}^{(n_1-k-1)}(t)$ *is absolutely continuous on* I_1.

If $k = n_1$, *then* $g_{n_12}(t)$ *is locally integrable on* I_1.

(ii) *For each* $y \in \mathscr{D}_1(\tau_1, p, I_1)$, $\lim_{(c,d)\to 0} \tilde{\Omega} = 0$.

Then, if $\phi_k(t)$ *is the function defined in* I_1 *as in* (ii) *of Part I, then* (4.17) *holds for every* $y \in \mathscr{D}_1(\tau_1, p, I_1)$.

COROLLARY 4.16. *Suppose, for a given expression* τ_1, *conditions* (i), (ii) *and* (iii) *of Theorem* 4.14 *hold. Suppose further that, for a given integer* k *with* $1 \leqq k \leqq n_1$, $g_{k1}(t)$ *and* $g_{k2}(t)$ *are functions in* $L_q(I_1)$ *satisfying* (i) *of Part II of the above theorem. Then, if we define a function* $\phi_k(t)$ *as in* (ii) *of Part I of the above theorem, it follows that*

$$(y^{(k)}|\phi_k)_{I_1} = (y|g_{k1})_{I_1} + (\tau_1 y|g_{k2})_{I_1}$$

for every $y \in \mathscr{D}_0(\tau_1, p, I_1)$.

Proof. Let $\mathscr{Y}$ denote the set of functions $f(t)$ defined on I_1 such that $f(t)$ is n_1 times continuously differentiable and has compact support on I_1^0. Then, in view of the definitions of $\tilde{\alpha}_k$, $\tilde{\beta}_k$, $\tilde{\gamma}_k$ and $\tilde{\delta}_k$ and (4.18) we see that

$$(y^{(k)}|\phi_k)_{I_1} = (y|g_{k1})_{I_1} + (\tau_1 y|g_{k2})_{I_1}$$

for every $y \in \mathscr{Y}$. Let $y \in \mathscr{D}_0(\tau_1, p, I_1)$ be given. Since $\mathscr{Y}$ is dense in the Banach space $\mathscr{D}_0(\tau_1, p, I_1)$ with $T_1(\tau_1, p, I_1)$-topology, there exists a sequence (y_i) in $\mathscr{Y}$ converging to $y \in \mathscr{D}_0(\tau_1, p, I_1)$ with respect to $T_1(\tau_1, p, I_1)$-topology. By Theorem 4.14, the map $y \to (y^{(k)}|\phi_k)_{I_1}$ is $T_1(\tau_1, p, I_1)$-continuous on $\mathscr{D}_0(\tau_1, p, I_1)$. Since the $T_1(\tau_1, p, I_1)$-topology is stronger than the norm topology in $\mathscr{D}_0(\tau_1, p, I_1)$ we see that

$$\begin{aligned}(y^{(k)}|\phi_k)_{I_1} = \lim_i (y_i^{(k)}|\phi_k)_{I_1} &= \lim_i \{(y_i^{(k)}|g_{k1})_{I_1} + (\tau_1 y|g_{k2})_{I_2}\} \\ &= (y^{(k)}|g_{k1})_{I_1} + (\tau_1 y|g_{k2})_{I_1}.\end{aligned}$$

This completes the proof.

Proof of Theorem 4.15. First we prove Part I. Take any $y \in \mathscr{D}_1(\tau_1, p, I_1)$ such that $y(t) = 0$ for $t \notin [c, d] \subset I_1^0$. Define

$$y^{(n_1)}(t) = h(t), \quad t \in [c, d].$$

Then y satisfies (2.3) with $y = g$. As we have seen in § 2, a simple calculation shows that

$$\begin{aligned}(\tau_1 y|g_{k2})_{I_1} &= \int_c^d h(s)\Big\{p_{10}(s)\bar{g}_{k2}(s) \\ &\quad + \sum_{\sigma=1}^{n_1} \int_s^d (p_{1\sigma}(t)(t-s)^{\sigma-1}/(\sigma-1)!)\bar{g}_{k2}(t)dt\Big\}ds; \\ (y|g_{k1})_{I_1} &= \int_c^d h(s)\Big\{\int_s^d (t-s)^{n_1-1}\bar{g}_{k1}(t)dt/(n_1-1)!\Big\}ds; \\ (y^{(k)}|\phi_k)_{I_1} &= \int_c^d h(s)\bar{\phi}_k(s)ds \quad \text{if } k = n_1, \\ &= -\int_c^d h(s)\Big\{\int_s^d (t-s)^{n_1-k-1}\bar{\phi}_k(t)dt/(n_1-k-1)!\Big\}ds \\ &\qquad \text{if } 1 \leqq k \leqq n_1 - 1.\end{aligned}$$

