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An Existence Theory for Incomplete Designs

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Abstract. An incomplete pairwise balanced design is equivalent to a pairwise balanced design with a distinguished block, viewed as a 'hole'. If there are v points, a hole of size w, and all (other) block sizes equal k, this is denoted $\mathrm{IPBD}((v;w),k)$. In addition to congruence restrictions on v and w, there is also a necessary inequality: $v > (k-1)w$. This article establishes two main existence results for $\mathrm{IPBD}((v;w),k)$: one in which w is fixed and v is large, and the other in the case $v > (k-1+\epsilon)w$ when w is large (depending on ϵ). Several possible generalizations of the problem are also discussed.

1 Introduction

Let v be a positive integer and let $K \subseteq \mathbb{Z}_{\geq 2} := \{2,3,4,\dots\}$. A *pairwise balanced design* $\mathrm{PBD}(v,K)$ is a pair $(V,\mathcal{B})$, where

- V is a v-element set of *points*,
- $\mathcal{B} \subseteq \cup_{k\in K}\binom{V}{k}$ is a family of subsets of V, called *blocks*, and
- every two distinct points appear together in exactly one block.

This object is sometimes also called a *linear space* with block sizes in K.

The case when $K = \{k\}$ is of primary interest. We then use the notation $\mathrm{PBD}(v,k)$ for consistency but should mention that the more standard notation is $(v,k,1)$-BIBD. More generally, pairwise balanced designs and (balanced incomplete) block designs permit an additional parameter λ and ask that every two distinct points appear together in exactly λ blocks. For the moment, though, our attention is restricted to $\lambda = 1$ and $K = \{k\}$. Recall that the case $k = 3$ yields Steiner triple systems.

In a $\mathrm{PBD}(v,k)$, note that there are $\binom{k}{2}$ pairs in each block, and that these must partition $\binom{V}{2}$. In addition, for any point $x \in V$, the remaining $v-1$ points must partition into $(k-1)$-element 'neighbourhoods' in the blocks incident with x. It is helpful to think of the resulting numerical restrictions as 'global' and 'local' conditions, respectively, and we record them below (in reverse order).

Proposition 1.1 *The existence of a* $\mathrm{PBD}(v,k)$ *implies*

$$v-1 \equiv 0 \pmod{k-1}, \tag{1.1}$$

$$v(v-1) \equiv 0 \pmod{k(k-1)}. \tag{1.2}$$

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The 'asymptotic sufficiency' of these conditions is a celebrated result due to Richard M. Wilson.

Theorem 1.2 (Wilson [17]) *Given any integer $k \geq 2$, there exist* PBD(v,k) *for all sufficiently large v satisfying* (1.1) *and* (1.2).

Theorem 1.2 lays the foundation for a rich existence theory for a variety of combinatorial structures, including PBD(v,K), graph decompositions, and resolvable designs; see [1, 5–7, 9, 18, 21] and the references therein. Our focus here is a basic (but challenging) extension.

Let $v \geq w$ be positive integers and $K \subseteq \mathbb{Z}_{\geq 2}$. An *incomplete pairwise balanced design* IPBD$((v;w),K)$ is a triple $(V,W,\mathcal{B})$ where

- V is a set of v points and $W \subset V$ is a *hole* of size w;
- $\mathcal{B} \subseteq \cup_{k\in K}\binom{V}{k}$ is a family of blocks;
- no two distinct points of W appear in a block; and
- every two distinct points not both in W appear together in exactly one block.

An equivalent notion is a PBD$(v, K \cup \{w^*\})$, where the star indicates that there is exactly one block of size w if $w \notin K$ and at least one block of size w if $w \in K$. Given an IPBD$((v;w),K)$, say $(V,W,\mathcal{B})$, the system $(V, \mathcal{B} \cup \{W\})$ is a PBD$(v, K \cup \{w^*\})$.

A closely related notion is that of a PBD(v,K), say $(V,\mathcal{B})$, containing a *subdesign* PBD(w,K), say $(W,\mathcal{A})$, where we have $W \subseteq V$ and $\mathcal{A} \subseteq \mathcal{B}$. We obtain an IPBD$((v;w),K)$ as $(V,W,\mathcal{B}\smallsetminus\mathcal{A})$. On the other hand, an IPBD with hole W can be 'filled' with a PBD (or another IPBD) on W, but only when this smaller design exists.

The case $w = v$ leads to $\mathcal{B} = \varnothing$ and we exclude this in what follows. The case $w = 1$ reduces to PBD(v,K), since such a hole contains no pairs. For simplicity, we again restrict our attention to the case $K = \{k\}$, although we have a few remarks about the general case later. By analogy with (1.1) and (1.2), there are naive divisibility conditions on the parameters.

Proposition 1.3 *The existence of an* IPBD$((v;w),k)$ *implies*

$$v-1 \equiv w-1 \equiv 0 \pmod{k-1}, \tag{1.3}$$

$$v(v-1) - w(w-1) \equiv 0 \pmod{k(k-1)}. \tag{1.4}$$

We say integers v and w are *admissible* (for IPBD of block size k) if (1.3) and (1.4) hold. There is another necessary condition taking the form of an inequality.

Proposition 1.4 *Every point in $V \smallsetminus W$ is incident with exactly $\frac{v-1}{k-1} - w$ blocks disjoint from W. Therefore, the existence of an* IPBD$((v;w),k)$ *with $v > w$ implies*

$$v \geq (k-1)w + 1. \tag{1.5}$$

Proof A point $x \in V \smallsetminus W$ is incident with $\frac{v-1}{k-1}$ blocks. Each such block covers at most one point in W, and conversely every point in W is together in exactly one block with x. It follows that, of the blocks containing x, exactly w of them intersect W. So $\frac{v-1}{k-1} \geq w$. ∎

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Remark Equality in (1.5) holds if and only if every block intersects the hole.

