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QUASICYCLIC SUBNORMAL SEMIGROUPS

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Introduction. Let $T(s)$, $s \geqq 0$, be a strongly continuous semigroup of bounded operators on a separable Hilbert space $\mathscr{H}$. $T(s)$ is said to be *quasicyclic* if there is a continuum of vectors $\{x_s\}_{s>0} \subset \mathscr{H}$ such that $T(s)x_t = x_{s+t}$ for all $s, t > 0$ and $\overline{\text{sp}}\{x_s\}_{s>0} = \mathscr{H}$. $T(s)$ is said to be *subnormal* if there is a semigroup of normal operators $N(s)$ acting on a Hilbert space $K \supset \mathscr{H}$ such that $N(s)|_{\mathscr{H}} = T(s)$ for all $s > 0$. In this paper we shall be concerned primarily with semigroups which have both of these properties.

In Section 1, we obtain a general representation for quasicyclic subnormal semigroups. A measure μ defined on a half plane $\Pi_\eta = \{z|x \geqq \eta\}$ is said to have *minimal exponential type* if e^{-sz} is μ-integrable for all $s > 0$. For any such measure μ, we denote by $H^2(\mu)$ the $L^2(\mu)$-closed span of the functions e^{-sz}, $s > 0$. It is shown that any quasicyclic subnormal semigroup is unitarily equivalent to the semigroup of multiplication by e^{-sz} on $H^2(\mu)$ for some measure μ having minimal exponential type. The result is a consequence of Sz.-Nagy's spectral representation of normal semigroups. See [**11**].

In Section 2, we study *weighted translation semigroups*, which are defined as follows. Let $w(t)$ be a positive continuous function defined on $(0, \infty)$, and let m_w be the measure defined on $(0, \infty)$ by $dm_w(t) = w(t)dt$. The space $L^2(m_w)$ carries a formal semigroup of forward translation operators $T_w(s)$, $s \geqq 0$, defined by $T_w(s)f(t) = f(t - s)$, it being agreed that $f(t - s) = 0$ for $t < s$. Set $M_w(s) = \sup_{t>0} w(t + s)/w(t)$. Then $T_w(s)$ is a strongly continuous semigroup of bounded operators if and only if $\limsup_{s\to\infty} \log M_w(s)/s = b_w < \infty$. In the special case when $w(t)$ is right hand continuous at 0, these semigroups have recently been studied systematically by Embry and Lambert [**3**; **4**] from a slightly different point of view. Sporadic examples have appeared from time to time in earlier literature. See, for example, Hille and Phillips [**8**]. In the present analysis, it is shown that any such semigroup is quasicyclic. In addition, we show that the spectrum of $T_w(s)$ is precisely the disk $\{z| \, |z| \leqq e^{\frac{1}{2}b_w s}\}$ for all $s > 0$.

In Section 3, we apply the result of Section 1 to study subnormal weighted translation semigroups. We show that $T_w(s)$ is a subnormal semigroup if and only if

$$w(t) = \int_{-\frac{1}{2}b_w}^{\infty} e^{-2tx} d\nu(x)$$

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for some positive Borel measure ν defined on the half line $x \geqq -\frac{1}{2}b_w$. In the case when $w(t)$ is right hand continuous at 0, ν is necessarily a finite measure, and the present theorem reduces to the result obtained by Embry and Lambert in [**4**]. In addition, we obtain a functional model for subnormal weighted translation semigroups. Given the measure ν, we define a measure μ on the half plane $\Pi_w = \{z|x \geqq -\frac{1}{2}b_w\}$ by $d\mu(x, y) = (2\pi)^{-1}d\nu(x)dy$. For any $p > 0$, let $F(z)$ be a function analytic for $x > -\frac{1}{2}b_w$, and assume that $F(u + z - \frac{1}{2}b_w)$ belongs to H^P of the half plane $x > 0$ for all $u > 0$. We then set

$$\mathscr{M}_p(u; F; \mu) = \left(\int_{\Pi_w} |F(u+z)|^P d\mu(z)\right)^{1/p}.$$

We denote by $\mathscr{E}^p(\mu)$ the collection of all such functions $F(z)$ satisfying the additional condition that

$$||F||_{p,\mu} = \sup_{u>0} \mathscr{M}_p(u; F; \mu) < \infty.$$

The space $\mathscr{E}^2(\mu)$ is seen to be a functional Hilbert space, and it is shown that the Laplace Transform is an isometric isomorphism of $L^2(m_w)$ onto $\mathscr{E}^2(\mu)$ which intertwines the semigroup $T_w(s)$ with the semigroup of multiplication by e^{-sz} on $\mathscr{E}^2(\mu)$.

1. A general result. Let $T(s)$, $s \geqq 0$, be a strongly continuous semigroup of bounded operators on a Hilbert space $\mathscr{H}$. $T(s)$ is said to be *cyclic* if there is a vector $x_0 \in \mathscr{H}$ such that $\overline{\text{sp}}\{T(s)x_0\}_{s\geqq 0} = \mathscr{H}$. x_0 is called a *cyclic vector* for $T(s)$. More generally, we say that $T(s)$ is *quasicyclic* if there is a family of vectors $\{x_s\}_{s>0} \subset \mathscr{H}$ such that $T(s)x_t = x_{s+t}$ for all $s, t > 0$ and $\overline{\text{sp}}\{x_s\}_{s>0} = \mathscr{H}$. The family $\{x_s\}_{s>0}$ is called a *quasicyclic family* for $T(s)$. Clearly if $T(s)$ is cyclic, then $T(s)$ is quasicyclic.

A strongly continuous semigroup $T(s)$ is said to be *subnormal* if there is a semigroup of normal operators $N(s)$, $s \geqq 0$, acting on a Hilbert space $K \supset \mathscr{H}$ such that $N(s)|_{\mathscr{H}} = T(s)$ for all $s \geqq 0$. It is known that $T(s)$ is a subnormal semigroup if and only if each individual operator $T(s)$ is subnormal. See Ito [**10**]. In this section, we obtain a concrete model for an arbitrary quasicyclic subnormal semigroup, analogous to Bram's result for a single subnormal operator with a cyclic vector. See [**1**].

We begin by introducing an appropriate class of measures. Let η be any real number, and let $\Pi_\eta = \{z|x \geqq \eta\}$. We define a mapping τ from Π_η to the half line $x \geqq \eta$ by $\tau(z) = x(= \text{Re } z)$. Given a measure μ defined on Π_η, we define a measure ν on the half line $x \geqq \eta$ by $d\nu(x) = d\mu(\tau^{-1}(x))$. By analogy with analytic functions, we say that μ has *minimal exponential type* if the Laplace-Stieltjes integral $\int_\eta^\infty e^{-sx}d\nu(x)$ converges for all $s > 0$. Given any such measure, we observe that $e^{-sy} \in L^2(\mu)$ for all $s > 0$. We denote by $H^2(\mu)$ the $L^2(\mu)$-closed span of these functions. We are now ready to proceed to the main result of this section.