In the case when $k = n_1$, (4.17) can be rewritten

$$\begin{aligned}0 = \int_c^d h(s)\Big\{&-\bar{\phi}_k(s) + p_{10}(s)\bar{g}_{k2}(s) + \int_s^d (t-s)^{n_1-1}\bar{g}_{k1}(t)dt/(n_1-1). \\ &+ \sum_{\sigma=1}^{n_1} \int_s^d p_{1\sigma}(t)(t-s)^{\sigma-1}\bar{g}_{k2}(t)dt/(\sigma-1)!\Big\}ds,\end{aligned}$$

so, as in § 2, there exists a polynomial $P_{n_1-1}(s)$ of degree $\leqq n_1 - 1$ such that for a.a. $s \in [c, d]$,

$$\begin{aligned}\phi_k(s) = \bar{p}_{10}(s)g_{n_12}(s) &+ \int_s^d (t-s)^{n_1-1}g_{n_11}(t)dt/(n_1-1)! \\ &+ \sum_{1\leqq\sigma\leqq n_1} \int_s^d \bar{p}_{1\sigma}(t)(t-s)^{\sigma-1}g_{n_12}(t)dt/(\sigma-1)! + P_{n_1-1}(s).\end{aligned}$$

For the case $1 \leqq k \leqq n_1 - 1$, formula (4.17) implies that

$$- \int_s^d (t-s)^{n_1-k-1}\phi_k(t)dt/(n_1-k-1)! + \bar{p}_{10}(s)g_{k2}(s)$$
$$+ \int_s^d \left\{(t-s)^{n_1-1}g_{k1}(t)/(n_1-1)! + \sum_{\sigma=1}^{n_1} \bar{p}_{1\sigma}(t)(t-s)^{\sigma-1}g_{k2}(t)dt/(\sigma-1)!\right\}dt$$

is a linear combination of $1, s, \cdots, s^{n_1-1}$ in $L_2[c, d]$. Thus, applying the same method used in § 2, we get (ii) of Part I. We now prove (4.18). Take any $y \in \mathscr{D}_1(\tau_1, p, I_1)$ and $[c, d] \subset I_1{}^0$. Then

$$\int_c^d y^{(k)}(s)\bar{\phi}_k(s)ds = \int_c^d y^{(k)}(s)\overline{(\tau_1{}^{*(n_1-k)}g_{k2})}(s) + \int_c^d y^{(k)}(s)\bar{p}_{k-1}(s)ds$$
$$+ \int_c^d y^{(k)}(s)\left\{\int_s^d \sum_{\sigma=n_1-k+1}^{n_1} p_{1\sigma}(t)(t-s)^{\sigma-n_1+k-1}g_{k2}(t)dt/(\sigma-n_1+k-1)!\right\}ds$$
$$+ \int_c^d y^{(k)}(s)\left\{\int_s^d (t-s)^{k-1}g_{k1}(t)dt/(k-1)!\right\}ds.$$

Interchanging the order of integration, and then integrating by parts (this can be justified by the use of Fubini's Theorem), a lengthy and tedious calculation shows that

$$\int_c^d y^{(k)}(s)(\bar{\tau}_1{}^{*(n_1-k)}g_{k2})(s)ds$$
$$= \int_c^d \sum_{0 \leqq \sigma \leqq n_1-k} p_{1\sigma}(t)y^{(n_1-\sigma)}(t)\bar{g}_{\sigma 2}(t)dt + \tilde{\beta}_k(c, d, y, g_{k2}),$$

$$\int_c^d y^{(k)}(s)\bar{P}_{k-1}(s)ds = \tilde{\delta}_k(c, d, P_{k-1}),$$

$$\int_c^d y^{(k)}(s)\left\{\int_s^d (t-s)^{k-1}\bar{g}_{k1}(t)dt/(k-1)!\right\}ds$$
$$= \int_c^d y(s)\bar{g}_{k1}(s)ds + \tilde{\gamma}_k(c, d, y, g_{k1}),$$

$$\int_c^d y^{(k)}(s)$$
$$\times \left\{\sum_{\sigma=n_1-k+1}^{n_1} \int_s^d p_{1\sigma}(t)(t-s)^{\sigma-n_1+k-1}g_{k2}(t)dt/(\sigma-n_1+k-1)!\right\}ds$$
$$= \sum_{\sigma=0}^{k-1} \int_c^d p_{1,n_1-\sigma}(s)y^{(\sigma)}(s)\bar{g}_{k2}(s)ds + \tilde{\alpha}_k(c, d, y, g_{k2}).$$

Therefore

$$\int_c^d y^{(k)}(s)\bar{\phi}_k(s)ds = \int_c^d (y\bar{g}_{k1} + (\tau_1 y)\bar{g}_{k2})ds + \tilde{\Omega}.$$

https://doi.org/10.4153/CJM-1978-053-7 Published online by Cambridge University Press

Taking $(c, d) \to I_1$ we see that $\tilde{\Omega} \to 0$. This proves Part I. Part II is obvious. This completes the proof.

Remark. Part I of Theorem 4.15 can be regarded as a generalization of the second part of Lemma XIII.2.9 of Dunford and Schwartz [**10**]. To see this we need only take $\phi_1 = 0$, $g_{11} = 0$, $k = 1$ in (4.17), and then use (ii) of Part I of Theorem 4.15. Note that the proof of Theorem 4.15 does not make use of any result from the theory of differential operators.