In what follows, the block size k is taken to be a fixed but arbitrary integer at least 3. Obviously, the case $k = 2$ is trivial. For $k = 3$, the necessary conditions (congruence and inequality) are known to be sufficient. This was originally proved in two separate cases: (i) the Doyen–Wilson Theorem [4] on Steiner triple systems containing subsystems; and (ii) Mendelsohn and Rosa's adaptation [11] to cases where v and w do not admit Steiner triple systems. Also, see Theorem 6.9 in Colbourn and Rosa's book [3] for a unified treatment.

Theorem 1.5 (Extended Doyen–Wilson Theorem [4, 11]) *An* $\mathrm{IPBD}((v;w),3)$ *with* $0 < w < v$ *exists if and only if* $v \geq 2w+1$*,* v *and* w *are both odd, and* $3 \mid v(v-1)-w(w-1)$.

Remarks There is an extension of Theorem 1.5 to higher λ; this is sometimes known as 'Stern's Theorem'. The case $k = 4$ with $\lambda = 1$ has also been solved; see [14].

The papers [13–15] are also key references on this problem. Unlike many earlier methods, these papers lean heavily on recursive design-theoretic techniques. The 'Doyen–Wilson' portion of Theorem 1.5 is re-proved in this way, and the case $k = 4$, $\lambda = 1$ is completely settled. In fact, those papers observe many powerful constructions (overlapping ours somewhat) for IPBDs. But, unfortunately, the lack of certain small ingredients for general k limits their use for our purposes. Even when a recursion is possible for some k, it is tricky to construct examples covering the most general forms of the global condition (1.4) and inequality (1.5).

Here, we overcome these challenges and establish two main results in the direction of an asymptotic existence theory for incomplete pairwise balanced designs. First, we obtain existence for all large admissible v relative to a fixed admissible hole size w.

Theorem 1.6 *Given fixed integers* $k \geq 2$ *and* $w \equiv 1 \pmod{k-1}$*, there exist* $\mathrm{IPBD}((v;w),k)$ *for all sufficiently large* v *satisfying* (1.3) *and* (1.4).

Second, we obtain existence for all large admissible v and w when the inequality (1.5) is weakened very slightly.

Theorem 1.7 *Let* $k \geq 2$ *be a fixed integer. For any real* $\epsilon > 0$*, there exists* $\mathrm{IPBD}((v;w),k)$ *for all sufficiently large* v *and* w *satisfying* (1.3)*,* (1.4)*, and* $v > (k-1+\epsilon)w$.

Remark In the above statement, 'sufficiently large' depends on ϵ.

Note that Theorem 1.6 also follows from the unpublished thesis of Gustavsson [7] and the preprint by Keevash [8]; however, our proof uses only basic constructions. Theorem 1.7 is far stronger than what the bound given from [7] would imply, but requires large w. It seems likely that the new random construction in [8] can be adapted to IPBDs, even with large holes. On the other hand, we feel there is still value in direct constructions. In fact, our proofs tie together many different types of combinatorial designs.

The outline of our presentation is as follows. We first prove a version of Theorem 1.6 in which the global condition (1.4) is replaced by the stronger hypothesis that $v \equiv w$ modulo a large period. This is a fairly standard use of 'transversal designs' and 'group divisible designs'. Next, we observe there is existence for all sufficiently large v, w which achieve equality in (1.5). This is immediate from a standard equivalence with 'resolvable designs'.

Now, a crucial next step is the use of 'incomplete group divisible designs' to realize an example with any prescribed congruence classes for v and w. (In fact, this holds for an arbitrary large modulus and with a control on v/w.) Together with the first step, this completes the proof of Theorem 1.6. To prove Theorem 1.7, we make use of (weighted) transversal designs and a 'postage stamp' calculation. Most of the previous results are needed as ingredients.

The conclusion discusses the remaining open cases and some possible extensions of the problem.

2 Transversal Designs and GDDs

Let T denote an integer partition of v. A *group divisible design* of *type* T with block sizes in K, denoted $\mathrm{GDD}(T, K)$, is a triple $(V, \Pi, \mathcal{B})$ such that

- V is a set of v points,
- $\Pi = \{V_1, \ldots, V_u\}$ is a partition of V into *groups* so that $T = \{|V_1|, \ldots, |V_u|\}$,
- $\mathcal{B} \subseteq \cup_{k \in K} \binom{V}{k}$ is a set of blocks meeting each group in at most one point, and
- any two points from distinct groups V_j appear together in exactly one block.

Often in this context, exponential notation such as n^u is used to abbreviate u parts or 'groups' of size n. For instance, a *transversal design* $\mathrm{TD}(k, n)$ is a $\mathrm{GDD}(n^k, k)$. In this case, the blocks are transversals of the partition. A $\mathrm{TD}(k, n)$ is equivalent to $k-2$ mutually orthogonal latin squares of order n, where two groups are reserved to index the rows and columns of the squares. Here is the famous existence theorem for TDs.

Theorem 2.1 (Chowla, Erdős, and Strauss [2]) *Given k, there exist $TD(k, n)$ for all sufficiently large integers n.*

In general, a group divisible design of type $T = g^u$ is called *uniform*. There is a satisfactory asymptotic existence result for such objects, stated here for later use.

Theorem 2.2 (Draganova [5] and Liu [10]) *Given g and $K \subseteq \mathbb{Z}_{\geq 2}$, there exist $GDD(g^u, K)$ for all sufficiently large u satisfying*

$$g(u-1) \equiv 0 \pmod{\alpha(K)} \quad \textit{and} \quad g^2 u(u-1) \equiv 0 \pmod{\beta(K)},$$

where $\alpha(K) = \gcd\{k-1 : k \in K\}$ *and* $\beta(K) = \gcd\{k(k-1) : k \in K\}$.

