Interval Pattern Avoidance for Arbitrary Root Systems

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Abstract. We extend the idea of interval pattern avoidance defined by Yong and the author for S_n to arbitrary Weyl groups using the definition of pattern avoidance due to Billey and Braden, and Billey and Postnikov. We show that, as previously shown by Yong and the author for GL_n, interval pattern avoidance is a universal tool for characterizing which Schubert varieties have certain local properties, and where these local properties hold.

1 Introduction

The purpose of this brief note is to extend the notions of interval pattern embedding and avoidance introduced by Yong and the author in type A [21] to Schubert varieties of arbitrary Lie type. This extension is the natural common generalization of the definition in type A and the definition of pattern avoidance coming from root subsystems, as introduced combinatorially by Billey and Postnikov [4] and explained geometrically via the pattern map by Billey and Braden [3]. (In type A, the pattern map was also implicit in work of Bergeron and Sottile [1].)

The main reason for our definition of interval pattern avoidance is that it gives a universal tool for describing local properties on Schubert varieties, in the sense that the set of points on all Schubert varieties satisfying any given local property (except for dimension) has a characterization using only interval pattern avoidance. The main example of such a property for which results are known is smoothness. The Schubert varieties which are smooth everywhere can be characterized by ordinary pattern avoidance [2, 4, 14]. The locus of singular points in any Schubert variety of type A was described independently in several papers [5, 10, 12, 16]; this description can be easily reformulated in terms of interval pattern embeddings [21, Theorem 6.1]. One purpose for our more general formulation of interval pattern embeddings is to provide the appropriate language for a similar description of the singular locus in other Lie types.

For geometric properties other than smoothness, ordinary pattern avoidance is sometimes insufficient even for characterizing which Schubert varieties have the given property globally. For example, the Schubert varieties which are everywhere Gorenstein [20] or everywhere factorial [7] cannot be characterized by ordinary pattern avoidance. However, Schubert varieties having these properties, or any local property preserved under products with affine space, can be characterized using interval pattern avoidance.

This universality is demonstrated by showing that interval pattern embeddings give an isomorphism of slices of different Schubert varieties. This isomorphism is proven using the pattern map of Billey and Braden; when written in coordinates, the proof becomes essentially the same as the one previously given for type A. However, this new proof shows the isomorphism extends to the Richardson varieties which are the closures of the slices, which is a new result even in type A.

2 Pattern Avoidance and Interval Pattern Avoidance

Fix a ground field $\Bbbk$. Let G be a connected semisimple linear algebraic group over $\Bbbk$, B a fixed Borel subgroup, and $T \subseteq B \subseteq G$ a maximal torus. Let Φ be the roots of G under the action of T, Φ_+ and Φ_- the positive and negative roots corresponding to our choice of Borel subgroup, and Δ_+ the set of simple positive roots. Let V be the inner product space spanned by the root lattice. The Weyl group W of G is the group generated by the reflections $\{s_\alpha \in \text{End}(V) \mid \alpha \in \Phi\}$, where $s_\alpha(v) = v - 2(v, \alpha)/(\alpha, \alpha)\alpha$. The *length* $\ell(w)$ of an element in a Weyl group is the minimal length of any expression $w = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_k}$, where each β_j is a simple root. The Weyl group can also be recovered from G as the group $N(T)/T$. Pattern avoidance depends not only on the abstract Weyl group but also on the root system it acts on; to emphasize this, in the remainder of the paper, we denote a Weyl group by the triple (W, Φ, V).

The variety G/B is known as the *flag variety*. The group G acts on G/B via multiplication on the left. To each element of $u \in (W, \Phi, V)$ (here considered as $N(T)/T$) we can associate the T-fixed point $e_u := uB$, and these are all the T-fixed points of G/B. There is a *Bruhat decomposition* of G/B into *Schubert cells* $X_w^\circ := Be_w B/B$, one for each $w \in W$, and the *Schubert variety* X_w is the closure of the Schubert cell X_w°. There is also a decomposition of G/B into *opposite Schubert cells* $\Omega_w^\circ = B_- e_w B/B$, where B_- is the Borel subgroup opposite to B; the closure of the opposite Schubert cell Ω_w° is called an *opposite Schubert variety* and is denoted Ω_w. The *Richardson variety* X_v^u is the intersection of Ω_u and X_v; Richardson showed that it is reduced and irreducible (when nonempty) [19]. The dimension of X_w and the codimension in G/B of Ω_w are both $\ell(w)$. The dimension of X_v^u is $\ell(v) - \ell(u)$.