5. Regular boundary problems in $L_p(I)$. In this section we assume that each I_j is a compact interval $[a_{j-1}, a_j]$ $(1 \leqq j \leqq l)$ and that each τ_j is regular on I_j. Thus $N_j = 2n_j$ and $N = 2(n_1 + \cdots + n_l)$, $\tilde{V}_j(y) = V_j(y)$, so we can take $V_j^T(y)$ to be the $1 \times 2n_j$ vector:

$$(y(a_{j-1}+),\ y'(a_{j-1}+),\ \cdots,\ y^{(n_j-1)}(a_{j-1}+),\ y(a_j-),\ y'(a_j-),\ \cdots,\ y^{(n_j-1)}(a_j-)),$$

so that $V^T(y) = (V_1^T(y), \cdots, V_l^T(y))$.

We are interested in the following problem: Given a (perhaps complex) number λ and a function f in $L_p(I)$ find a function $y \in \mathscr{D}(T)$ $(1 \leqq p \leqq \infty)$ such that

$$(5.19)\quad (T - \lambda)y = f$$

where $Ty = \mathscr{L}y$ and

$$\mathscr{D}(T) = \{y \in \mathscr{D}_1(\tau, p, I) \colon P[V(y) + C^{-1}(\tau)(y|\tilde{D}^T\tilde{\chi})] = 0\}.$$

Here $\mathscr{L}$ is defined in (0.1) and P is an $M \times N$ constant matrix of rank $M \leqq N$.

This problem is still too general to handle, because B is an arbitrary bounded operator on $L_p(I)$. We shall consider two cases: B of finite dimensional range, and B a multiplication operator.

Case 5—(i). B is of finite dimensional range.

Suppose that the dimension of the range of B is d. Thus we can find a $d \times 1$ column vector function $\psi(s)$ whose d rows are in $L_p(I)$ and linearly independent there, and a $d \times 1$ column vector function $\tilde{\psi}$ whose d rows are in $L_q(I)$ (the d rows need not be linearly independent) such that

$$(5.20)\quad By = (y|\tilde{\psi}^T)\psi,\quad y \in L_p(I).$$

Let us define a $1 \times (N/2)$ matrix function $S(t, \lambda)$ defined for $t \in I$ and a function $K(t, s, \lambda)$ defined for $(t, s) \in I \times I$ as follows: If $t \in I_j$ then $S(t, \lambda)$ is the $1 \times (N/2)$ matrix function

$$(0, \cdots, 0, S_j(t, \lambda), 0, \cdots, 0)\quad \text{if } t \in I_j,$$

where there are $n_1 + \cdots + n_{j-1}$ zeros before S_j and $n_{j+1} + \cdots + n_l$ zeros after; $S_j(t, \lambda)$ is a $1 \times n_j$ fundamental matrix solution of the differential equation $\tau_j y = \lambda y$ on the interval I_j. Let $\tilde{S}_j(s, \lambda)$ denote the cofactor of the row vector $S_j(t, \lambda)$ in the matrix (cf. section 1):

$$\begin{bmatrix} S_j(s, \lambda) \\ S_j'(s, \lambda) \\ \cdot \\ \cdot \\ \cdot \\ S_j^{(n_j-2)}(s, \lambda) \\ S_j(t, \lambda) \end{bmatrix}.$$

Then $K(t, s, \lambda)$ is defined as follows:

$$\begin{aligned} K(t, s, \lambda) &= \frac{S_j(t, \lambda)\tilde{S}_j^T(s, \lambda)}{P_{j0}(s)W(S_j(s, \lambda))} \quad \text{if } (t, s) \in I_j \times I_j \text{ and } s < t. \\ &= 0 \quad \text{if } (t, s) \in I_j \times I_j \text{ and } s > t. \\ &= 0 \quad \text{if } (t, s) \in I_k \times I_j \text{ and } k \neq j. \end{aligned}$$

Here $W(S_j(s, \lambda))$ is the Wronskian of $S_j(s, \lambda)$.

Then using the variation constants formula, (5.19) can be rewritten as:

$$\begin{aligned} (5.21) \quad y(t) &= S(t, \lambda)b - (K(t, \cdot, \lambda)|\psi^*)(y|\tilde{\psi}) \\ &\quad - (K(t, \cdot, \lambda)|\chi^*)[A(\tau y|\phi) + \iota DV(y)] + (K(t, \cdot, \lambda)|\tilde{f}) \end{aligned}$$

for $t \in I$ where b is an $(n_1 + n_2 + \cdots + n_l) \times 1$ constant column vector depending only on y. First let us define an $(M + N + d + \tilde{m} + r) \times (N/2 + N + d + \tilde{m} + r)$ constant matrix $\Lambda(\lambda) = (\Lambda_{kj}(\lambda))$ with the (k, j)-matrix entry $\Lambda_{kj}(\lambda)$ as follows:

$$\Lambda_{11}(\lambda) = -V(S(\cdot, \lambda)), \ \Lambda_{12}(\lambda) = I_N + \iota V\left(\int_I K(\cdot, s, \lambda)\chi^T(s)ds\right)D,$$

$$\Lambda_{13}(\lambda) = V\left(\int_I K(\cdot, s, \lambda)\chi^T(s)ds\right)A,$$

$$\Lambda_{14}(\lambda) = V\left(\int_I K(\cdot, s, \lambda)\psi^T(s)ds\right),$$

$$\Lambda_{15}(\lambda) = 0_{N\times\tilde{m}}, \quad \Lambda_{21}(\lambda) = 0_{M\times N/2}, \quad \Lambda_{22}(\lambda) = P, \quad \Lambda_{23}(\lambda) = 0_{M\times r},$$