In a $\mathrm{PBD}(v, k)$, it is easy to see that every point belongs to the same number $r = \frac{v-1}{k-1}$ of blocks. This quantity is called the *replication number*. That r is an integer follows from (1.1). In fact, a restatement of Theorem 1.2 is that PBDs exist for all sufficiently large and numerically allowed replication numbers. More precisely, let us define $r_0(k)$ such that there exist $\mathrm{PBD}((k-1)r+1, k)$ for all $r \geq r_0(k)$ satisfying $r(r-1) \equiv 0 \pmod{k}$. From this, we note that there exist PBDs with arbitrary large and consecutive replication numbers $0, 1 \pmod{k}$. This is helpful in what follows.

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One motivation for GDDs is that their groups act as a partition into holes; each can be 'filled' with PBDs (or smaller GDDs). For example, a PBD(v,k) is equivalent both to a GDD$(1^v,k)$ and also to a GDD$((k-1)^r,k)$. The second equivalence follows in one direction from deleting a point with all incident blocks, and in the other direction by adding a new point and filling groups. There are similar equivalences under the presence of a hole.

Proposition 2.3 *The following are equivalent:*

- IPBD$((v;w),k)$*;*
- GDD$(1^{v-w}w^1,k)$*;*
- GDD$((k-1)^{\frac{v-w}{k-1}}(w-1)^1,k)$.

Another feature of GDDs is that they admit a natural 'expansion' of their groups. This is made precise in the next construction. The idea is simply to independently replicate the points of a 'master' GDD, replacing its blocks by 'ingredient' GDDs of the right type.

Lemma 2.4 (Wilson's fundamental construction [20]) *Suppose there exists a GDD $(V,\Pi,\mathcal{B})$, where $\Pi=\{V_1,\dots,V_u\}$. Let $\omega\colon V\to\mathbb{Z}_{\geq 0}$, assigning nonnegative weights to each point in such a way that for every $B\in\mathcal{B}$ there exists a GDD$([\omega(x):x\in B],K)$. Then there exists a GDD(T,K), where*

$$T=\Big[\sum_{x\in V_1}\omega(x),\dots,\sum_{x\in V_u}\omega(x)\Big].$$

Combining this construction with the earlier remark about consecutive replication numbers is a powerful combination. One application we use is a construction of certain non-uniform GDDs.

Lemma 2.5 *Let $m\geq r_0(k)$ with $m\equiv 0 \pmod{k}$. There exists a constant $s_0=s_0(m,k)$ such that, for all integers $s\geq s_0$ and any integer t satisfying $0\leq t\leq s$ and $s\equiv t\equiv 0 \pmod{k-1}$, there exist* GDD$(s^m t^1,k)$.

Proof By assumption on m, there exist PBDs of block size k and consecutive replication numbers $m,m+1$. Equivalently, we have GDD$((k-1)^m,k)$ and GDD$((k-1)^{m+1},k)$.

By Theorem 2.1, there exist TD$(m+1,\frac{s}{k-1})$ for s sufficiently large with $s\equiv 0 \pmod{k-1}$. Truncate one group of such a TD (that is, give weight zero to some points) so that there are $\frac{t}{k-1}$ points remaining, $0\leq t\leq s$ with $t\equiv 0 \pmod{k-1}$. The result is a

$$\text{GDD}\Big(\Big(\tfrac{s}{k-1}\Big)^m\Big(\tfrac{t}{k-1}\Big)^1,\{m,m+1\}\Big).$$

Using this as the master, apply Lemma 2.4 with constant weight $\omega=k-1$. The ingredients exist by choice of m, and so we obtain a GDD$(s^m t^1,k)$. ∎

We are now ready to prove a preliminary 'sparse' result on fixed hole sizes.

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Proposition 2.6 *Let $m \geq r_0(k)$ with $m \equiv 0 \pmod{k}$. For any $w \equiv 1 \pmod{k-1}$, there exist* IPBD$((v;w),k)$ *for all sufficiently large $v \equiv w \pmod{mk(k-1)}$.*

Proof Suppose $v-w \equiv 0 \pmod{mk(k-1)}$. Appealing to Lemma 2.5, we can choose a large enough so that there exists a GDD$((ak(k-1))^m(w-1)^1,k)$ and also so that $ak \geq r_0(k)$. Fill the m large groups with copies of a GDD$((k-1)^{ak},k)$. This results in a GDD$((k-1)^{amk}(w-1)^1,k)$, which, by Proposition 2.3, is equivalent to an IPBD$((v;w),k)$. Note that a can be incremented, and the result follows. ∎

Remark Although the modulus depends only on k, the 'sufficiently large' depends on w.

3 Resolvable Designs

A PBD(v,k) is *resolvable* if its blocks $\mathcal{B}$ can be resolved into partitions of V, each of which is called a *parallel class*. The number of parallel classes must of course agree with the replication number r, since every point is in exactly one parallel class for each incident block.

Combining $k|v$ with (1.1), the necessary numerical condition for resolvable PBD(v,k) is

$$v \equiv k \pmod{k(k-1)}. \tag{3.1}$$

The reader can easily verify that the global condition (1.2) is automatically implied.

For example, resolvable PBD$(v,2)$ are equivalent to one-factorizations of the complete graph of order v, for which (3.1) – that v be even – is well known to be sufficient.

The asymptotic existence of resolvable designs (of fixed block size) was a celebrated achievement.

Theorem 3.1 (Ray Chaudhuri and Wilson [12]) *Given any integer $k \geq 2$, there exists resolvable* PBD(v,k) *for all sufficiently large v satisfying* (3.1).

It is an important observation that resolvable designs are in some sense equivalent to incomplete designs with 'large' holes. Let us assume here and in what follows that $k \geq 3$; the case of block size two is trivial for IPBD.

Proposition 3.2 *There exists an* IPBD$((v;w),k)$ *with $v=(k-1)w+1$ if and only if there exists a resolvable* PBD$(v-w,k-1)$.