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THEOREM 1. *Let $T(s)$, $s \geqq 0$, be a quasicyclic subnormal semigroup of operators on a Hilbert space $\mathscr{H}$. Then there is a measure μ defined on a half plane Π_η and having minimal exponential type, such that the semigroup $T(s)$, $s \geqq 0$, is unitarily equivalent to the semigroup of multiplication by e^{-sz}, $s \geqq 0$, on $H^2(\mu)$.*

Proof. Let $N(s)$ be the minimal normal semigroup extension of $T(s)$, acting on some Hilbert space $K \supset \mathscr{H}$. Let $\{x_s\}_{s>0}$ be a quasicyclic family for $T(s)$. The minimality of $N(s)$ clearly implies that $\mathscr{H} = \overline{\text{sp}}\{N(t)^*x_s\}_{s,t>0}$. By a classic result of Sz-Nagy [**11**], $N(s)$ has a spectral representation of the form

$$(1)\qquad N(s) = \int_\Pi e^{-sz}dE(z),$$

where $\Pi = \Pi_\eta = \{z|x \geqq \eta\}$ for some real number η, and $E(z)$ is the spectral resolution of the infinitesimal generator of the semigroup $N(s)$. For each $s, t > 0$, we define a complex measure $\mu_{s,t}$ on Π by

$$(2)\qquad d\mu_{s,t}(z) = \langle dE(z)x_s, x_t\rangle.$$

If $s = t$, we write $\mu_{t,t} = \mu_t$. We assert that for any $s, t, u, v > 0$,

$$(3)\qquad d\mu_{s+u,t+v}(z) = e^{-(uz+v\bar{z})}d\mu_{s,t}(z).$$

To see this, let f be any continuous function with compact support in Π, and note that

$$\begin{aligned}\int_\Pi f(z)d\mu_{s+u,t+v}(z) &= \int_\Pi f(z)\ \langle dE(z)x_{s+u}, x_{t+v}\rangle \\ &= \int_\Pi f(z)\ \langle dE(z)N(v)^*N(u)x_s, x_t\rangle \\ &= \left\langle \int_\Pi f(z)dE(z) \int_\Pi e^{-(uw+v\bar{w})}dE(w)x_s, x_t\right\rangle \\ &= \left\langle \int_\Pi f(z)e^{-(uz+v\bar{z})}dE(z)x_s, x_t\right\rangle \\ &= \int_\Pi f(z)e^{-(uz+v\bar{z})}d\mu_{s,t}(z).\end{aligned}$$

The assertion follows. In particular, for any $s, t > 0$, $d\mu_{s+t}(z) = e^{-2sx}d\mu_t(z)$. Consequently, we may unambiguously define a measure μ on Π by

$$(4)\qquad d\mu(z) = e^{2sx}d\mu_s(z)$$

for any $s > 0$. Since each measure μ_s is finite, it follows easily that μ has minimal exponential type.

Now let $\mathscr{K}_0 = \text{sp}\{N(t)^*x_s\}_{s,t>0}$. We define a transformation U from $\mathscr{K}_0$ to $L^2(\mu)$ by setting

$$(5)\qquad U(N(t)^*x_s) = e^{-(sz+t\bar{z})}$$

for any s, $t > 0$, and extending linearly to $\mathcal{K}_0$. We claim that U is isometric. To see this, note that for any s, t, u, $v > 0$, we have

$$\begin{aligned}\langle N(t)^*x_s, N(v)^*x_u\rangle_{\mathscr{H}} &= \langle N(t)^*N(v)x_s, x_u\rangle_{\mathscr{H}} \\ &= \int_{\Pi} e^{-(t\bar{z}+vz)} \langle dE(z)x_s, x_u\rangle_{\mathscr{H}} \\ &= \int_{\Pi} e^{-(t\bar{z}+vz)} e^{-(sz+u\bar{z})} d\mu(z) \\ &= \int_{\Pi} e^{-(sz+t\bar{z})} \overline{e^{-(uz+v\bar{z})}} d\mu(z) \\ &= \langle e^{-(sz+t\bar{z})}, e^{-(uz+v\bar{z})}\rangle_{L^2(\mu)}.\end{aligned}$$

From this, the assertion follows easily. Thus U extends to an isometric mapping from $\mathcal{K}$ to $L^2(\mu)$. In particular, U carries $\mathscr{H}$ isometrically onto $H^2(\mu)$, and evidently U intertwines the semigroup $T(s)$, $s \geqq 0$, and the semigroup of multiplication by e^{-sz}, $s \geqq 0$, acting on $H^2(\mu)$. The theorem follows.

Remarks. (a) We point out that if the measure μ is finite, then $T(s)$ will be cyclic, a cyclic vector being $U^*(1)$. The converse is false, as we shall see later.

(b) The model of Theorem 1 would be a bit tidier if we could say that U maps $\mathcal{K}$ *onto* $L^2(\mu)$, thus providing a simple concrete realization of $N(s)$. At present, however, we do not know if this is true. The generalized Stone-Weierstrass theorem would appear to be relevant, but the noncompactness of Π seems to render it ineffective.

2. Weighted translation semigroups. Let $w(t)$ be a positive, continuous function on the open half line $(0, \infty)$, and let m_w be the measure defined on $(0, \infty)$ by $dm_w(t) = w(t)dt$. The Hilbert space $L^2(m_w)$ carries a semigroup of forward translation operators $T_w(s)$, $s \geqq 0$, defined formally by

(6) $\quad T_w(s) : f(t) \rightarrow f(t - s),$

it being understood that $f(t - s) = 0$ for $t < s$. It is evident that (6) is at least meaningful when f has compact support contained in $(0, \infty)$. A simple computation with characteristic functions shows that (6) defines a bounded operator on $L^2(m_w)$ if and only if

(7) $$M_w(s) = \sup_{t>0} \frac{w(t+s)}{w(t)} < \infty,$$

and in this case $\|T_w(s)\| = M_w(s)^{\frac{1}{2}}$. We shall therefore assume in all that follows that $M_w(s) < \infty$ for all $s \geqq 0$. We point out for the record that the semigroup $T_w(s)$ is unitarily equivalent to the semigroup $\tilde{T}_w(s)$ defined on $L^2(m_1)$ by

(8) $$\tilde{T}_w(s): f(t) \rightarrow \sqrt{\frac{w(t)}{w(t-s)}} f(t-s).$$

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Sporadic examples of semigroups of this type have appeared previously in the literature. See, for example, Hille and Philips [**8**, Ch. 19]. The first systematic study of such semigroups seems to have been made by Embry and Lambert [**3**; **4**]. They use the form (8) with the additional restriction that $w(t)$ be right hand continuous at 0. In the present study, we shall use the form (6) throughout, with no additional assumptions about $w(t)$. In this section, we list some general properties of these semigroups which will be needed in the sequel. We begin by giving simple conditions which are necessary and sufficient to insure that $T_w(s)$ be strongly continuous.