$$\Lambda_{24}(\lambda) = 0_{M\times d}, \quad \Lambda_{25}(\lambda) = PC^{-1}(\tau)\tilde{D}^*, \quad \Lambda_{31}(\lambda) = -\overline{(\tilde{\psi}(\cdot)|S(\cdot, \lambda))},$$

$$\Lambda_{32}(\lambda) = \iota\overline{(\tilde{\psi}|(K(\lambda)|\chi^*))}D, \quad \Lambda_{33} = \overline{(\tilde{\psi}|(K(\lambda))|\chi^*))}A,$$

$$\Lambda_{34}(\lambda) = I_d + \overline{(\tilde{\psi}|(K(\lambda)|\psi^*))}, \ \Lambda_{35}(\lambda) = 0_{d\times\tilde{m}}, \ \Lambda_{41}(\lambda) = -\overline{(\tilde{\chi}|S(\lambda))},$$

$$\Lambda_{42}(\lambda) = \iota\overline{(\tilde{\chi}|(K(\lambda)|\chi^*))}D, \quad \Lambda_{43}(\lambda) = \overline{(\tilde{\chi}|(K(\lambda)|\chi^*))}A,$$

$$\Lambda_{44}(\lambda) = \overline{(\tilde{\chi}|(K(\lambda)|\psi^*))}, \quad \Lambda_{45}(\lambda) = I_{\tilde{m}}, \quad \Lambda_{51}(\lambda) = -\lambda\overline{(\phi|S(\lambda))},$$

$$\Lambda_{52}(\lambda) = \iota\overline{\left(\phi|\int_I \chi^T(s)\tau_t K(\cdot, s, \lambda)ds\right)}D,$$

$$\Lambda_{53}(\lambda) = I_r + \overline{\left(\phi|\int_I \psi^T(s)\tau_t K(\cdot, s, \lambda)ds\right)}A,$$

$$\Lambda_{54}(\lambda) = \overline{\left(\phi|\int_I \psi^T(s)\tau_t K(\cdot, s, \lambda)ds\right)}, \quad \Lambda_{55}(\lambda) = 0_{r\times\tilde{m}}.$$

Here $\tau_t K(t, s, \lambda) = \tau k(t)$ with $k(t) = K(t, s, \lambda)$. If we put $g(t) = (f|\bar{K}(t, \cdot, \lambda))$, then we have the following $M + d + r + N + \tilde{m}$ equations with $(N/2 + d + r + N + \tilde{m})$ unknowns:

(i) $\Lambda_{31}(\lambda)b + \Lambda_{32}(\lambda)V(y) + \Lambda_{33}(\lambda)(\tau y|\phi) + \Lambda_{34}(\lambda)(y|\tilde{\psi}) = (g|\tilde{\psi})$.

(ii) $\Lambda_{11}(\lambda)b + \Lambda_{12}(\lambda)V(y) + \Lambda_{13}(\lambda)(\tau y|\phi) + \Lambda_{14}(\lambda)(y|\tilde{\psi}) = V(g)$.

(iii) $\Lambda_{51}(\lambda)b + \Lambda_{52}(\lambda)V(y) + \Lambda_{53}(\lambda)(\tau y|\phi) + \Lambda_{54}(\lambda)(y|\tilde{\psi}) = \tau(g|\phi)$.

(iv) $\Lambda_{22}(\lambda)V(y) + \Lambda_{25}(\lambda)(y|\tilde{\chi}) = 0$.

(v) $\Lambda_{41}(\lambda)b + \Lambda_{42}(\lambda)V(y) + \Lambda_{43}(\lambda)(\tau y|\phi)$
$+ \Lambda_{44}(\lambda)(y|\tilde{\psi}) + \Lambda_{45}(\lambda)(y|\tilde{\chi}) = (g|\tilde{\chi})$.

The equations (i)–(v) can be rewritten

$$\Lambda(\lambda)\begin{bmatrix} b \\ V(y) \\ (\tau y|\phi) \\ (y|\tilde{\psi}) \\ (y|\tilde{\chi}) \end{bmatrix} = \begin{bmatrix} V(g) \\ 0_{M\times 1} \\ (g|\tilde{\psi}) \\ (g|\tilde{\chi}) \\ (\tau g|\phi) \end{bmatrix}.$$

Thus we have

THEOREM 5.17. *Let T be the same as in* (5.19) *and B be the same one as in* (5.20).

(i) *If $M < N/2$ then any $\lambda \in \mathbf{C}$ is an eigenvalue for T.*

(ii) *If $M = N/2$ then $\lambda \in \mathbf{C}$ is not an eignevalue for T if and only if* det $\Lambda(\lambda) \neq 0$. *In this case the inverse operator $R_\lambda = (T - \lambda)^{-1}$ is a compact integral operator defined everywhere in $L_p(I)$ and is given by*

$$(R_\lambda f)(t) = \int_I K(t, s, \lambda)f(s)ds +$$
$$- \begin{bmatrix} -S(t, \lambda)^T \\ ((K(t, \cdot, \lambda)|\chi^*)\iota D)^T \\ ((K(t, \cdot, \lambda)|\chi^*)A)^T \\ ((K(t, \cdot, \lambda)|\psi^*))^T \\ 0_{\tilde{m}\times 1} \end{bmatrix}^T \Lambda^{-1}(\lambda) \begin{bmatrix} \int_I V(K(\cdot, s, \lambda))f(s)ds \\ 0_{M\times 1} \\ (f|\overline{(K(\lambda)|\tilde{\psi})}) \\ (f|\overline{(K(\lambda)|\tilde{\chi})}) \\ \lambda(f|\overline{(K(\lambda)|\phi)}) \end{bmatrix}$$