Proof Let $(V,W,\mathcal{B})$ be an IPBD achieving equality in (1.5). Then every block intersects W. Truncating W results in a PBD$(v-w,k-1)$, say with blocks $\mathcal{B}'$. These blocks resolve into $w=\frac{v-w}{k-2}$ parallel classes of the form $\mathcal{R}_x = \{B-x : x \in B \in \mathcal{B}\}$, one class for every point in W.

This process is reversible. Given a resolvable PBD$(n,k-1)$, just add $w=\frac{n-1}{k-2}$ new points and extend the blocks of each parallel class with a common new point. The result is an IPBD$((n+w;w),k)$. ∎

When combined with Theorem 3.1, we get a straightforward asymptotic solution for IPBDs in the case of maximum holes.

Corollary 3.3 *For w sufficiently large and $w \equiv 1 \pmod{k-1}$, there exist* $\mathrm{IPBD}((v;w),k)$ *for* $v = (k-1)w+1$.

Remark The v constructed here satisfies $v \equiv 1 - w \pmod{k}$. This is our first example outside of the class $v \equiv w \pmod{k}$, and, as we will see later, this is enough to 'seed' a construction for any admissible congruence classes.

Suppose we wish to let v be slightly larger than $(k-1)w+1$. If, for instance, $v = (k-1)(w+1)+1$, then every point in $V \setminus W$ is incident with exactly one block disjoint from W. This can be achieved with one parallel class of blocks of size k and the rest of size $k-1$. To this end, we are led to consider resolvability in group divisible designs.

Naturally, we declare a GDD with blocks $\mathcal{B}$ and points V to be *resolvable* if $\mathcal{B}$ can be resolved into partitions of V, here also called *parallel classes.* An asymptotic existence result for resolvable GDDs of uniform type was recently proved.

Theorem 3.4 ([1]) *Given $g \geq 1$ and $k \geq 2$, there exist resolvable GDD(g^u,k) for all sufficiently large integers u satisfying*

$$gu \equiv 0 \pmod{k} \quad \textit{and}$$
$$g(u-1) \equiv 0 \pmod{k-1}.$$

In general, if there exists a $\mathrm{PBD}(g,k)$, then groups of a resolvable $\mathrm{GDD}(g^u, k-1)$ can be filled to produce $\mathrm{IPBD}((gu+w;w),k)$, where $w = \frac{g(u-1)}{k-2}$. We have $v = gu + w = (k-1)(w+r)+1$, where $r = \frac{g-1}{k-1}$ is the replication number of the hypothesized $\mathrm{PBD}(g,k)$.

4 Incomplete Group Divisible Designs

An *incomplete group divisible design*, or IGDD, is a quadruple $(V,\Pi,\Xi,\mathcal{B})$ such that V is a set of v points, $\Pi = \{V_1,\dots,V_u\}$ is a partition of V into 'groups', $\Xi = \{W_1,\dots,W_u\}$ with $W_i \subseteq V_i$ called 'holes' for each i, and $\mathcal{B} \subseteq \binom{V}{\geq 2}$ is a set of blocks such that

- two points get covered by a block (exactly one block) if and only if they come from different groups, say V_i and V_j, $i \neq j$, **and** they do not both belong to the corresponding holes W_i and W_j.

As with GDDs, the type of an IGDD can be written by listing, using exponential notation when appropriate, the pairs $(|V_i|;|W_i|)$ of group size and corresponding hole size. A (uniform) *incomplete group divisible design* of type $(g;h)^u$ with block size k is abbreviated $\mathrm{IGDD}((g;h)^u,k)$. Note that this is equivalent to an edge-decomposition of the multipartite graph $\overline{u \cdot K_g} - \overline{u \cdot K_h}$ into cliques K_k.

We state the necessary 'divisibility' conditions on uniform IGDDs.

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Proposition 4.1 *The existence of an IGDD$((g;h)^u,k)$ implies*

$$g(u-1) \equiv h(u-1) \equiv 0 \pmod{k-1}, \tag{4.1}$$

$$(g^2-h^2)u(u-1) \equiv 0 \pmod{k(k-1)}. \tag{4.2}$$

We say that integers g, h, and u are *admissible* (for IGDD of block size k) if (4.1) and (4.2) hold.

Also, a similar counting argument as for Proposition 1.4 give

$$\frac{hu(g-h)(u-1)}{k-1}\binom{k-1}{2} \le (g-h)^2\binom{u}{2},$$

or simply

$$g \ge (k-1)h. \tag{4.3}$$

From the theory of 'edge-colored graph decompositions' (see [9]), we have an asymptotic existence result (in u) for uniform IGDDs. The proof is a straightforward extension of that of Theorem 2.2, with details now included in [16]. For completeness, we sketch a proof in Section 5.

Theorem 4.2 *Given integers g,h,k with $k \ge 2$ and $g \ge (k-1)h$, there exists an* IGDD$((g;h)^u,k)$ *whenever u is sufficiently large satisfying* (4.1) *and* (4.2).

As expected, groups of an IGDD can be filled with examples of IPBD. The excess points on each group get identified.

Construction 4.3 *Suppose there exists an* IGDD$((g;h)^u,k)$ *and an* IPBD$((x;y),k)$ *with $g-h = x-y$ and $y \ge h$. Then there exists an* IPBD$((v;w),k)$ *with $v-w = u(x-y)$ and $w = (u-1)h+y$.*

Remark If a subdesign on one group is deleted from the resulting IPBD, we obtain an instance of a $\diamond$-IPBD, in which there are two intersecting holes. Such objects are defined and used in [15].

