J. Inst. Math. Jussieu (2023), **22**(1), 335–381
doi:10.1017/S1474748021000256

CONNECTIVITY AND PURITY FOR LOGARITHMIC MOTIVES

FEDERICO BINDA [1] AND ALBERTO MERICI [2]

[1] *Dipartimento di Matematica "Federigo Enriques", Università degli Studi di Milano Via Cesare Saldini 50, Milano 20133, Italy*
(federico.binda@unimi.it)
[2] *Institut für Mathematik, Universität Zurich, Winterthurerstr. 190, CH-8057 Zürich, Switzerland*
(alberto.merici@math.uzh.ch)

(Received 21 December 2020; revised 10 May 2021; accepted 11 May 2021; first published online 14 June 2021)

Abstract The goal of this article is to extend the work of Voevodsky and Morel on the homotopy t-structure on the category of motivic complexes to the context of motives for logarithmic schemes. To do so, we prove an analogue of Morel's connectivity theorem and show a purity statement for $(\mathbf{P}^1,\infty)$-local complexes of sheaves with log transfers. The homotopy t-structure on $\mathbf{logDM}^{\mathrm{eff}}(k)$ is proved to be compatible with Voevodsky's t-structure; that is, we show that the comparison functor $R^{\overline{\square}}\omega^*\colon \mathbf{DM}^{\mathrm{eff}}(k)\to \mathbf{logDM}^{\mathrm{eff}}(k)$ is t-exact. The heart of the homotopy t-structure on $\mathbf{logDM}^{\mathrm{eff}}(k)$ is the Grothendieck abelian category of strictly cube-invariant sheaves with log transfers: we use it to build a new version of the category of reciprocity sheaves in the style of Kahn-Saito-Yamazaki and Rülling.

Keywords: Motives, Logarithmic Schemes, Cohomology Theories

2020 Mathematics Subject Classification: Primary 14F08
Secondary 18F35; 14F42; 18G80

Contents

1. Introduction

Voevodsky's category of motivic complexes over a perfect field k is based on a simple idea: most cohomology theories for smooth k-schemes are insensitive to the affine line; that is, they satisfy $\mathbf{A}^1$-homotopy invariance. This observation led Voevodsky to introduce as a building block of his theory of motives the category of homotopy-invariant sheaves with transfers $\mathbf{HI}_{\mathrm{Nis}}(k)$; that is, sheaves F for the Nisnevich topology defined on the category of finite correspondences over k such that $F(X\times\mathbf{A}^1)\xrightarrow{\simeq}F(X)$ for every smooth k-scheme X. These sheaves enjoy many nice properties: the category $\mathbf{HI}_{\mathrm{Nis}}(k)$ is a Grothendieck abelian subcategory of the category $\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k)$ of Nisnevich sheaves with transfers, closed under extensions and equipped with a (closed) symmetric monoidal structure $\otimes_{\mathbf{HI}}$. Moreover, a celebrated theorem of Voevodsky shows that the cohomology presheaves $H^n_{\mathrm{Nis}}(-,F)$ of a homotopy-invariant sheaf with transfers F are still $\mathbf{A}^1$-homotopy invariant. In fact, $\mathbf{HI}_{\mathrm{Nis}}(k)$ can be identified with the heart of a certain t-structure on the triangulated category $\mathbf{DM}^{\mathrm{eff}}(k)$, induced by the standard t-structure on the derived category $\mathbf{D}(\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$ and called by Voevodsky the *homotopy t-structure*. The $\mathbf{A}^1$-invariance of the cohomology of homotopy-invariant sheaves can be rephrased by saying that a sheaf $F\in\mathbf{HI}_{\mathrm{Nis}}(k)$, seen as object of $\mathbf{D}(\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$, is *local* with respect to the Bousfield localisation of $\mathbf{D}(\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$ over the complexes $(\mathbf{A}^1_X)[n]\to X[n]$ for $X\in\mathbf{Sm}(k)$.