Proof. (i) and (ii) are an obvious consequence of (5.22). The assertion concerning the form R_λ follows from (5.21) and (5.22), since $\tau g = \lambda g$ and

$$V\left(\int_I K(\cdot, s, \lambda)f(s)ds\right) = \int_I V(K(\cdot, s, \lambda))f(s)ds.$$

Thus R_λ is an integral operator with a kernel of Hilbert-Schmidt type, and thus a compact operator. This completes the proof.

Case 5—(ii). *By* = *hy*.

Here h is Lebesgue measurable and essentially bounded in I. To investigate this case, we merely define another $\tilde{\tau}$ by $\tilde{\tau}y = \tau y + hy$. Thus the case 5—(ii) can be reduced to case 5—(i).

If we allow the matrices A, D, $\tilde{D}$ to vanish identically and $B \equiv 0$, then we get ordinary regular multiple boundary problems (in fact, more general than that because we deal with direct sum operators). Let $\theta(\lambda)$ ($\lambda \in \mathbf{C}$) denote the $(N + M) \times (N + N/2)$ constant matrix

$$\begin{bmatrix} -V(S(\cdot, \lambda)), & I_N \\ 0_{M\times N/2}, & P_{M\times N} \end{bmatrix}$$

where P is an $M \times N$ constant matrix of rank $M \leqq N$. Then the previous theorem yields

Remark. Suppose $1 \leqq p \leqq \infty$ and T is the operator defined by $Ty = \bigoplus_1^l \tau_j y$ on $\{y | y \in \bigoplus_1^l \mathscr{D}_1(\tau_j, p, I_j), PV(y) = 0\}$.

(I) *If* $M < N/2$, then any point $\lambda \in \mathbf{C}$ is an eigenvalue for T.

(II) If $M = N/2$, then λ is not an eigenvalue for T if and only if $\det \theta(\lambda) \neq 0$, and in this case the inverse operator $(T - \lambda)^{-1}$ is given by:

$$(T - \lambda)^{-1}f = (S(t, \lambda) : 0_{1\times N})\theta^{-1}(\lambda)\begin{bmatrix} \int_I V(K(\cdot, s, \lambda))f(s)ds \\ 0_{M\times 1} \end{bmatrix} + \int_I K(t, s, \lambda)f(s)ds.$$

6. Resolvents of self-adjoint perturbations. The variation of parameter method used in § 5 is not satisfactory for computing resolvents for singular cases. Here we use a perturbation technique combined with the variation of parameter method. The idea came from discussions with R. R. D. Kemp. In the case when $l = 1$, Coddington and Dijksma [**7**] have obtained generalized resolvents of self-adjoint subspace extensions, which depend on the kernels obtained by approximating self-adjoint operators on compact intervals (see § 6 in [**7**], § 5 in [**6**], p. 179 in [**4**]; see also [**23**]). Here we do not use such approximations.

In this section we assume that $p = q = 2$, $\tau_j = \tau_j^*$ for each j.

Thus $\tilde{V}_j = V_j$. Let z be an arbitrary but fixed non-real complex number. Let c_j be a fixed point in $I_j^0 = (a_{j-1}, a_j)$. Thus $-\infty \leqq a_0 < a_l \leqq \infty$. Let $\gamma_j(z) = \dim \mathscr{N}(T_1(\tau_j, 2, I_j) - zI)$, $\alpha_j(z) = \dim \mathscr{N}(T_1(\tau_j, 2, (a_{j-1}, c_j)) - zI)$ and $\beta_j(z) = \dim \mathscr{N}(T_1(\tau_j, 2, (c_j, a_j)) - zI)$. Here $\mathscr{N}$ denotes "the null space of" and I denotes the identity operator. Let $S_j(t, z)$ denote the $1 \times n_j$ fundamental matrix solution

$$(S_{j1}(t, z), S_{j2}(t, z), S_{j3}(t, z)), \quad t \in I_j$$

of the differential equation $\tau_j y = zy$ in the interval I_j such that (i) $S_{j1}(t, z)$ is

https://doi.org/10.4153/CJM-1978-053-7 Published online by Cambridge University Press

a $1 \times \gamma_j(z)$ row vector with entries in $L_2(I_j)$, (ii) $S_{j2}(t, z)$ is a $1 \times (\alpha_j(z) - \gamma_j(z))$ row vector with entries in $L_2(a_{j-1}, c_j)$, (iii) $S_{j3}(t, z)$ is a $1 \times (n_j - \alpha_j(z))$ row vector with entries in $L_2(c_j, a_j)$. This can obviously be done, because $\alpha_j(z) + \beta_j(z) = \gamma_j(z) + n_j$ (see, for example, [**22**]). Let $\tilde{S}_{j1}(s, z)$, $\tilde{S}_{j2}(s, z)$ and $\tilde{S}_{j3}(s, z)$ denote the cofactors of the row vectors $S_{j1}(t, z)$, $S_{j2}(t, z)$ and $S_{j3}(t, z)$ respectively in the matrix:

$$\begin{bmatrix} S_j(s, z) \\ S_j'(s, z) \\ \cdot \\ \cdot \\ \cdot \\ S_j^{(n_j-2)}(s, z) \\ S_j(t, z) \end{bmatrix}.$$