Let us use the extremal IPBDs in Corollary 3.3 as ingredients, so that we are taking $x = (k-1)y+1$. Substituting this and $y = w-(u-1)h$ into $v-w = u(x-y)$, one obtains an identity connecting the parameters u and h. That is, we have

$$u\big[(k-2)(w-(u-1)h)+1\big] = v-w, \tag{4.4}$$

or equivalently, after some simplification,

$$u(u-1)(k-2)h = u(k-2)w+w+u-v. \tag{4.5}$$

Note that this equation is linear in h for each u. Another equivalent expression to be used later is

$$\begin{aligned}(u-1)(g+h) &= (u-1)(g-h)+2h(u-1)\\ &= (u-1)(x-y)+2(w-y)\\ &= v+w-x-y\\ &= v+w-1-ky. \end{aligned}\tag{4.6}$$

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Given v and w, if we have a solution to (4.5) such that g, h, u are admissible for IGDD and with $y \geq h$, then Construction 4.3 can be invoked to obtain $\mathrm{IPBD}((v; w), k)$. Observe, though, that only finitely many values of v are constructible in this way for a given fixed w.

Fortunately, Construction 4.3 is sufficiently general if we only care about achieving desired congruence classes for v and w. We are specifically interested in the modulus $M = mk(k-1)$ from Proposition 2.6, but the statement below applies modulo any multiple of $k(k-1)$.

Proposition 4.4 *Suppose $k \geq 2$ is an integer, and we are given integers v_0, w_0 satisfying the necessary divisibility conditions* (1.3) *and* (1.4). *Let $M \equiv 0 \pmod{k(k-1)}$ be a positive modulus. Then there exists an* $\mathrm{IPBD}((v_1; w_1), k)$ *for infinitely many $v_1 \equiv v_0$ and $w_1 \equiv w_0 \pmod{M}$.*

Proof The idea is as follows. First, pick h and u strategically in certain congruence classes $(\mathrm{mod}\ M)$ as a function of k, v_0, w_0. This selection is done separately based on prime power divisors of M, appealing to the Chinese remainder theorem for a simultaneous selection $(\mathrm{mod}\ M)$. Then, working from h and u, we compute a large integer $y \equiv w_0 - (u-1)h \pmod{M}$, followed by $x = (k-1)y + 1$ and $g = h + x - y$. This set-up allows us to invoke Construction 4.3 to produce $\mathrm{IPBD}((v_1; w_1), k)$ with $w_1 = (u-1)h + y$ and $v_1 - w_1 = u(x - y)$. Each of w_1 and v_1 can be made arbitrarily large by increasing the choice for y.

The essential step remaining is a selection of h and u so that the needed IGDDs for the construction are admissible. So consider a congruence version of (4.5), namely,

$$u(u-1)(k-2)h \equiv u(k-2)w_0 + w_0 + u - v_0 \pmod{p^t},$$

where $p^t \parallel M$ is a prime power exact divisor. It is convenient to choose u such that the right-hand side becomes independent of w_0. To this end, put

$$u \equiv \begin{cases} -(k-2)^{-1} & \text{if } \gcd(p, k-2) = 1, \\ (v_0 - w_0)((k-2)w_0 + 1)^{-1} & \text{otherwise,} \end{cases} \tag{4.7}$$

where the congruence is modulo p^t. For such integers u, the congruence on h becomes much simpler. In the case $p \mid k-2$, we have simply $u(u-1)(k-2)h \equiv 0$, which is of course solved by $h \equiv 0$. In the case $\gcd(p, k-2) = 1$, we have $(u-1)h \equiv v_0 - u$. This admits the solution

$$h \equiv \frac{(k-2)v_0 + 1}{1-k},$$

since by (1.3), $k-1$ divides the numerator and leads to an integer expression. Modulo M, the Chinese remainder theorem gives a simultaneous solution for u and h. See Table 1 for a summary of the choice of parameters.

We now check the necessary conditions for IGDDs. Note that $u-1 \equiv -(k-2)^{-1} - 1 \equiv -(-1) - 1 = 0 \pmod{p^t}$ for any $p^t \parallel k-1$ from (4.7). So (4.1) holds. To verify (4.2),

	$\gcd(p,k-2)=1$	$p \mid k-2$
$h \equiv$	$((k-2)v_0+1)/(1-k)$	0
$u \equiv$	$-(k-2)^{-1}$	$(v_0-w_0)((k-2)w_0+1)^{-1}$

$$\begin{aligned} y &\equiv w_0-(u-1)h \\ x &\equiv (k-1)y+1 \\ g &\equiv h+x-y. \end{aligned}$$

Table 1: Choice of parameters for hitting congruence classes mod $p^t \parallel M$.

we compute

$$\begin{aligned} (g-h)(g+h)u(u-1) &\equiv u(x-y)\cdot(u-1)(g+h) \\ &\equiv (v_0-w_0)\cdot(v_0+w_0-1-2ky) \qquad \text{by (4.4) and (4.6)}, \\ &\equiv v_0(v_0-1)-w_0(w_0-1) \qquad \text{using (1.3)}, \\ &\equiv 0 \pmod{k(k-1)} \qquad \text{by (1.4)}. \end{aligned}$$

Therefore, Theorem 4.2 ensures the ingredient IGDD exists for sufficiently large such u. It follows that Construction 4.3 yields IPBD$((v_1;w_1),k)$ with $w_1=h(u-1)+y\equiv w_0$ and $v_1-w_1=u(x-y)\equiv v_0-w_0 \pmod{M}$. ■

Remark It is interesting that the choice of congruence class for h in Table 1 is independent of w_0. It can be chosen as a function of k and v_0, and before a value of u is specified. In this way, it is possible to strengthen Proposition 4.4 so that w and v also land near prescribed integers that nearly satisfy equality in (1.5). Details are omitted, since we do not make use of this in what follows.