The Schubert variety X_w is a union of Schubert cells. We define the *Bruhat order* on (W, Φ, V) by declaring that $u \leq v$ if $X_u^\circ \subseteq X_v$. Alternatively, Bruhat order can be defined combinatorially by declaring it to be the reflexive transitive closure of the relation $\prec$ under which $u \prec v$ if both $u = s_\alpha v$ for some $\alpha \in \Phi$ and $\ell(u) < \ell(v)$. This combinatorial definition has a geometric explanation; when u and v are so related, the curve $\overline{U_\alpha \cdot e_v}$ is a $\mathbb{P}^1$ inside X_v connecting e_u and e_v. Here, U_α is the root subgroup of B corresponding to the root α. The Richardson variety X_v^u is nonempty if and only if $u \leq v$.

Now we recall the definitions of pattern embeddings and pattern avoidance found in [3], [4]. Let (W', Φ', V') and (W, Φ, V) be Weyl groups. A *subsystem embedding i* of (W', Φ', V') into (W, Φ, V) is an embedding of V' as a subspace of V so that $\Phi' \cong \Phi \cap i(V')$; this induces an embedding of W' into W as the subgroup generated by the reflections $\{s_\alpha \mid \alpha \in i(\Phi')\}$.

Define the *flattening map* ϕ_i from (W, Φ, V) to (W', Φ', V') as follows. An ele-

ment $w \in (W, \Phi, V)$ is uniquely determined by its inversion set $I(w) = \Phi_+ \cap w(\Phi_-)$. Therefore we can define $\phi_i(w)$ as the element of (W', Φ', V') whose inversion set is $i^{-1}\big(I(w) \cap i(\Phi'^+)\big)$. Then i *(pattern) embeds* $v \in (W', \Phi', V')$ in $w \in (W, \Phi, V)$ if $\phi_i(w) = v$. The Weyl group element w is said to *(pattern) avoid* v if $\phi_i(w) \neq v$ for every embedding i of (W', Φ', V') into (W, Φ, V).

Our definition of interval pattern avoidance is now as follows. Let $u \leq v \in (W', \Phi', V')$ and $x \leq w \in (W, \Phi, V)$, where $\leq$ denotes the Bruhat order. Let i be a subsystem embedding of (W', Φ', V') into (W, Φ, V). We say i *(interval pattern) embeds* $[u, v]$ *in* $[x, w]$ if the following three conditions are all satisfied.

(i) $\phi_i(w) = v$ and $\phi_i(x) = u$.

(ii) x and w are in the same right $i(W')$ coset.

(iii) $[u, v]$ and $[x, w]$ are isomorphic as intervals in Bruhat order.

The third condition implies in particular that $\ell(v) - \ell(u) = \ell(w) - \ell(x)$. This equality in lengths is actually sufficient to imply the third condition, given the first two; a combinatorial proof of this fact is possible, but the geometry below also shows it.

Note that the first two conditions imply that $x = i(uv^{-1})w$. Since x is determined by u, v, w, and i, we will say that w *(interval pattern) avoids* $[u, v]$ if, for every subsystem embedding i of (W', Φ', V') into (W, Φ, V), $[u, v]$ does not embed in $[i(uv^{-1})w, w]$.

In Type A, where W and W' are respectively S_n and S_m for some $n \geq m$, arguments of Billey and Braden [3, Section 2.3] show that the notion of interval pattern avoidance found here is the same (up to the Dynkin diagram automorphism realized by conjugation by the longest element w_0) as the previous definition of Yong and the author [21]. When $u = v$ and $x = w$, our notion is the same as that introduced by Billey and Postnikov [4].

3   Main Theorem and Corollary

Our main theorem can now be stated as follows.

Theorem 3.1 *Suppose there is some subsystem embedding i which embeds $[u, v]$ in $[x, w]$. Then the Richardson varieties X_v^u and X_w^x are isomorphic. This isomorphism sends $\Omega_\sigma^\circ \cap X_\tau^\circ$ to $\Omega_{\phi_i(\sigma)}^\circ \cap X_{\phi_i(\tau)}^\circ$ for every $\sigma, \tau \in [x, w]$.*

The main application of this theorem we have in mind is to the study of singularities of Schubert varieties. Call a local property $\mathcal{P}$ *semicontinuously stable* if it is preserved under products with affine space, and the $\mathcal{P}$-locus on any variety is closed. Examples include being singular, being non-Gorenstein, or having multiplicity greater than some fixed number k. In addition, the dimension of the i-th local intersection homology group and hence the i-th coefficient of the Kazhdan–Luzstig polynomial being greater than a fixed number k is not in general semicontinuously stable, but behaves as such on Schubert varieties by a theorem of Irving [11].