Let $S(t, z)$ denote the $1 \times (\gamma_1(z) + \cdots + \gamma_l(z))$ row vector defined for $t \in I$ by:

$$S(t, z) = (0, \cdots, 0, S_j(t, z), 0, \cdots, 0) \quad \text{if } t \in I_j.$$

where there are $\gamma_1(z) + \cdots + \gamma_{j-1}(z)$ zeros before S_j, and $\gamma_{j+1}(z) + \cdots + \gamma_l(z)$ zeros after. Note here that this $S(t, z)$ is different from that in § 5. Let $\mathscr{K}(t, s, z)$ be a function of $(t, s) \in I \times I$ defined as follows:

$$\mathscr{K}(t, s, z) = \frac{S_{j1}(t, z)\tilde{S}_{j1}{}^T(s, z) + S_{j3}(t, z)\tilde{S}_{j3}{}^T(s, z)}{P_{j0}(s)W(S_j(s, z))}$$

$$\text{if } (t, s) \in I_j \times I_j,\ s < t.$$

$$= \frac{-S_{j2}(t, z)\tilde{S}_{j2}{}^T(s, z)}{P_{j0}(s)W(S_j(s, z))} \quad \text{if } (t, s) \in I_j \times I_j,\ s > t.$$

$$= 0 \quad \text{if } (t, s) \in I_k \times I_j,\ k \neq j.$$

Suppose now that T is the self-adjoint operator defined by

$$\text{(6.22)} \quad Ty = \tau y + By + \iota\chi^T DV(y),$$
$$D(T) = \{y \in \mathscr{D}_1(\tau, 2, I): P[V(y) + C^{-1}(\tau)D^*(y|\chi)] = 0\}.$$

Here P is an $M \times N$ $(N = 2M)$ constant matrix of rank M such that $PC^{-1}(\tau)P^* = 0$, and B satisfies:

$$(B^* - B)y = \iota\chi^T DC^{-1}(\tau)(y|D^T\chi), \quad y \in L_2(I).$$

Thus by Theorem 3.11, $T = T^*$. We shall compute $(T - z)^{-1}$.

Let us define an operator $\hat{T}$ by

$$\hat{T}y = \tau y, \quad \mathscr{D}(\hat{T}) = \{y \in \mathscr{D}_1(\tau, 2, I): PV(y) = 0\}.$$

This is a self-adjoint operator because $PC^{-1}(\tau)P^* = 0$, and $N = 2M$. First we note that

$$\sum_{j=1}^{l} \gamma_j(z) = \sum_{j=1}^{l} \gamma_j(\bar{z}) = M = N/2.$$

This follows from the fact that $\hat{T}$ is a self-adjoint extension of the direct sum operator $\bigoplus_1^l T_0(\tau_j, 2, I_j)$, and the fact that

$$\left(\sum_{j=1}^{l} \gamma_j(z_0), \sum_{j=1}^{l} \gamma_j(\bar{z}_0)\right) \quad (\operatorname{Im} z > 0)$$

are the deficiency indices of the direct sum operator. First we shall compute $\hat{R}_z = (\hat{T} - z)^{-1}$ and then use this to compute $R_z = (T - z)^{-1}$.

Take any $f \in L_2(I)$ such that for each I_j, $f|_{I_j}$ has compact support in I_j^0, and let $g = \hat{R}_z f$. Then, using the same method as in [**23**], we see that

$$g = S(t, z)b + \int_I \mathscr{K}(t, s, z)f(s)ds$$

for some $(\gamma_1(z) + \cdots + \gamma_l(z)) \times 1$ constant column vector b. Since $b|_{I_j}$ has compact support in I_j^0,

$$V\left(\int_I \mathscr{K}(\cdot, s, z)f(s)ds\right) = \int_I V(\mathscr{K}(\cdot, s, z))f(s)ds.$$

Hence, since $g \in \mathscr{D}(\hat{T})$,

$$PV(S(\cdot, z))b = -\int_I V(\mathscr{K}(\cdot, s, z))f(s)ds.$$

The $M \times M$ matrix $PV(S(\cdot, z))$ $(N = 2M)$ cannot be singular. Therefore, if we define

$$\hat{\mathscr{G}}(t, s, z) = \mathscr{K}(t, s, z) - S(t, z)(PV(S(\cdot, z)))^{-1}V(\mathscr{K}(\cdot, s, z))$$

for $(t, s) \in I \times I$, then

$$g = \hat{R}_z f = \int_I \hat{\mathscr{G}}(t, s, z)f(s)ds \tag{6.23}$$

for every $f \in L_2(I)$ such that for each I_j, $f|_{I_j}$ has compact support on I_j^0. It is easy to see that the above also holds for every $f \in L_2(I)$ (cf. Lee [**23**]).