LEMMA 1. *Let $w(t)$ be a positive, continuous function on $(0, \infty)$. Then the following conditions are equivalent:*

(i) *$T_w(s)$ is a strongly continuous semigroup.*

(ii) *There exists a real number b and a constant $M > 0$ such that $M_w(s) \leqq Me^{bs}$ for all $s \geqq 0$.*

This result is obtained by Embry and Lambert in [**3**]. We present an alternative proof, for the sake of completeness.

Proof. It follows from [**8**, Theorem 10.6.2, p. 323] that (i) is satisfied if and only if $M_w(s) = ||T(s)||^2 = O(1)$ as $s \to 0+$. Thus there is some $\delta > 0$ and a constant $K > 0$ such that $M(s) \leqq K$ for $0 \leqq s \leqq \delta$. Set $b = (2/\delta) \log K$. If $s > \delta$, then there is an integer $n \geqq 2$ and a number s' with $\delta - \delta/n < s' < \delta$ such that $s = ns'$. Then

$$\begin{aligned} M_w(s) = ||T_w(s)||^2 \leqq ||T_w(s')||^{2n} &= M_w(s')^n \\ &\leqq e^{b\delta n} \\ &\leqq e^{b\delta}e^{bs}. \end{aligned}$$

The lemma follows.

On account of this result, we assume in all that follows that $w(t)$ satisfies (ii). We denote by b_w the infinum of all numbers b for which (ii) is satisfied. A straightforward argument shows that b_w is given by

$$(9) \qquad b_w = \limsup_{s\to\infty} \frac{\log M_w(s)}{s}.$$

Our next result shows that any semigroup of the form (6) for which $b_w < \infty$ is quasicyclic.

LEMMA 2. *Let $w(t)$ be a positive, continuous function on $(0, \infty)$ such that $b_w < \infty$. For any $b > \frac{1}{2}b_w$, let $g_b(t) = e^{-bt}$. Then $g_b(t - s) \in L^2(m_w)$ for every $s > 0$, and $\{g_b(t - s)\}_{s>0}$ is a quasicyclic family for $T_w(s)$.*

Proof. Given b, choose b' with $b > b' > \frac{1}{2}b_w$. Then for any $s > 0$,

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$$\int_s^\infty |g_b(t-s)|^2 w(t)dt = \int_0^\infty e^{-2bt} w(t+s)\, dt$$
$$= \sum_{n=0}^\infty \int_n^{n+1} e^{-2bt} w(t+s)\, dt$$
$$= \sum_{n=0}^\infty e^{-2bn} \int_0^1 e^{-2bt} w(t+n+s)\, dt$$
$$\leqq M' \sum_{n=0}^\infty e^{-2bn} e^{2b'n} \int_0^1 e^{-2bt} w(t+s)\, dt$$
$$= \text{const.} \sum_{n=0}^\infty e^{-2(b-b')n}.$$

Since the series on the right converges, it follows that $g_b(t-s) \in L^2(m_w)$. To prove that $\{g_b(t-s)\}_{s>0}$ is a quasicyclic family, suppose $f \in L^2(m_w)$ satisfies $\langle f, g_b(t-s)\rangle = 0$ for all $s > 0$. Then for all $s > 0$,

$$\int_s^\infty e^{-bt} f(t)w(t)dt = e^{-bs} \int_s^\infty g_b(t-s)f(t)w(t)dt = 0.$$

Differentiating with respect to s, we then have $f(s) = 0$ a.e. The lemma follows.

We point out that if $w(t)$ is right hand continuous at 0, the proof of Lemma 2 implies that $e^{-bt} \in L^2(m_w)$ for any $b > \frac{1}{2}b_w$, and that e^{-bt} is a cyclic vector for $T_w(s)$. If $w(t)$ is not right hand continuous at 0, this need not happen. For example, if $w(t) = 1/t$, then $b_w = 0$, but $e^{-bt} \notin L^2(m_w)$ for any real number b.

We turn next to an examination of the spectra of the semigroups $T_w(s)$. If one considers these semigroups to be the natural continuous analogue of weighted shifts, it is reasonable to expect that for any given w, the spectrum of the operator $T_w(s)$ should be a disk for any $s > 0$. Our next result bears out this expectation.

THEOREM 2. *Let $w(t)$ be a positive, continuous function on $(0, \infty)$ with $b_w < \infty$. Then for each $s > 0$, $\sigma(T_w(s)) = \{z \mid |z| \leqq e^{\frac{1}{2}b_w s}\}$.*

Proof. We show first that the spectral radius of $T_w(s)$ is $e^{\frac{1}{2}b_w s}$. Note that

$$r(T_w(s)) = \limsup_{n\to\infty} M_w(ns)^{1/2n},$$

which is equivalent to saying that

$$\log r(T_w(s)) = \frac{1}{2} \limsup_{n\to\infty} \frac{1}{n} \log M_w(ns).$$

In view of this, it suffices to show that

$$\limsup_{n\to\infty} \frac{1}{n} \log M_w(ns) = b_w s.$$

Evidently,

$$\limsup_{n\to\infty} \frac{1}{n} \log M_w(ns) \leqq b_w s.$$

Now given any positive number u, let $n = [u]$. Then for any positive number $b > b_w$, we have

$$\begin{aligned} M_w(su) = ||T_w(s(u-n))T_w(sn)||^2 &\leqq ||T_w(s(u-n))||^2||T_w(sn)||^2 \\ &= M_w(s(u-n))M_w(sn) \\ &\leqq Me^{bs(u-n)}M_w(sn) \\ &\leqq Me^{bs}M_w(sn). \end{aligned}$$

Thus

$$\frac{1}{u}\log M_w(su) \leqq \frac{1}{n}[\log M + bs + \log M_w(sn)],$$

from which the reverse inequality follows easily.

Next suppose that λ belongs to the resolvent set of $T_w(s)$. Then $\lambda I - T_w(s)$ is bounded below, and maps $L^2(m_w)$ onto itself. So given any $h \in L^2(m_w)$, there is a unique function $g \in L^2(m_w)$ such that $\lambda g(t) - g(t-s) = h(t)$. It can be proved by mathematical induction that

$$g(t+ns) = \lambda^{-(n+1)} \sum_{k=0}^{n} \lambda^k h(t+ks)$$

for $0 \leqq t < s$ and any integer $n \geqq 0$.