Example 4.5 We illustrate the selection of parameters in the case $k=6$, and $M=k(k-1)=30$. There are twelve numerically admissible congruence class pairs (v_0,w_0). The selection of parameters dictates

$$\begin{aligned} &u \equiv -4^{-1} \equiv 11 \pmod{15}, && u \equiv v_0-w_0 \pmod{2}, \text{ and} \\ &h \equiv -(4v_0+1)/5 \pmod{15}, && h \equiv 0 \pmod{2}. \end{aligned}$$

In the specific case $(v_0,w_0)\equiv(26,11)$, we compute $u\equiv 11$, $h\equiv 24$, $g\equiv 9$, $y\equiv 11$, $x\equiv 26 \pmod{M}$. After the construction, we obtain an IPBD$((v_1;w_1),6)$ with $w_1=(u-1)h+y\equiv 11$ and $v_1=w_1+u(x-y)\equiv 26$, as desired. Note that the needed IGDDs exist, since $u\equiv 1 \pmod{5}$ and $(g+h)(g-h)u(u-1)\equiv 0 \pmod{30}$.

5 Proofs

We first consider Theorem 4.2, which constructs IGDD$((g;h)^u,k)$ for all large admissible u. This is a straightforward extension of Lamken and Wilson's argument [9, §8] for uniform GDDs (in which $h=0$). Further details for IGDDs (general h) can be found in [16], and so we only sketch the setup and calculations.

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Proof of Theorem 4.2. For a positive integer n, we abbreviate $\{1,2,\ldots,n\}$ by $[n]$. Define the 'color set' $S = S_{g,h} := [g]^2 - [h]^2$. Consider also the family of functions $f\colon[k] \to [g]$ with at most one element in the range belonging to $[h]$. There are $(g-h)^k + kh(g-h)^{k-1}$ such functions. Each such function f induces an edge-colouring of the complete (bi-directed) graph K_k by colors in S. Specifically, the arc (x,y), for $x,y \in [k]$ receives color $(f(x),f(y))$. Let $\mathcal{G}_{g,h,k}$ be the family of all such edge-colored copies of K_k induced by some f.

Consider the complete S-colored (directed) graph, which we denote by $K_u^\dagger$. We claim that an IGDD$((g;h)^u,k)$ is equivalent to a decomposition of $K_u^\dagger$ into graphs in $\mathcal{G}_{g,h,k}$. In this correspondence, each group of the IGDD is represented with a vertex of $K_u^\dagger$, and each 'level' in a group is an element of $[g]$. A legal block in the IGDD$((g;h)^u,k)$ (with group levels given by f, say) is equivalent to a placement of some graph in $\mathcal{G}_{g,h,k}$ (with coloring induced by f) as a block on some k of the vertices in $K_u^\dagger$. Note that the condition that f maps at most one vertex into $[h]$ ensures that pairs of points from the hole are not covered by such a block.

Next, we define some vectors indexed by the color set S. For $G \in \mathcal{G}_{g,h,k}$, define the 'edge-vector' $\mu(G) \in \mathbb{Z}^{S_{g,h}}$ as the number of edges of G of each color. For $G \in \mathcal{G}_{g,h,k}$ and a vertex x of G, define the degree-vectors $\tau^+(G)$ (and $\tau^-(G)$) as the number of outgoing (respectively incoming) arcs of each color incident with x.

We now observe that the all-ones vector $\mathbf{1} \in \mathbb{Z}^S$ is a positive rational combination of the $\mu(G)$, $G \in \mathcal{G}$. To see this, we first partition S as a disjoint union $S_1 \cup S_2$, where $S_1 = ([g]-[h])^2$, and S_2 contains colors using $[h]$. If we consider the average of $\mu(G)$ over all G avoiding the hole, every element of S_1 appears equally often and every element of S_2 appears exactly 0 times. If we consider instead the average of $\mu(G)$ over all G intersecting the hole, we compute that every color in S_1 appears exactly

$$(c_1,c_2) := \Big(\frac{(k-1)(k-2)}{(g-h)^2}, \frac{k-1}{(g-h)h}\Big)$$

times as an element of S_1 and S_2, respectively. Note that (4.2) is equivalent to $c_1 \le c_2$. Hence, some positive combination of these averages equals $\mathbf{1}$, as desired.

Finally, we let α denote the generator of the ideal $\{A : A(\mathbf{1};\mathbf{1}) \in \sum_G(\tau^+(G);\tau^-(G))\mathbb{Z}\}$ and let β denote the generator of the ideal $\{B : B\mathbf{1} \in \sum_G \mu(G)\mathbb{Z}\}$. By Lamken and Wilson's asymptotic existence theory for edge-colored decompositions [9], it follows that we have the needed decomposition of $K_u^\dagger$ into $\mathcal{G}_{g,h,k}$ for sufficiently large integers u satisfying $u-1 \equiv 0 \pmod{\alpha}$ and $u(u-1) \equiv 0 \pmod{\beta}$. We refer the reader to [16] for the calculations showing that these conditions on u are equivalent to (4.1) and (4.2). ■

We now turn to Theorem 1.6. Since in this case we do not care about inequality (1.5), the proof can simply fill holes in two steps.

Proof of Theorem 1.6. Let $w \equiv 1 \pmod{k-1}$, and take a large v satisfying (1.3) and (1.4). We would like to construct an IPBD$((v;w),k)$.

Let $M = mk(k-1)$, from the end of Section 2. By Proposition 4.4, there exist IPBD$((v_1;w_1),k)$ with $v_1 \equiv v$ and $w_1 \equiv w \pmod{M}$. We may assume that both v and w_1 are large enough so that, by Proposition 2.6, there exist IPBD$((w_1;w),k)$ and IPBD$((v;v_1),k)$. The required design exists by filling. ■

Before proving Theorem 1.7, we require two lemmas on ingredient designs. These are both essentially extracted from earlier results. Note various parameters are recycled with a slightly different use.