Now define a poset on the set of all intervals in all Weyl groups (where, as throughout, the root system is considered part of the data of the Weyl group) by taking the reflexive transitive closure of the following two relations.

(i) $[u, v] \prec [x, w]$ if there is some embedding of $[u, v]$ into $[x, w]$.

(ii) $[u, v] \prec [u', v]$ if $u \leq u'$.
   Now we can state our corollary.

Corollary 3.2   *Let $\mathcal{P}$ be a semicontinuously stable property. Then the set of intervals such that $\{[u, v] \mid \mathcal{P}$ holds at e_u on $X_v\}$ is an upper order ideal on the aforementioned poset. The set $\{w \mid \mathcal{P}$ holds on no points of $X_w\}$ is the set of w avoiding some list of intervals $[u, v]$.*

Notice that this corollary holds separately for different ground fields, in that the order ideal for the same property may depend on $\Bbbk$. The list of intervals to be avoided may be infinite, but we hope that for any particular property it has a nice form.

Proof   The point e_u has a neighborhood $u \cdot \Omega_{\mathrm{id}}^\circ$ in G/B, so $u \cdot \Omega_{\mathrm{id}}^\circ \cap X_v$ is a neighborhood of e_u on X_v. This neighborhood is isomorphic to $(\Omega_u^\circ \cap X_v) \times \mathbb{A}^{\ell(u)}$ [13, Lemma A.4]. Therefore, any semicontinuously stable property $\mathcal{P}$ depends only on $\Omega_u^\circ \cap X_v$, which is commonly called the *slice* of X_v at e_u. Our theorem now shows that $\mathcal{P}$ is preserved under going up in our poset by the first type of generating relation, since $\Omega_u^\circ \cap X_v$ is isomorphic to $\Omega_x^\circ \cap X_w$.

As for the second type of generating relation, we can by induction on Bruhat order reduce to the case where $u' = s_\alpha u$. In that case, $U_\alpha \cdot e_u$ is a curve in X_v all of whose points have neighborhoods isomorphic to the neighborhood at e_u (since X_v has a B-action). The closure of $U_\alpha \cdot e_u$ includes the additional point $e_{u'}$. Since the set at which $\mathcal{P}$ holds is closed, $\mathcal{P}$ is also preserved under going up by the second type of generating relation.

The last statement follows by taking a generating set for the order ideal.   ∎

We also have the following corollary about Kazhdan–Luzstig polynomials, generalizing a lemma of Polo [17, Lemma 2.6]. (See also [3, Theorem 6].) This corollary can also be deduced from the algorithm introduced by Braden and Macpherson in [8] to calculate intersection cohomology from moment graphs, as a subsystem embedding gives an isomorphism of intervals in moment graphs and not merely an isomorphism of intervals in Bruhat order.

Corollary 3.3   *Suppose a subsystem embedding embeds $[u, v]$ into $[x, w]$. Then the Kazhdan–Luzstig polynomials $P_{u,v}(q)$ and $P_{x,w}(q)$ are equal.*

It is conjectured that $P_{u,v}(q) = P_{x,w}(q)$ whenever $[u, v]$ and $[x, w]$ are isomorphic as intervals, and this theorem confirms a very special case of this conjecture. Kazhdan–Luzstig polynomials and this conjecture are discussed with further references in [6].

4   The Pattern Map

To prove the theorem, we use the geometric pattern map introduced by Billey and Braden [3]. Let T_0 be a one parameter subgroup of T which is generic among subgroups satisfying $\alpha(T_0) = 1$ for every $\alpha \in i(\Phi')$. (Recall that roots are actually characters of T, which are homomorphisms from T to $\Bbbk^\times$.) Let G' be the centralizer $Z_G(T_0)$ of T_0. The Weyl group and roots of G' are then $i(W')$ and $i(\Phi')$. In G' we fix the Borel subgroup $B' = G' \cap B$.

Now Billey and Braden define a map $\psi \colon (G/B)^{T_0} \to (G'/B')$ as follows. There is a bijection between points of G/B and Borel subgroups of G given by associating to the coset gB the Borel subgroup gBg^{-1}. Now define $\psi(gB)$ to be the point in G'/B' associated with the Borel subgroup $gBg^{-1} \cap G'$. This is a Borel subgroup of G' whenever gB is fixed by T_0 [18, Theorem 6.4.7]. Billey and Braden prove the following theorem.