Now since $\lambda I - T_w(s)$ is bounded below, there is a constant $K > 0$ independent of h, such that

$$\begin{aligned} \int_0^\infty |h(t)|^2 w(t)dt &\geqq K \int_0^\infty |g(t)|^2 w(t)dt \\ &= K \sum_{n=0}^{\infty} \int_0^s |g(t+ns)|^2 w(t+ns)dt \\ &= K \sum_{n=0}^{\infty} |\lambda|^{-2(n+1)} \int_0^s \left| \sum_{k=0}^{n} \lambda^k h(t+ks) \right|^2 w(t+ns)dt. \end{aligned}$$

Given any $u > 0$, let m be the unique nonnegative integer for which $ms \leqq u < (m+1)s$. Choose $\epsilon > 0$ such that $u + \epsilon < (m+1)s$, and let $h(t)$ be the characteristic function of the interval $[u, u+\epsilon]$. Then we have

$$\begin{aligned} \int_u^{u+\epsilon} w(t)dt &\geqq K \sum_{n=m}^{\infty} |\lambda|^{2(m-n-1)} \int_{u-ms}^{u-ms+\epsilon} w(t+ns)dt \\ &= K \sum_{n=m}^{\infty} |\lambda|^{2(m-n-1)} \int_u^{u+\epsilon} w(t+ns-ms)\, dt \\ &= K \sum_{k=0}^{\infty} |\lambda|^{-2(k+1)} \int_u^{u+\epsilon} w(t+ks)\, dt. \end{aligned}$$

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Thus for any $u > 0$ and any integer $k \geqq 0$, we have

$$|\lambda|^{-2(k+1)} \int_u^{u+\epsilon} w(t + ks)\, dt \leqq \text{const.} \int_u^{u+\epsilon} w(t)\, dt.$$

Dividing by ϵ and letting $\epsilon \to 0$, we obtain

$$|\lambda|^{-2(k+1)} w(u + ks) \leqq \text{const. } w(u).$$

It follows that for any $k \geqq 0$,

$$M_w(ks) \leqq \text{const. } |\lambda|^{2(k+1)}.$$

So also

$$e^{\frac{1}{2}b_w s} = \limsup_{k\to\infty} M_w(ks)^{1/2k} \leqq |\lambda|.$$

Since the resolvent set is open, this inequality must actually be strict. The result follows.

We conclude this section with some analysis of the point spectrum of $T_w(s)^*$ which will be vital in what follows. Evidently, the point spectrum of $T_w(s)^*$ always contains 0. More importantly, suppose that α is a real number for which

$$(10) \quad \int_0^\infty e^{-2\alpha t} \frac{dt}{w(t)} < \infty.$$

Then for any complex number $\zeta = \alpha + i\beta$, the function $g(t) = e^{-\zeta t}/w(t) \in L^2(m_w)$, and it is a simple matter to show that $T_w(s)^* g(t) = e^{-\zeta s} g(t)$ for any $s > 0$. Thus $e^{-\zeta s}$ belongs to the point spectrum of $T_w(s)^*$. We define a formal Laplace integral $\varphi_w(z)$ by

$$(11) \quad \varphi_w(z) = \int_0^\infty e^{-zt} \frac{dt}{w(t)}.$$

The foregoing discussion shows that if σ_w is the abscissa of convergence of $\varphi_w(z)$, then $e^{-\zeta s}$ belongs to the point of $T_w(s)^*$ whenever $\operatorname{Re} \zeta > \frac{1}{2}\sigma_w$. Moreover, it follows from Schwarz' inequality that if $f \in L^2(m_w)$, the Laplace Transform

$$(12) \quad \hat{f}(z) = \int_0^\infty e^{-tz} f(t)\, dt$$

converges absolutely and uniformly in the half plane $x \geqq \eta$ for any $\eta > \frac{1}{2}\sigma_w$. Indeed, we may observe that

$$(13) \quad |\hat{f}(z)| \leqq \varphi_w(2x)^{\frac{1}{2}} \|f\|$$

for any $f \in L^2(m_w)$ and any z with $x > \frac{1}{2}\sigma_w$. If we denote by $\mathscr{L}^2(w)$ the linear space of Laplace transforms of functions in $L^2(m_w)$, we observe that the mapping $f \to \hat{f}$ is a linear isomorphism of $L^2(m_w)$ onto $\mathscr{L}^2(w)$. We may use this isomorphism to transfer the metric structure of $L^2(m_w)$ to $\mathscr{L}^2(w)$. With this understanding, $\mathscr{L}^2(w)$ becomes a functional Hilbert space. It is easily shown that the reproducing kernel of $\mathscr{L}^2(w)$ is given by $K_w(\zeta, z) = \varphi_w(\bar{\zeta} + z)$.

3. Subnormal weighted translation semigroups. In this section, we study the problem of when a semigroup of the form (6) will be subnormal. This question has previously been considered by Embry and Lambert [**4**] in the special case when $w(t)$ is right hand continuous at 0. Our analysis proceeds from Theorem 1 together with the results of the previous section, and eliminates this restriction. We obtain an extension of Embry and Lambert's result, together with a realization of subnormal weighted translation semigroups as multiplication semigroups on certain Hilbert spaces of Laplace integrals.

We first introduce one additional scrap of terminology. Given a positive Borel measure ν defined on a half line $x \geqq \eta$, we set $a_\nu = \inf\{a|\nu([a, c]) > 0$ for all $c > a\}$. Then the support of ν is contained in $[a_\nu, \infty)$. We can now state