Much work has been done around the homotopy t-structure, including Déglise's extension to the noneffective version of $\mathbf{DM}^{\mathrm{eff}}(k)$ and the identification of its heart with the category of Rost's cycle modules [9] and Morel's work on the stable homotopy category $\mathbf{SH}(k)$ [24]. In informal terms, we can interpret the existence of the homotopy t-structure as a manifestation of the interplay between the Postnikov truncation functors $\tau_{\leq n}$ and the $\mathbf{A}^1$-localisation functor on the derived category $\mathbf{D}(\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$. This interplay is precisely expressed by Morel's connectivity theorem.

Voevodsky's category of motives over a field has been recently extended to the setting of logarithmic algebraic geometry in [7]. The basic objects in this context are no longer smooth k-schemes but rather fine and saturated log schemes, log smooth over a base considered with trivial log structure (typically, the base is a perfect field). The Nisnevich topology on the underlying schemes defines naturally a topology, called the *strict Nisnevich topology*, sNis for short. This topology is not enough to guarantee that the resulting category of motives satisfies a number of nice properties and needs to be replaced with a subtle variant, the *dividing Nisnevich topology*, dNis for short, with additional covers given by certain blow-ups with center in the support of the log structure. The affine line $\mathbf{A}^1$ is replaced by its compactified avatar; that is, the log scheme $\overline{\square}=(\mathbf{P}^1,\infty)$ obtained by considering the compactifying log structure along the embedding $\mathbf{A}^1\hookrightarrow\mathbf{P}^1$. The category of log motives $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$ (with transfers) is then defined as the homotopy category of the $(\mathrm{dNis},\overline{\square})$-local model structure on the category of (unbounded) chain complexes of presheaves with logarithmic transfers, $\mathbf{C}(\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda))$, for Λ a ring of coefficients. See [7, 4-5] and Section 2 for more details. The variant without transfers will be denoted $\mathbf{logDA}^{\mathrm{eff}}(k,\Lambda)$, and it is obtained as Bousfield localisation of the category of (unbounded) chain complexes of presheaves $\mathbf{C}(\mathbf{PSh}^{\mathrm{log}}(k,\Lambda))$.

The goal of this article is to develop in the logarithmic context the analogue of Voevodsky's homotopy t-structure and to derive some consequences from this. As discussed above, the homotopy t-structure on (usual) motives is induced by the standard t-structure on the derived category of sheaves. In order to restrict this t-structure to the subcategory of local objects, one needs to understand how much connectivity (with respect to the homology sheaves) is lost after taking a fibrant replacement for the $(\mathbf{A}^1,\mathrm{Nis})$-local model structure. This is the content of Morel's connectivity theorem [24, Theorem 6.1.8].

Our first main result is the following logarithmic variant.

Theorem 1.1 (see Theorem 3.2)**.** *Assume that k is a perfect*[1] *field and let $\tau \in \{\mathrm{sNis},\mathrm{dNis}\}$. Let $C \in \mathbf{Cpx}(\mathbf{PSh}^{\log}(k,\Lambda))$ be locally n-connected for the τ-topology. Then any $(\tau,\overline{\square})$-fibrant replacement $C \to L$ is locally n-connected.*

A complex of presheaves is said to be locally n-connected with respect to a topology τ if the homology sheaves $a_\tau H_i(C)$ vanish below n. For the proof of Theorem 1.1 we follow the pattern given by Ayoub in his adaptation of Morel's argument to the $\mathbf{P}^1$-local theory, developed in [5]. In particular, the statement can be reduced to a purity result for local complexes.

Theorem 1.2 (see Theorem 4.4)**.** *Let X be a connected fs log smooth k-scheme that is essentially smooth over k (in particular, the underlying scheme $\underline{X}$ is an essentially smooth k-scheme) such that $\underline{X}$ is a Henselian local scheme. Then the map*

$$H_i(C(X)) \to H_i(C(\eta_X,triv))$$

is injective for every $(\mathrm{sNis},\overline{\square})$-fibrant complex of presheaves $C \in \mathbf{Cpx}(\mathbf{PSh}^{\log}(k,\Lambda))$.