Lemma 5.1 *For sufficiently large m with $m \equiv -1 \pmod{k}$ and $m \equiv 1 \pmod{k-2}$, there exist both $\mathrm{GDD}((k-1)^m r^1, k)$ and $\mathrm{GDD}((k-1)^{m+1} r^1, k)$, where*

$$r = \frac{(k-1)(m-1)}{k-2}.$$

Proof Start with a resolvable $\mathrm{PBD}((k-1)m, k-1)$. Delete a parallel class of blocks to obtain a $\mathrm{GDD}((k-1)^m, k-1)$, which has r parallel classes. Extend each class by a single extra point, and the result is a $\mathrm{GDD}((k-1)^m r^1, k)$.

Next, take a resolvable $\mathrm{GDD}(k^{(k-1)(m+1)/k}, k-1)$, which is seen to exist for the stated values of m by Theorem 3.4 and a short calculation. There are $\frac{(k-1)m}{k-2} = r+1$ parallel classes; extend each one and turn the groups into blocks. The result is an $\mathrm{IPBD}(((k-1)(m+1)+r+1; r+1), k)$, or equivalently, a $\mathrm{GDD}((k-1)^{m+1} r^1, k)$. ■

Remark It is important to note that the congruences on m admit solutions, even if $2|k$.

Lemma 5.2 *Let s be an integer with $s \equiv 0 \pmod{k-1}$ and $s \equiv -1 \pmod{k}$. There exist both $\mathrm{GDD}((k-1)^m s^1, k)$ and $\mathrm{GDD}((k-1)^{m+1} s^1, k)$ for all sufficiently large $m \equiv -1 \pmod{k}$.*

Proof This follows from Theorem 1.6 after verifying admissibility of the relevant parameters. ■

We are now ready for the proof of our main result.

Proof of Theorem 1.7 Suppose we are given parameters v, w satisfying (1.3), (1.4) and $v > (k-1+\epsilon)w$. For convenience, we instead construct $\mathrm{GDD}((k-1)^a b^1, k)$, where $b = w-1$ and $(k-1)a = v-w$. In terms of the new parameters, the inequality becomes $a > (k-2+\epsilon)(b+1)/(k-1)$. So, it is sufficient to prove the existence of $\mathrm{GDD}((k-1)^a b^1, k)$ for all sufficiently large integers a and b with $b \equiv 0 \pmod{k-1}$, $a(a-2b-1) \equiv 0 \pmod{k}$, and $a > (k-2+\epsilon)b/(k-1)$.

Let m satisfy both lemmas, so that there exist $\mathrm{GDD}((k-1)^m x^1, k)$ and $\mathrm{GDD}((k-1)^{m+1} x^1, k)$ for each $x \in R := \{k-1, k^2-1, r\}$. We may also demand that m is of order $1/\epsilon$. Observe that $r \in R$ is taken as in Lemma 5.1, while the two small values in R meet the conditions for s in Lemma 5.2.

Let z denote the least residue of b modulo $k(k-1)$. By Theorem 1.6, there is an integer $u_0(z,k)$ so that there exists $\mathrm{GDD}((k-1)^u z^1, k)$ for all admissible $u \geq u_0(z,k)$. We need only consider k possible congruence classes for z; hence, we have $u_0(k) = \max\{u_0(z,k) : 0 \leq z < k(k-1), k-1|z\}$ as a universal value independent of the class. Put $y = b-z$, so that $y \equiv 0 \pmod{k(k-1)}$.

By Theorem 2.1, there exists $\mathrm{TD}(m+2, n)$ for $n \geq n_0(m)$. Now, for sufficiently large a, we can write $a = mn+p$, where $k|n$, $n \geq n_0(m)$, and both $n, p \geq u_0(k)$. Truncating all but p points from one group of the TD, we obtain a $\mathrm{GDD}(n^m p^1 n^1, \{m+1, m+2\})$.

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Apply Wilson's fundamental construction to this GDD. To each point in the first $m+1$ groups, give weight $k-1$; in the last group, assign weights in R. As ingredients, we use $\mathrm{GDD}((k-1)^m x^1, k)$ and $\mathrm{GDD}((k-1)^{m+1} x^1, k)$ for $x \in R$.

The result is a GDD with block size k having m groups of size $(k-1)n$, one group of size $(k-1)p$, and one group size running through $n * R$, the set of n-fold sums of integers chosen from R. We now analyze this set. Since the least two elements of R have difference $k(k-1)$ and since the largest element r is independent of n, it follows after a calculation that $n * R$ covers

$$(k-1)\Big\{ n, n+k, n+2k, \dots, n\Big(\frac{m-1}{k-2}-1\Big)\Big\} \tag{5.1}$$

for large n. Let $y_{\max}$ denote the right endpoint of the arithmetic progression (5.1); we estimate $y_{\max}$ later. For large n, we can express any $y \in k(k-1)\mathbb{Z}$ with $(k-1)n \le y \le y_{\max}$ as a combination in $n * R$. In particular, this holds for $y = b - z$ as chosen above.

Now, fill each of the first m groups with $\mathrm{GDD}((k-1)^n z^1, k)$ and fill the $(m+1)$-st group with $\mathrm{GDD}((k-1)^p z^1, k)$. Identify all the lone groups of size z, and include these points in the $(m+2)$-nd group. We obtain a $\mathrm{GDD}((k-1)^{mn+p}(y+z)^1, k)$. With $a = mn + p$, $b = y + z$, the construction realizes values with a/b as small as

$$\frac{(m+1)n}{y_{\max}} < \Big(\frac{k-2}{k-1}\Big)\big(1 + O(1/m)\big) < \frac{k-2+\epsilon}{k-1},$$