Theorem 4.1 ([3, Theorem 10])
(i)   *The map ψ restricts to an isomorphism on each connected component of $(G/B)^{T_0}$.*
(ii)   *For any $w \in (W, \Phi, V)$, the restriction of ψ is an isomorphism between $X_w^\circ \cap (G/B)^{T_0}$ and $X_{\phi_i(w)}^\circ$ taking e_w to $e_{\phi_i(w)}$.*

Their proof of part (ii) (which is a combination of parts (ii) and (iii) of their original statement) also shows that ψ restricts to an isomorphism between $\Omega_w^\circ \cap (G/B)^{T_0}$ and $\Omega_{\phi_i(w)}^\circ$.

As remarked by Billey and Braden [3] (see also [15, Proposition 4.2]), this geometric pattern map explains why ordinary pattern avoidance characterizes singular Schubert varieties. Given any one parameter torus $T_0 \cong \Bbbk^\times$ acting on a Schubert variety, if the T_0 fixed locus is singular, the entire Schubert variety must be singular. If there is a pattern embedding i of v into w, and X_v is singular, then ψ gives an isomorphism between $X_w \cap (G/B)^{T_0}$ and X_v, showing that X_w is singular.

5   Proof of Main Theorem

We will show that X_v^u and X_w^x are isomorphic under the map ψ. First we show that e_w and e_x are in the same connected component of $(G/B)^{T_0}$, as follows. Since x and w are in the same right W' coset, we can successively multiply x on the left by reflections s_α, with $\alpha \in \Phi'$, to get w. If $\sigma = s_\alpha \tau$ for some $\alpha \in \Phi'$, then e_σ and e_τ are in the same connected component of $(G/B)^{T_0}$ since the points are connected by the Schubert curve $\overline{U_\alpha \cdot e_\sigma}$ (assuming $\sigma \geq \tau$), and U_α is T_0 fixed as $\alpha \in \Phi'$.

Now, by part (ii) of the theorem, $X_w^\circ \cap (G/B)^{T_0}$ and $\Omega_x^\circ \cap (G/B)^{T_0}$ are connected, so, given that e_w and e_x are in the same connected component of $(G/B)^{T_0}$, $(X_w^\circ \cap \Omega_x^\circ) \cap (G/B)^{T_0}$ is contained in a single component of $(G/B)^{T_0}$. Therefore, ψ is an isomorphism when restricted to $X_w^\circ \cap \Omega_x^\circ \cap (G/B)^{T_0}$.

We show that the image of $X_w^\circ \cap \Omega_x^\circ \cap (G/B)^{T_0}$ is $X_v^\circ \cap \Omega_u^\circ$. If $p \in X_v^\circ \cap \Omega_u^\circ$, then $p = \psi(p_1)$ for some $p_1 \in X_w^\circ \cap (G/B)^{T_0}$, and $p = \psi(p_2)$ for some $p_2 \in \Omega_x^\circ \cap (G/B)^{T_0}$. Since $X_w^\circ \cap (G/B)^{T_0}$ and $\Omega_x^\circ \cap (G/B)^{T_0}$ lie in the same connected component of $(G/B)^{T_0}$, by part (i) of the theorem, $p_1 = p_2 \in X_w^\circ \cap \Omega_x^\circ \cap (G/B)^{T_0}$. Conversely, for $p \in X_w^\circ \cap \Omega_x^\circ \cap (G/B)^{T_0}$, $\psi(p) \in \psi\big(X_w^\circ \cap (G/B)^{T_0}\big) \cap \psi\big(\Omega_x^\circ \cap (G/B)^{T_0}\big) = X_v^\circ \cap \Omega_u^\circ$.

In particular, $X_w^\circ \cap \Omega_x^\circ \cap (G/B)^{T_0}$ has dimension $\ell(v) - \ell(u)$. Since we have a pattern embedding from $[u, v]$ to $[x, w]$, $\ell(v) - \ell(u) = \ell(w) - \ell(x)$, so the dimension of $X_w^\circ \cap \Omega_x^\circ \cap (G/B)^{T_0}$ is the same as the dimension of $X_w^\circ \cap \Omega_x^\circ$. As the latter is known to be irreducible [19] (or [9, Proposition 1.3.2]), $X_w^\circ \cap \Omega_x^\circ$, and therefore its closure X_w^x, must be pointwise T_0-fixed.