Here, we write η_X for the generic point of $\underline{X}$ and (η_X,triv) for η_X seen as a log scheme with trivial log structure. The proof is quite long, for which we use in an essential way the results developed in [7], such as the existence of a number of distinguished triangles in $\mathbf{logDA}^{\mathrm{eff}}(k)$ and a description of the motivic Thom spaces [7, 7.4]: in particular, new ingredients (compared to the argument given by Morel or Ayoub) are required when the log structure on X is not trivial.

We remark that the original formulation of Morel's connectivity theorem was given for the $\mathbf{A}^1$-localisation of presheaves of S^1-spectra, rather than presheaves of chain complexes. The arguments given in this article can be easily adapted to that context. Because our main application is about the motivic category introduced in [7], we decided to state the results for $\mathbf{Cpx}(\mathbf{PSh}^{\log}(k,\Lambda))$.

Having the analogue of Morel's connectivity theorem at disposal, it is possible to characterise $\overline{\square}$-local complexes of sheaves.

Corollary 1.3 (see Corollary 5.5)**.** *Let $C \in \mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\mathrm{t}}(k,\Lambda))$ where $t \in \{\mathrm{log},\mathrm{ltr}\}$. Then the following are equivalent:*

(a) *C is $\overline{\square}$-local.*

---

[1] If $\mathrm{ch}(k)$ is invertible in Λ, this assumption can be relaxed because $\mathbf{lCor}(k,\Lambda) \cong \mathbf{lCor}(k^{perf},\Lambda)$

(b) *The homology sheaves* $a_{\mathrm{dNis}}H_iC$ *are strictly* $\overline{\square}$*-invariant for every* $i \in \mathbb{Z}$*; that is, their cohomology presheaves are* $\overline{\square}$*-invariant.*

We can then consider the inclusions

$$\mathbf{logDA}^{\mathrm{eff}}(k,\Lambda) \hookrightarrow \mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\mathrm{log}}(k,\Lambda))$$
$$\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda) \hookrightarrow \mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda))$$

that identify $\mathbf{logDA}^{\mathrm{eff}}(k,\Lambda)$ and $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$ with the subcategories of $\overline{\square}$-local complexes. Using Theorem 1.1 it is easy to show that the truncation functors $\tau_{\leq n}$ and $\tau_{\geq n}$ preserve the categories of $\overline{\square}$-local complexes and therefore that the standard t-structures on the categories of (pre)sheaves induce the desired homotopy t-structure on log motives. We denote by $\mathbf{CI}^{\mathrm{log}}_{\mathrm{dNis}}$ (and by $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$ for the variant with transfers) its heart, which is then identified with the category of strictly $\overline{\square}$-invariant dNis-sheaves. It follows from the fact that the t-structures are compatible with colimits (in the sense of [20]) that $\mathbf{CI}^{\mathrm{log}}_{\mathrm{dNis}}$ and $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$ are Grothendieck abelian categories. See Theorem 5.7. In particular, the inclusions

$$i\colon\ \mathbf{CI}^{\mathrm{log}}_{\mathrm{dNis}} \hookrightarrow \mathbf{Shv}^{\mathrm{log}}_{\mathrm{dNis}}(k,\Lambda)$$
$$i^{\mathrm{tr}}\colon\ \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}} \hookrightarrow \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\Lambda)$$

admit both a left adjoint and a right adjoint. Objects of $\mathbf{CI}^{\mathrm{log}}_{\mathrm{dNis}}$ and of $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$ satisfy the following *purity* property.