THEOREM 3. *Let $w(t)$ be a positive, continuous function on $(0, \infty)$, with $b_w < \infty$. Then in order for the semigroup $T_w(s)$ to be subnormal it is necessary and sufficient that there exist a positive Borel measure ν defined on a half line $x \geqq \eta$ such that*

$$(14) \quad w(t) = \int_\eta^\infty e^{-2tx}\, d\nu(x), \quad t > 0.$$

In this case, $a_\nu = -\frac{1}{2}b_w$. Furthermore, for any $b > \frac{1}{2}b_w$ the semigroup $T_w(s)$ is unitarily equivalent to the semigroup of multiplication by e^{-sz} on $H^2(\mu_b)$, where

$$(15) \quad d\mu_b(x, y) = \frac{1}{2\pi} \frac{d\nu(x)dy}{(x+b)^2 + y^2}.$$

Proof. First suppose that $T_w(s)$ is a subnormal semigroup. By Lemma 2, for any $b > \frac{1}{2}b_w$, $\{g_b(t-s)\}_{s>0}$ is a quasicyclic family for $T_w(s)$, where $g_b(t) = e^{-bt}$. Thus by Theorem 1, there is a measure $\tilde{\mu}_b$ defined on a half plane $\Pi_\eta = \{z|x \geqq \eta\}$ and having minimal exponential type, such that the semigroup $T_w(s)$ is unitarily equivalent to the semigroup of multiplication by e^{-sz} on $H^2(\tilde{\mu}_b)$ via the map $\tilde{U}_b$ which sends $g_b(t-s)$ to e^{-sz}, $s > 0$. Let $d\tilde{\nu}_b(x) = d\tilde{\mu}_b(\tau^{-1}(x))$, and let $d\nu(x) = 2(x+b)d\tilde{\nu}_b(x)$. We assert that ν satisfies (14). To see this, note that for any $s, h > 0$,

$$\int_0^\infty [g_b(t-s)^2 - g_b(t-s-h)^2]\, w(t)\, dt = \int_{\Pi_\eta} [e^{-2sx} - e^{-2(s+h)x}] d\tilde{\mu}_b(z)$$
$$= \int_\eta^\infty e^{-2sx} [1 - e^{-2hx}]\, d\tilde{\nu}_b(x).$$

Therefore

$$\frac{1}{h}\int_s^{s+h} e^{-2b(t-s)}\, w(t)dt + \frac{1}{h}[1 - e^{2bh}] \int_{s+h}^\infty e^{-2b(t-s)}\, w(t)dt$$
$$= \frac{1}{h}\int_\eta^\infty e^{-2sx}\, [1 - e^{-2hx}]\, d\tilde{\nu}_b(x).$$

Letting $h \to 0$ and applying dominated convergence, we obtain

$$w(s) - 2b \int_s^\infty e^{-2b(t-s)}\, w(t)dt = 2 \int_\eta^\infty x\, e^{-2sx}\, d\tilde{\nu}_b(x).$$

Thus

$$\begin{aligned} w(s) &= 2b \int_0^\infty g_b(t-s)^2 w(t)dt + 2 \int_\eta^\infty x\, e^{-2sx}\, d\tilde{\nu}_b(x) \\ &= 2b \int_\eta^\infty e^{-2sx}\, d\tilde{\nu}_b(x) + 2 \int_\eta^\infty x\, e^{-2sx}\, d\tilde{\nu}_b(x) \\ &= \int_\eta^\infty e^{-2sx}\, d\nu(x), \end{aligned}$$

as claimed. In particular, we observe that the measure ν is independent of b.

We show next that if $w(t)$ satisfies (14), then $a_\nu = -\frac{1}{2}b_w$. First note that for any $s,\, t > 0$,

$$\begin{aligned} w(t+s) = \int_{a_\nu}^\infty e^{-2(t+s)x}\, d\nu(x) &\leqq e^{-2a_\nu s} \int_{a_\nu}^\infty e^{-2tx}\, d\nu(t) \\ &= e^{-2a_\nu s}\, w(t). \end{aligned}$$

It follows easily that $a_\nu \leqq -\frac{1}{2}b_w$. On the other hand, for any $a > a_\nu$, we have

$$w(t) = \int_{a_\nu}^\infty e^{-2tx}\, d\nu(x) \geqq \int_{a_\nu}^a e^{-2tx}\, d\nu(x) \geqq \nu([a_\nu, a])e^{-2at}.$$

Therefore $w(t)^{-1} = O(e^{2at})$ as $t \to \infty$, for any $a > a_\nu$. It follows easily that the Laplace integral $\varphi_w(z)$ defined by (11) converges whenever $\operatorname{Re} z > 2a_\nu$. So for any such z, $e^{-sz} \in \sigma(T_w(s))$. It then follows from Theorem 2 that $a_\nu \geqq -\frac{1}{2}b_w$.

Next we assert that if $w(t)$ satisfies (14), then for any $b > \frac{1}{2}b_w$, the linear mapping U_b that sends $g_b(t-s)$ to e^{-sz} extends to an isometric isomorphism of $L^2(m_w)$ onto $H^2(\mu_b)$. To prove this, it evidently suffices to show that

$$\int_0^\infty g_b(u-s)\, g_b(u-t)\, w(u)du = \int_{\Pi_\eta} e^{-(sz+t\bar{z})}\, d\mu_b(z),$$

for all $s \geqq t > 0$. We have

$$\begin{aligned} \int_0^\infty g_b(u-s)\, g_b(u-t)\, w(u)du &= \int_s^\infty e^{-b(u-s)}\, e^{-b(u-t)}\, w(u)du \\ &= e^{-b(s-t)} \int_s^\infty e^{-2b(u-s)}\, w(u)du \\ &= e^{-b(s-t)} \int_{a_\nu}^\infty d\nu(x) \int_s^\infty e^{-2b(u-s)}\, e^{-2xu}\, du \\ &= e^{-b(s-t)} \int_{a_\nu}^\infty e^{-2sx} \frac{d\nu(x)}{2(x+b)}\,. \end{aligned}$$

On the other hand,

$$\int_{\Pi_\eta} e^{-(sz+t\bar{z})}\, d\mu_b(z) = \int_{a_\nu}^{\infty} e^{-x(s+t)} \frac{d\nu(x)}{2(x+b)} \frac{(x+b)}{\pi} \int_{-\infty}^{\infty} \frac{e^{-iy(s-t)}dy}{y^2+(x+b)^2}$$
$$= \int_{a_\nu}^{\infty} e^{-x(s+t)} \frac{d\nu(x)}{2(x+b)} \frac{(x+b)}{\pi} \int_{-\infty}^{\infty} \frac{h(iy)dy}{y^2+(x+b)^2},$$

where $h(z) = e^{-(s-t)y}$. By the Poisson representation,

$$\frac{x+b}{\pi} \int_{-\infty}^{\infty} \frac{h(iy)dy}{y^2+(x+b)^2} = h(x+b) = e^{-(s-t)(x+b)}.$$

Therefore

$$\int_{\Pi_\eta} e^{-(sz+t\bar{z})}\, d\mu_b(z) = \int_{a_\nu}^{\infty} e^{-x(s+t)}\, e^{-(s-t)(x+b)} \frac{d\nu(x)}{2(x+b)}$$
$$= e^{-b(s-t)} \int_{a_\nu}^{\infty} e^{-2sx} \frac{d\nu(x)}{2(x+b)}.$$

Thus U_b extends to an isometric isomorphism of $L^2(m_w)$ onto $H^2(\mu_b)$, and clearly U_b intertwines the semigroup $T_w(s)$ with the semigroup of multiplication by e^{-sz} on $H^2(\mu_b)$. The theorem follows.