We now find $(T - z)^{-1}$. As before, take any $f \in L_2(I)$ such that on each $I_j, f|_{I_j}$ has compact support on I_j^0 and put $g = (T - z)^{-1}f$. Thus $(T - z)g = f$. Therefore

$$(\tau - z)g = f - Bg - \iota\chi^T(t)DV(g).$$

Thus, since $(\tau - z)g \in L_2(I)$,

$$\hat{R}_z(\tau - z)g = \hat{R}_z f - (\hat{R}_z \circ B)g - \iota(\hat{R}_z\chi^T)DV(g).$$

Let $\hat{R}_z(\tau - z)g = h$. Thus $(\tau - z)(h - g) = 0$. Since $h - g \in L_2(I)$ we must have $h - g = S(t, z)b$ for some $(\gamma_1(z) + \cdots + \gamma_l(z)) \times 1$ constant column vector b. Therefore

$$(6.24)\quad g = -S(t, z)b + \hat{R}_z f - (\hat{R}_z \circ B)g - \iota(\hat{R}_z\chi^T)DV(g).$$

We must determine b, $(\hat{R}_z \circ B)g$ and $V(g)$ in the above expressions. The operator B is too general to handle. Thus, as in § 5, we assume that B is of finite dimensional range. Assume further that B has the form as (5.20), where in this case $p = 2$. Thus (6.24) can be rewritten:

$$(6.25)\quad g = -S(t, z)b + \hat{R}_z f - (g|\tilde{\psi}^T)(\hat{R}_z\psi) - \iota(\hat{R}_z\chi^T)DV(g).$$

Let us define the $(M + N + d + m) \times (M + N + d + m)$ matrix $\delta(z) = (\delta_{kj}(z))$ with matrix (k, j)-entry $\delta_{kj}(z)$ as follows:

$$\delta_{11}(z) = P,\quad \delta_{12}(z) = 0_{M\times M},\quad \delta_{14}(z) = 0_{M\times d},\quad \delta_{13}(z) = PC^{-1}(\tau)D^*,$$
$$\delta_{21}(z) = I_N + \iota V(\hat{R}_z\chi^T)D,\quad \delta_{22}(z) = V(S(\cdot, z)),$$
$$\delta_{23}(z) = 0_{N\times m},\quad \delta_{24}(z) = V(\hat{R}_z\psi^T),$$
$$\delta_{31}(z) = (\hat{R}_z\chi|\tilde{\psi}^T)^T D,\quad \delta_{32}(\bar{z}) = \overline{(\tilde{\psi}|S(\cdot, z))},$$
$$\delta_{33}(z) = 0_{d\times m},\quad \delta_{34}(z) = (\hat{R}_z\psi|\tilde{\psi}^T) + I_d,$$
$$\delta_{41}(z) = I_m + \overline{\iota(\chi|(\hat{R}_z\chi^T))}D^T,$$
$$\delta_{42}(z) = \overline{(\chi|S(\cdot, z))},\quad \delta_{43}(z) = I_m,$$
$$\delta_{44}(z) = \overline{(\chi|\hat{R}_z\psi^T)}.$$

Then using (6.25) and in view of $\mathscr{D}(T)$ we have the following equations:

$$\delta_{11}(z)V(g) + \delta_{13}(z)(g|\chi) = 0.$$
$$\delta_{21}(z)V(g) + \delta_{22}(z)b + \delta_{24}(z)(g|\tilde{\psi}) = V(\hat{R}_z f).$$
$$\delta_{31}(z)V(g) + \delta_{32}(z)b + \delta_{34}(z)(g|\tilde{\psi}) = (\hat{R}_z f|\tilde{\psi}).$$
$$\delta_{41}(z)V(g) + \delta_{42}(z)b + \delta_{43}(z)(g|\chi) + \delta_{44}(z)(g|\tilde{\psi}) = (\hat{R}_z f|\chi).$$

The above equation implies that $\delta(z)$ cannot be singular. If we define

$$Q(s) = V(\hat{\mathscr{G}}(\cdot, s, z)),$$

then, since each $f|_{I_j}$ has compact support in I_j^0, we see that $V(\hat{R}_z f) = (f|\bar{Q})$. Thus (6.25) can be rewritten

$$(6.26)\quad R_z f = (\hat{R}_z f)(t)$$
$$- [\iota(\hat{R}_z\chi^T)(t)D\colon S(t, z)\colon 0_{1\times m}\colon (\hat{R}_z\psi^T)(t)]\delta^{-1}(z)\begin{bmatrix} 0_{M\times 1} \\ (f|\bar{Q}) \\ (f|(\hat{R}_z)^*\tilde{\psi} \\ (f|(\hat{R}_z)^*\chi \end{bmatrix}$$

https://doi.org/10.4153/CJM-1978-053-7 Published online by Cambridge University Press

Let us define a function $\mathscr{G}(t, s, z)$ for $(t, s) \in I \times I$ by

$$(6.27) \quad \mathscr{G}(t, s, z) = \hat{\mathscr{G}}(t, s, z) - [\iota(\hat{R}_z\chi^T)(t)D \colon S(t, z) \colon 0_{1\times m} \colon (\hat{R}_z\psi^T)(t)]\delta^{-1}(z)\begin{bmatrix} 0_{M\times 1} \\ \overline{Q(s)} \\ \overline{((\hat{R}_z)^*\tilde{\psi})}(s) \\ \overline{((\hat{R}_z)^*\chi)}(s) \end{bmatrix}$$

Then

$$(6.28) \quad R_z f = \int_I \mathscr{G}(t, s, z) f(s) ds$$

for every $f \in L_2(I)$ such that each $f|_{I_j}$ has compact support in I_j^0. Since $T = T^*$, we can show that $\mathscr{G}(t, s, z) = \hat{\mathscr{G}}(s, t, \bar{z})$. From this we can conclude that (6.28) holds for every $f \in L_p(I)$. Thus we have the following.