as required. ∎

Example 5.3 We sketch the construction in the case of block size $k = 4$ for which a much more complete analysis is given in [14]. Let $m \equiv 3 \pmod 4$; we work from six ingredient GDDs, having types $3^m x^1$ and $3^{m+1} x^1$ for $x \in \{3, 15, 3(m-1)/2\}$. In the first pass, take $m = 11$; these specific GDDs reduce to just four and are not hard to construct. Now, for $n \ge 7222$, there exist $\mathrm{TD}(13, n)$ and, after truncation, a $\mathrm{GDD}(n^{11} p^1 n^1, \{12, 13\})$. Apply Wilson's fundamental construction and we get a $\mathrm{GDD}((3n)^{11}(3p)^1 y^1, 4)$, where y is any sum of n terms from $\{3, 15\}$. It is clear that such y cover the range $3n$, $3n + 12, \dots, 15n$. In the last step of the construction, fill groups of size $3n$ and $3p$ with the non-hole points of $\mathrm{IPBD}((3n+z; z), 4)$ and $\mathrm{IPBD}((3p+z; z), 4)$ for various $z \in \{1, 4, 7, 10\}$. The result is a design with $33n + 3p + y + z$ points and hole size $y + z$, where y is as large as $15n$. We thus obtain existence for large admissible v, w with $v/w \ge (48n + 3p + z)/(15n + z) \gtrapprox 16/5$. For examples with this limit nearing 3, we choose larger m, impacting n and the 'sufficiently large' cut-off for v, w.

6 Concluding Remarks

We have seen that a combination of resolvable designs, transversal designs, and (incomplete) GDDs results in a nearly complete existence theory for incomplete pairwise balanced designs.

We are hopeful that, perhaps with some new constructions, the bound in Theorem 1.7 can be improved to settle existence for almost every pair in the 'admissible cone'.

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Conjecture 6.1 *Let $k \geq 2$ be an integer. There exists an* $\mathrm{IPBD}((v;w),k)$ *for all sufficiently large pairs v, w satisfying* (1.3), (1.4), *and* (1.5).

A certain pair of extra small designs, if they exist, can enrich the proof of Theorem 1.7. Specifically, suppose there exist $\mathrm{PBD}(v,k)$ for $v = k(k-1)(k-2)+1$ and $v = (k-1)^3+1$. (They are admissible and presently known to exist for $k = 3,4,5,6$.) Let g and h, respectively, denote these two values of v. From Theorem 3.4, we have the existence of resolvable $\mathrm{GDD}(g^{(k-1)m/g}, k-1)$ and resolvable $\mathrm{GDD}(h^{(k-1)(m+1)/h}, k-1)$ for certain large integers m. One can compute that the number of parallel classes of each is $r_- := r - k(k-1)$, where r is as in Lemma 5.1. By hypothesis, we can fill the groups with $\mathrm{PBD}(g,k)$ and $\mathrm{PBD}(h,k)$. Then, extending each parallel class, one obtains $\mathrm{GDD}((k-1)^m r_-^1, k)$ and $\mathrm{GDD}((k-1)^{m+1} r_-^1, k)$. Now, let $R' = \{k-1, k^2-1, r_-, r\}$, and observe that $n * R'$ is now the full arithmetic progression from $n(k-1)$ to nr for sufficiently large n. The only remaining 'slack' in the argument occurs in filling the group of size $(k-1)p$. In any case, this enhanced technique is a possible approach to proving Conjecture 6.1 in certain cases, and it offers a different proof for the case $k = 3$ avoiding many more technical constructions.

Recall that an $\mathrm{IPBD}((v;w),k)$ is equivalent to a $\mathrm{GDD}(1^{v-w}w^1, k)$. In more generality, one can ask for an asymptotic existence theory for $\mathrm{GDD}(g^u w^1, k)$ for some fixed parameter g. Early investigations on this can be found in [19]. The necessary divisibility conditions are

$$gu \equiv w - g \equiv 0 \pmod{k-1} \quad \text{and}$$
$$g(gu(u+w-1)) \equiv 0 \pmod{k(k-1)}.$$

Thinking of w as 'large', its upper bound is given by $g(u-1) \geq (k-2)w$. Extending Proposition 3.2 in the natural way, the case of maximal w is obtained through resolvable $\mathrm{GDD}(g^u, k-1)$.

Proposition 6.2 *There exists a* $\mathrm{GDD}(g^u w^1, k)$ *with* $g(u-1) = (k-2)w$ *if and only if there exists a resolvable* $\mathrm{GDD}(g^u, k-1)$.

For this case of maximal holes, there are only a finite number of possible exceptions for each g by the main result of [1] on resolvable GDDs. For fixed g, we see no serious obstacle in extending our techniques to obtain a similar result as Theorem 1.7.

Another direction of interest is the study of $\mathrm{IPBD}((v;w),K)$ where K is a set of block sizes. We expect that Theorem 1.6, for fixed hole sizes, has a straightforward extension to this more general case. On the other hand, it is not even clear what to aim for if both w and v are allowed to grow. The inequality (1.5) simply becomes

$$v \geq (\min K - 1)w + 1. \tag{6.1}$$

However, some congruence classes may require a variety of block sizes and forbid sharpness in (6.1). In the thesis [16], existence of $\mathrm{IPBD}((v;w),K)$ has been shown for large v, w satisfying a much stronger inequality than (6.1).

Consider the case where K contains three consecutive block sizes. Note that the necessary divisibility conditions disappear, since $\alpha(K) = 1$ and $\beta(K) = 2$. So any integers v and w satisfying (6.1) are admissible for $\mathrm{IPBD}((v;w),K)$. In this case, it is

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straightforward to adapt the proof of Theorem 1.7 to get a strong partial result. Details are left to future work.

Finally, the 'higher λ' version of this problem is noteworthy in that it motivates study of 'thickly-resolvable' designs. Rather than blocks resolving into partitions, we may ask that blocks resolve into families covering every point exactly λ times. Although this appears at first glance to simply be a weakening of the resolvability property, the necessary condition $k|v$ is also weakened to $k|\lambda v$ for these objects.

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