We remark in passing that if $w(t)$ is as in (14), then $w(t)$ is right hand continuous at 0 if and only if ν is a finite measure. This observation yields Embry and Lambert's result [**4**].

One of the most unfortunate aspects of the $H^2(\mu_b)$ model for $T_w(s)$ is its dependence upon b. We shall therefore replace $H^2(\mu_b)$ by a model which eliminates this dependence. To do this, we set $d\mu(x, y) = (2\pi)^{-1}d\nu(x)dy$, and observe that the linear mapping V which sends $g_b(t-s)$ to $e^{-sz}/(z+b)$ extends to an isometric mapping of $L^2(m_w)$ onto the $L^2(\mu)$-closed linear span of the functions $e^{-sz}/(z+b)$, $s > 0$, which we denote by $\mathscr{H}^2(\mu)$. Moreover, V intertwines the semigroup $T_w(s)$ with the semigroup of multiplication by e^{-sz} on $\mathscr{H}^2(\mu)$. It will be shown in the sequel that the space $\mathscr{H}^2(\mu)$ is actually independent of the choice of b.

We now focus our attention upon the space $\mathscr{L}^2(w)$. We shall give an alternative description of $\mathscr{L}^2(w)$, and establish a concrete connection between $\mathscr{L}^2(w)$ and $\mathscr{H}^2(\mu)$. To motivate what follows, we point out that in the special case $w(t) \equiv 1$, $\mathscr{L}^2(w)$ is actually H^2 of the half plane $x > 0$, and $\mathscr{H}^2(w)$ the space of boundary functions. See Hoffman [**9**], Duren [**2**].

In general, suppose that ν is a positive Borel measure defined on a half line $x \geqq \eta$, and assume that the function $w(t)$ defined by (14) is finite valued for all $t > 0$. Let the measure μ be defined on the half plane Π_η by $d\mu(x, y) = (2\pi)^{-1}d\nu(x)dy$. Let $p > 0$, and let $F(z)$ be a function analytic in the half plane $\Pi = \{z|x > a_\nu\}$. We make the assumption that for any $u > 0$ there exists a constant $M_u > 0$ such that

(16) $$\mathcal{M}_p(u+x;F) = \left(\int_{-\infty}^{\infty} |F(u+x+iy)|^p dy\right)^{1/p} \leqq M_u$$

for all $x > a_\nu$. For any such function $F(z)$, we set

(17) $$\mathcal{M}_p(u;F;\mu) = \left(\int_{\Pi} |F(u+z)|^p d\mu(z)\right)^{1/p},$$

$u > 0$. We then denote by $\mathcal{E}^p(\mu)$ the collection of all functions $F(z)$ analytic in Π and satisfying (16) for which

(18) $$||F||_{p,\mu} = \sup_{u>0} \mathcal{M}_p(u;F;\mu) < \infty.$$

We remark that $\mathcal{E}^p(\mu)$ is always nontrivial, and is a normed linear space for $p \geqq 1$. We are primarily concerned with the case $p = 2$. Specifically, we prove

THEOREM 4. *If μ and w are as defined above, then $\mathcal{L}^2(w)$ and $\mathcal{E}^2(\mu)$ are identical.*

Proof. We note first that the proof of Theorem 3 guarantees us that $\mathcal{L}^2(w)$ is a Hilbert space of functions analytic in the half plane Π. If $f \in L^2(m_w)$, let $f_u(t) = e^{-ut}f(t)$ for any $u > 0$, and observe that $\hat{f}(u+z) = \hat{f}_u(t)$. Now

$$\int_0^\infty |f_u(t)|^2 w(t)dt = \int_{a_\nu}^\infty d\nu(x) \int_0^\infty e^{-2tx}|f_u(t)|^2 dt < \infty.$$

Thus

$$\int_0^\infty e^{-2tx}|f_u(t)|^2 dt < \infty \text{ for all } x > a_\nu.$$

So by Plancherel's Theorem, we have

$$\int_0^\infty e^{-2t(x+u)}|f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^\infty |\hat{f}(u+x+iy)|^2 dy$$

for all $x > a_\nu$. Clearly, then, $\hat{f}(z)$ satisfies (16). Integrating with respect to ν, we obtain,

$$\mathcal{M}_2(u;\hat{f};\mu)^2 = \int_{\Pi} |\hat{f}(u+z)|^2 d\mu(z) = \int_0^\infty e^{-2ut}|f(t)|^2 w(t)dt$$
$$\leqq \int_0^\infty |f(t)|^2 w(t)dt.$$

Thus $\hat{f} \in \mathcal{E}^2(\mu)$.

Conversely, suppose $F \in \mathcal{E}^2(\mu)$. Then by (16), $F(u+z+a_\nu)$ belongs to H^2 of the right half plane $x > 0$ for all $u > 0$. Thus by the Paley-Wiener theorem, for any $u > 0$, there is a function $f_u \in L^2(m_1)$ such that

$$F(u+z+a_\nu) = \int_0^\infty e^{-zt}f_u(t)dt$$

for $x > 0$. Evidently, for any u, $v > 0$, we have $f_{u+v}(t) = e^{-vt}f_u(t)$ almost

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everywhere. Therefore we may unambiguously define a measurable function $f(t)$ on $(0, \infty)$ by setting

$$f(t) = e^{(a_\nu + u)t} f_u(t)$$

for any $u > 0$. Then

$$F(z) = \int_0^\infty e^{-tz} f(t) dt$$

for any $z \in \Pi$, and by Plancherel's Theorem,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |F(u + x + iy)|^2 dy = \int_0^\infty e^{-2(u+x)t} |f(t)|^2 dt$$

for all $u > 0$ and $x > a_\nu$. Integrating with respect to ν, we obtain

$$\int_0^\infty e^{-2ut} |f(t)|^2 w(t) dt = \mathscr{M}_2(u; F; \mu)^2 \leqq \text{const.}$$

for all $u > 0$. It follows easily from monotone convergence that $f \in L^2(m_w)$. The theorem is proved.