THEOREM 6.18. *If B is the same as in* (5.20) *and if T is the self-adjoint operator in* (6.22) *then the resolvent R_z of T is an integral operator with the kernel $\mathscr{G}(t, s, z)$ given in* (6.27).

Remark. We do not treat expansion theorems in this paper. In the case when $l = 1$, $\tau_1 = \tau_1^*$ and B is of finite dimensional range, the corresponding expansion theorems are given in Dijksma and De Snoo [**9**], where point-wise convergence is allowed.

Acknowledgement. I am very grateful to Professor J. W. Macki for many helpful discussions of these results.

REFERENCES

1. N. I. Akhiezer, and I. M. Glazman, *Theory of linear operators in Hilbert spaces*, Vol. II (Frederick Ungar Publ., New York, 1963).

2. F. V. Atkinson, *Discrete and continuous boundary problems* (Academic Press, New York, 1964).

3. R. C. Brown, *The operator theory of generalized boundary value problems*, Can. J. Math. *28*, (1976), 486–512.

4. E. A. Coddington, *The spectral matrix and Green's function for singular self-adjoint boundary value problems*, Can. J. Math. *6* (1954), 169–185.

5. ——— *Self-adjoint subspaces extensions of nondensely defined symmetric operators*, Advances in Math. *14* (1974), 309–332.

6. ——— *Self-adjoint problems for nondensely defined ordinary differential operators and their eigenfunction expansions*, Advances in Math. *15* (1975), 1–40.

7. E. A. Coddington, and A. Dijksma, *Self-adjoint subspaces and eigenfunction expansions for ordinary differential subspaces*, J. Differential Equations *20* (1976), 473–526.

8. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations* (McGraw-Hill, New York, 1955).

9. A. Dijksma and H. S. V. DeSnoo, *Eigenfunction expansions for non-densely defined differential operators*, J. Differential Equations *17* (1975), 198–219.

10. N. Dunford and J. T. Schwartz, *Linear operators*, Part II (Interscience Publ., New York, 1963).

11. S. Goldberg, *Unbounded linear operators; Theory and application* (McGraw-Hill, New York, 1966).

12. I. Halperin and H. R. Pitt, *Integral inequalities connected with differential operators*, Duke Math. J. *4* (1938), 613–625.

13. C. S. Hönig, *The Green's function of a linear differential equation with a lateral condition*, Bull. Amer. Math. Soc. *79* (1973), 587–593.

14. ——— *Volterra-Stieltjes integral equations with linear constraints and discontinuous solutions*, Bull. Amer. Math. Soc. *81* (1975), 593–598.

15. ——— *Volterra-Stieltjes integral equations*, North-Holland Mathematics Studies Series *16* (North-Holland Publ., New York, 1975).

16. R. R. D. Kemp, *On a class of singular differential operators*, Can. J. Math. *13* (1961), 316–330.

17. R. R. D. Kemp and S. J. Lee, *Finite dimensional perturbations of differential expressions*, Can. J. Math. *28* (1976), 1082–1104.

18. T. B. Kim, *The adjoint of a differential-boundary operator with an integral boundary condition on a semi-axis*, J. Math. Anal. Appl. *44* (1973), 436–446.

19. A. M. Krall, *The development of general differential and general differential boundary systems*, Rocky Mountain J. Math. *5* (1975), 493–542.

20. ——— *Stieltjes differential-boundary operators III, Multivalued operators—linear relations*, Pacific J. Math. *59* (1975), 125–133.

21. ——— *Nth order Stieltjes differential-boundary operators and Stieltjes-boundary systems*, J. Differential Equations (to appear).

22. S. J. Lee, *On boundary conditions for ordinary linear differential operators*, J. London Math. Soc. (2) *12* (1976), 447–454.

23. ——— *A note on generalized resolvents for ordinary differential operators*, Proc. Amer. Math. Soc. *57* (2) (1976), 279–282.

24. ——— *Operators generated by countably many differential operators*, J. Diff. Equations, to appear.

25. G. C. Rota, *Extension theory of differential operators I*, Comm. Pure and Appl. Math. *11* (1958), 23–65.

26. M. Tvrdý, *Boundary value problems for linear generalized differential equations and their adjoints*, Czechoslovak Math. J. *23* (1973), 183–217.

27. ——— *Linear boundary value type problems for functional-differential equations and their adjoints*, Czechoslovak Math. J. *25* (1975), 37–65.

28. M. Tvrdý and O. Vejvoda, *General boundary value problem for an integrodifferential system and its adjoints*, Câs. Pês. Matematiky *97* (1972), 399–419.

University of Alberta,
Edmonton, Alberta