Since X_w^x is connected and pointwise T_0-fixed, it must be isomorphic to its image under ψ. This image is the closure of $X_v^\circ \cap \Omega_u^\circ$, which is X_v^u.

The calculation of the image can be repeated for every $\sigma, \tau \in [x, w]$, proving the second statement.

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References

[1]   N. Bergeron and F. Sottile, *Schubert polynomials, the Bruhat order, and the geometry of flag manifolds*. Duke Math. J. **95**(1998), 373–423.   doi:10.1215/S0012-7094-98-09511-4

[2]   S. Billey, *Pattern avoidance and rational smoothness of Schubert varieties*. Adv. Math. **139**(1998), 141–156.   doi:10.1006/aima.1998.1744

[3]   S. Billey and T. Braden, *Lower bounds for Kazhdan–Lusztig polynomials from patterns*. Transform. Groups **8**(2003), 321–332.   doi:10.1007/s00031-003-0629-x

[4]   S. Billey and A. Postnikov, *Smoothness of Schubert varieties via patterns in root systems*. Adv. in Appl. Math. **34**(2005), 447-466.   doi:10.1016/j.aam.2004.08.003

[5]   S. Billey and G. Warrington, *Maximal singular loci of Schubert varieties on* SL$(n)/B$. Trans. Amer. Math. Soc. **355**(2003), 3915–3945.   doi:10.1090/S0002-9947-03-03019-8

[6]   A. Björner and F. Brenti, *Combinatorics of Coxeter groups*. Graduate Texts in Math. **231**, Springer-Verlag, New York–Heidelberg, 2005.

[7]   M. Bousquet-Mélou and S. Butler, *Forest-like permutations*. Ann. Comb. **11**(2007), no. 3–4, 335–354.   doi:10.1007/s00026-007-0322-1

[8]   T. Braden and R. Macpherson, *From moment graphs to intersection cohomology*. Math. Ann. **321**(2001), 533–551.   doi:10.1007/s002080100232

[9]   M. Brion, *Lectures on the geometry of flag varieties*. In: Topics in cohomological studies of algebraic varieties, Trends Math., Birkhäuser, Basel, 2005, 33–85.

[10] A. Cortez, *Singularités génériques et quasi-résolutions des variétés de Schubert pour le groupe linéaire*. Adv. Math. **178**(2003), 396–445.   doi:10.1016/S0001-8708(02)00081-6

[11] R. Irving, *The socle filtration of a Verma module*. Ann. Sci. École Norm. Sup. Sér. 4 **21**(1988), 47–65.

[12] C. Kassel, A. Lascoux and C. Reutenauer, *The singular locus of a Schubert variety*. J. Algebra **269**(2003), 74–108.   doi:10.1016/S0021-8693(03)00014-0

[13] D. Kazhdan and G. Lusztig, *Representations of Coxeter Groups and Hecke Algebras*. Invent. Math. **53**(1979), 165–184.   doi:10.1007/BF01390031

[14] V. Lakshmibai and B. Sandhya, *Criterion for smoothness of Schubert varieties in* SL$(n)/B$. Proc. Indian Acad. Sci. Math. Sci. **100**(1990), 45–52.   doi:10.1007/BF02881113

[15] C. Lenart, S. Robinson and F. Sottile, *Grothendieck Polynomials via permutation patterns and chains in the Bruhat order*. Amer. J. Math. **128**(2006), 805–848.   doi:10.1353/ajm.2006.0034

[16] L. Manivel, *Le lieu singulier des variétés de Schubert*. Internat. Math. Res. Notices **16**(2001), 849–871.

[17] P. Polo, *Construction of arbitrary Kazhdan–Luzstig polynomials in symmetric groups*. Represent. Theory **3**(1999), 90–104 (electronic).   doi:10.1090/S1088-4165-99-00074-0

[18] T. Springer, *Linear algebraic groups*. Second edition, Birkhäuser, Boston, MA, 1998.

[19] R. W. Richardson, *Intersections of double cosets in algebraic groups*. Indag. Math. (N.S.) **3**(1992), 69–77.   doi:10.1016/0019-3577(92)90028-J

[20] A. Woo and A. Yong, *When is a Schubert variety Gorenstein?*, Adv. Math. **207**(2006), 205–220. doi:10.1016/j.aim.2005.11.010

[21] _____, *Governing singularities of Schubert varieties*. J. Algebra **320**(2008), no. 2, 459–520. doi:10.1016/j.jalgebra.2007.12.016

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