On account of this result, we observe that the Laplace Transform is an isometric isomorphism of $L^2(m_w)$ onto $\mathscr{E}^2(\mu)$ which intertwines the semigroup $T_w(s)$ with the semigroup of multiplication by e^{-sz} on $\mathscr{E}^2(\mu)$. On the other hand, for any given $b > \frac{1}{2}b_w$ we have an isometric isomorphism of $L^2(m_w)$ onto $\mathscr{H}^2(\mu)$ which carries $g_b(t - s)$ onto $e^{-sz}/(z + b)$, $s > 0$. By composing this mapping with the inverse Laplace Transform, we obtain an isometric mapping of $\mathscr{E}^2(\mu)$ onto $\mathscr{H}^2(\mu)$. Given $F \in \mathscr{E}^2(\mu)$, we denote the image of F in $\mathscr{H}^2(\mu)$ by $\tilde{F}$. In the case when ν has no mass at a_ν, it is not difficult to see that $\tilde{F} = F \mu -$ a.e. On the other hand, if ν has mass at a_ν, then $F(z + a_\nu)$ belongs to H^2 of the right half plane $x > 0$ for any $F \in \mathscr{E}^2(\mu)$. Thus in this case, $\tilde{F}(a_\nu + iy) = \lim_{u\to 0} F(u + a_\nu + iy)\, dy -$ a.e., and $\tilde{F}(z) = F(z)\, \mu -$ a.e. on supp $\mu \cap \Pi$. On account of these correspondences, $\mathscr{E}^2(\mu)$ is naturally embedded as a subspace of $L^2(\mu)$. Loosely speaking, a function in $\mathscr{E}^2(\mu)$ may be identified with its representative in $\mathscr{H}^2(\mu)$ in much the same way as an H^2 function is identified with its boundary function.

The spaces $\mathscr{E}^p(\mu)$, $p > 0$, appear to be the natural half plane analogue of the spaces $E^p(\sigma)$, $p > 0$, defined and studied in [**5**; **6**] in connection with subnormal weighted shifts. For the convenience of the reader, we recall the definition of these latter spaces. Let ρ be a finite positive Borel measure defined on $[0, \infty)$, having compact support and no mass at 0. We set $c_\rho = \sup\{c | \nu([a, c]) > 0$ for all $0 \leqq a < c\}$, and let $D = \{z |\, |z| < c_\rho\}$. We then define a measure σ on $\bar{D}$ by $d\sigma(r, \theta) = (2\pi)^{-1} d\rho(r) d\theta$. If $f(z)$ is any function analytic in D, for any $p > 0$ and any $0 \leqq r < 1$, we write

$$(19) \quad M_p(r; f; \sigma) = \left(\int_{\bar{D}} |f(r\zeta)|^2 d\sigma(\zeta) \right)^{1/p}.$$

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We then denote by $E^p(\sigma)$ the collection of all functions $f(z)$ analytic in D such that

$$(20) \quad \|f\|_{p,\sigma} = \sup_{0 \leqq r < 1} M_p(r; f; \sigma) < \infty.$$

It was proved in [**5**] that $E^p(\sigma)$ is a complete linear metric space for all $p > 0$, a Banach space for $p \geqq 1$, and a functional Hilbert space for $p = 2$. The spaces $E^2(\sigma)$ provide a functional model for subnormal weighted shifts.

In the classical case of H^p spaces, it is well known that H^p of the disk is isometrically isomorphic to H^p of the half plane, the isomorphism being induced by a conformal mapping of the half plane to the disk. Unfortunately, in general no such simple relationship can exist between the spaces $\mathscr{E}^p(\mu)$ and $E^p(\sigma)$. However, it is possible to obtain a variety of nonisometric embeddings of any $E^p(\sigma)$ space into an $\mathscr{E}^p(\mu)$ space, as the following result serves to indicate.

THEOREM 5. *Let ρ and σ be as described above, and let ν be the measure defined on a half line by $d\nu(x) = -d\rho(e^{-x})$. Let $d\mu(x, y) = (2\pi)^{-1}d\nu(x)dy$. Then for any $b > \frac{1}{2}b_w$, any $p > 0$, and any $f \in E^p(\sigma)$, the function $F(z) = f(e^{-z})/(z + b)^{2/p}$ belongs to $\mathscr{E}^p(\mu)$, and*

$$(21) \quad \|F\|_{p,\mu} \leqq \left[\frac{\coth (a_\nu + b)}{a_\nu + b}\right]^{1/p} \|f\|_{p,\sigma}.$$

Proof. We note first that ν is a finite positive Borel measure, so that $w(t)$ is finite valued for all $t \geqq 0$. Hence $\mathscr{E}^p(\mu)$ is meaningful. We also point out for the record that $b_w = -2a_\nu = 2 \log c_\rho$.

Now if $f \in E^p(\sigma)$, then for any $u > 0$ and $x > a_\nu$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |F(u + x + iy)|^p dy &= \int_{-\infty}^{\infty} \frac{|f(e^{-u}e^{-z})|^p}{(x + b)^2 + y^2} dy \\
&= \sum_{n=-\infty}^{\infty} \int_{(n-1)\pi}^{(n+1)\pi} \frac{|f(e^{-u}e^{-z})|^p\, dy}{(x + b)^2 + y^2} \\
&= \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{|f(e^{-u}e^{-z})|^p\, dy}{(x + b)^2 + (y + n\pi)^2} \\
&\leqq \left[\sum_{n=-\infty}^{\infty} \frac{1}{(a_\nu + b)^2 + (n - 1)^2\pi^2}\right] \\
&\qquad \times \int_{-\pi}^{\pi} |f(e^{-u}e^{-z})|^p\, dy \\
&= \left[\frac{\coth (a_\nu + b)}{a_\nu + b}\right] \int_{-\pi}^{\pi} |f(e^{-u}e^{-z})|^p\, dy.
\end{aligned}$$

It follows easily that $F(z)$ satisfies (16). Multiplying by $(2\pi)^{-1}$ and integrating with respect to ν, we obtain

$$\mathscr{M}_p(u; F; \mu)^p \leqq \left[\frac{\coth(a_\nu + b)}{a_\nu + b}\right] \frac{1}{2\pi} \int_{a_\nu}^{\infty} d\nu(x) \int_{-\pi}^{\pi} |f(e^{-u}e^{-x}e^{-iy}|^p\, dy$$
$$= \left[\frac{\coth(a_\nu + b)}{a_\nu + b}\right] \frac{1}{2\pi} \int_0^{c_\rho} d\rho(t) \int_{-\pi}^{\pi} |f(e^{-u}te^{iy})|^p\, dy$$
$$= \left[\frac{\coth(a_\nu + b)}{a_\nu + b}\right] M_p(e^{-u}; f; \sigma)^p.$$

The result follows.

As a particular consequence of Theorem 5, we note that if ν is a finite positive Borel measure defined on a half line $x \geqq \eta$, and if the periodic Dirichlet series $G(z) = \sum_{n=0}^{\infty} a_n e^{-nz}$ satisfies $\sum_{n=0}^{\infty} w(n)|a_n|^2 < \infty$, then the function $F(z) = G(z)/(z + b)$ belongs to $\mathscr{E}^2(\mu)$ for any $b > \frac{1}{2}b_w$, and

$$(22) \quad ||F||_{2,\mu}^2 \leqq \left[\frac{\coth(a_\nu + b)}{a_\nu + b}\right] \sum_{n=0}^{\infty} w(n)|a_n|^2.$$

We conclude this section by listing some curious examples to illustrate the foregoing material

(a) We consider the case when $w(t)$ is a Dirichlet series with positive coefficients and abscissa of convergence $\leqq 0$. Thus

$$w(t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} = \int_{\frac{1}{2}\lambda_1}^{\infty} e^{-2tx}\, d\nu(x),$$

where $a_n > 0$ for all $n \geqq 1$, $\lambda_n \to \infty$ as $n \to \infty$, and the measure ν is the completely atomic measure defined by $\nu(\{\frac{1}{2}\lambda_n\}) = a_n$, $n \geqq 1$. Clearly $b_w = -\lambda_1$. For any $p > 0$, $\mathscr{E}^p(\mu)$ is the collection of all functions $F(z)$ analytic for $x > \lambda_1$ and satisfying

$$(23) \quad \sup_{u>0} \sum_{n=1}^{\infty} a_n \mathscr{M}_p(u + \tfrac{1}{2}\lambda_n; F)^p < \infty.$$

In particular, if $\lambda_1 = 0$, we observe that $\mathscr{E}^p(\mu)$ is contained in H^p of the half plane $x > 0$. The containment is proper if and only if $\sum a_n = \infty$. For if $\sum a_n < \infty$, then

$$a_1 \int_{-\infty}^{\infty} |F(u + iy)|^p\, dy \leqq \sum_{n=1}^{\infty} a_n \int_{-\infty}^{\infty} |F(u + \tfrac{1}{2}\lambda_n + iy)|^p\, dy$$
$$\leqq \left(\sum_{n=1}^{\infty} a_n\right) \int_{-\infty}^{\infty} |F(u + iy)|^p\, dy,$$

so that $F \in \mathscr{E}^p(\mu)$ if and only if $F(z)$ belongs to H^p of the half plane $x > 0$. On the other hand, if $\sum a_n = \infty$, then $(z + b)^{-2/p} \notin \mathscr{E}^p(\mu)$ for all $b > 0$.

As a special case, these observations may be applied to the series

$$w(t) = \sum_{n=1}^{\infty} \frac{1}{n} e^{-t\log n} = \zeta(t + 1),$$

where $\zeta(s)$ is Riemann's zeta function. In this case (23) reduces to

$$(24)\quad \sup_{u>0} \sum_{n=1}^{\infty} \frac{1}{n} \mathscr{M}_p(u + \log\sqrt{n}; F)^p < \infty .$$

In particular, a measurable function $f(t)$ on $(0, \infty)$ satisfies $\int_0^\infty |f(t)|^2\zeta(t+1)dt < \infty$ if and only if $\hat{f}(z)$ satisfies (24) with $p = 2$. By monotone convergence,

$$(25)\quad 2\pi \int_0^\infty |f(t)|^2\zeta(t+1)dt = \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{\infty} |\hat{f}(\log\sqrt{n} + iy)|^2\, dy.$$

(b) Next we consider the example $w(t) = t^{-\alpha}e^{1/t}$, $\alpha > 0$. In this case,

$$w(t) = \int_0^\infty e^{-tu}\, u^{(\alpha-1)/2}\, I_{\alpha-1}\,(2\sqrt{u})\, du$$
$$= 2^{(\alpha+1)/2} \int_0^\infty e^{-2tx}\, x^{(\alpha-1)/2}\, I_{\alpha-1}\,(2\sqrt{2x})\, dx,$$

where $I_\nu(z)$ is a modified Bessel function. See Watson [**12**]. Therefore $d\nu(x) = 2^{(\alpha+1)/2}x^{(\alpha-1)/2}I_{\alpha-1}(2\sqrt{2x})dx$ for $x > 0$. The space $\mathscr{E}^p(\mu)$ consists of all functions $F(z)$ analytic in the half plane $x > 0$ and there satisfying (16), such that

$$(26)\quad \sup_{u>0} \int_0^\infty \mathscr{M}_p(u + x; F)^p\, x^{(\alpha-1)/2}\, I_{\alpha-1}\,(2\sqrt{2x})\, dx < \infty .$$

In particular, a measurable function $f(t)$ on $(0, \infty)$ satisfies $\int_0^\infty |f(t)|^2t^{-\alpha}e^{1/t}dt < \infty$ if and only if $\hat{f}(z)$ satisfies (16) and (26) with $p = 2$. Moreover,

$$(27)\quad \pi \int_0^\infty |f(t)|^2t^{-\alpha}e^{1/t}dt = 2^{(\alpha-1)/2} \int_0^\infty \mathscr{M}_2(x; F)^2x^{(\alpha-1)/2}\, I_{\alpha-1}\,(2\sqrt{2x})\, dx.$$

We can also explicitly compute the reproducing kernel of $\mathscr{E}^2(\mu)$. Indeed,

$$(28)\quad \varphi_w(z) = \int_0^\infty e^{-zt-1/t}\, t^\alpha dt = 2z^{-(\alpha+1)/2}\, K_{-(\alpha+1)}(2\sqrt{z}),$$

where $K_\nu(z)$ is a modified Bessel function. See Watson [**12**, p. 183]. Therefore $K_w(\zeta, z) = 2(\bar{\zeta} + z)^{-(\alpha+1)/2}K_{-(\alpha+1)}(2\sqrt{\bar{\zeta} + z})$.

(c) As a final illustration, we consider the simple example $w(t) = t^{-2}$. In this case,

$$w(t) = 4 \int_0^\infty e^{-2tx}\, x\, dx.$$

Thus $d\nu(x) = 4xdx$ for $x > 0$. The space $\mathscr{E}^2(\mu)$ is of particular interest. To see this, we have only to note that the mapping $f(t) \to -tf(t)$ is an isometric isomorphism of $L^2(m_1)$ onto $L^2(m_w)$. It then follows that $G \in \mathscr{E}^2(\mu)$ if and only if $G(z) = F'(z)$ for some $F(z)$ in H^2 of the right half plane. Moreover, for any $F(z)$ in H^2 of the right half plane,

$$(29)\quad \int_{-\infty}^{\infty} |F(x)|^2 dx = \frac{2}{\pi} \int_0^{\infty} x\, dx \int_{-\infty}^{\infty} |F'(x+iy)|^2\, dy.$$

We note for the record that

$$(30)\quad \varphi_w(z) = \int_0^{\infty} e^{-zt}\, t^2\, dt = \frac{2}{z^3}.$$

Thus the reproducing kernel of $\mathscr{E}^2(\mu)$ is $K_w(\zeta, z) = 2(\bar{\zeta} + z)^{-3}$.

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