Theorem 1.4 (see Theorem 5.10)**.** *Let* $F \in \mathbf{CI}^{log}_{\mathrm{dNis}}$ *(respectively* $F \in \mathbf{CI}^{ltr}_{\mathrm{dNis}}$*). Then for all* $X \in \mathbf{SmlSm}(k)$ *(see the notation below) and* $U \subseteq X$ *an open dense, the restriction* $F(X) \to F(U)$ *is injective.*

In [7], a comparison functor

$$R^{\overline{\square}}\omega^*\colon\ \mathbf{DM}^{\mathrm{eff}}(k,\Lambda) \to \mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$$

has been constructed. Under resolution of singularities, it is known that $R^{\overline{\square}}\omega^*$ is fully faithful, and it identifies $\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)$ with the subcategory of $(\mathbf{A}^1,\mathrm{triv})$-local objects in $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$ (see [7, Theorem 8.2.16] and the results quoted there). Even without knowing that $R^{\overline{\square}}\omega^*$ is a full embedding, we can show that it is t-exact with respect to the homotopy t-structures on both sides. In fact, when $R^{\overline{\square}}\omega^*$ is an embedding, it is straightforward to conclude that Voevodsky's homotopy t-structure is induced by the t-structure on $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$ via $R^{\overline{\square}}\omega^*$. See Proposition 5.11.

The good properties of the category of strictly $\overline{\square}$-invariant sheaves $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$, deduced from the identification with the heart of the homotopy t-structure, allow us to make a further comparison with the category $\mathbf{RSC}_{\mathrm{Nis}}$ of *reciprocity sheaves* of Kahn-Saito-Yamazaki. This is an abelian subcategory of the category of Nisnevich sheaves with transfers $\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k)$, whose objects satisfy a certain restriction on their sections inspired by the Rosenlicht-Serre theorem on reciprocity for morphisms from curves to commutative algebraic groups [30, III]. See [18] and the recollection paragraph below.

In [29], S. Saito constructed an exact and fully faithful functor

$$\mathcal{L}og\colon \mathbf{RSC}_{\mathrm{Nis}}(k) \to \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\mathbb{Z}) \tag{1.4.1}$$

having as essential image a subcategory of $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$. In Section 6 we study its pro-left adjoint $\mathcal{R}sc$: $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\mathbb{Z}) \to$ *pro-*$\mathbf{RSC}_{\mathrm{Nis}}$ and in particular its behavior with respect to the lax symmetric monoidal structure $(-,-)_{\mathbf{RSC}_{\mathrm{Nis}}}$ constructed in [27]. See Theorem 6.11 and Corollary 6.12.

The category of reciprocity sheaves $\mathbf{RSC}_{\mathrm{Nis}}$ is defined in terms of the auxiliary category of *modulus pairs*, building block of the theory of motives with modulus as developed in [14], [15] and [16]. In fact, Saito's functor (1.4.1) is itself defined by first 'lifting' a reciprocity sheaf to the category of (semipure) sheaves on modulus pairs and then applying another functor landing in $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\mathbb{Z})$. It turns out that such a detour is not necessary, at least if k admits resolution of singularities.

In fact, we can look at the composite functor

$$\omega^{\log}_{\mathbf{CI}}\colon\ \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}} \xhookrightarrow{i^{tr}} \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\mathbb{Z}) \xrightarrow{\omega_\sharp} \mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k,\mathbb{Z}) \tag{1.4.2}$$

where $\omega_\sharp$ is the left Kan extension of the restriction functor from smooth log schemes to smooth k-schemes $\omega\colon\ \mathbf{lSm}(k) \to \mathbf{Sm}(k)$, sending $X \in \mathbf{lSm}(k)$ to X^o, the open subscheme of the underlying scheme $\underline{X}$ of X where the log structure is trivial. Using a comparison result from [7] (which relies on the resolution of singularities) and our purity Theorem 5.10 we can show that $\omega^{\log}_{\mathbf{CI}}$ in (1.4.2) is fully faithful and exact (Proposition 7.3). We denote by **LogRec** its essential image: it is a Grothendieck abelian category that contains $\mathbf{RSC}_{\mathrm{Nis}}$ as full subcategory; see Theorem 7.6. Thanks to the purity property for strictly $\overline{\square}$-invariant sheaves, its objects satisfy global injectivity; that is, for every $F \in \mathbf{LogRec}$ and $U \subset X$ dense open subset of $X \in \mathbf{Sm}(k)$, the restriction map

$$F(X) \hookrightarrow F(U)$$

is injective. See [18] for a similar statement for reciprocity sheaves (relying on [28]). In fact, we can show that the cohomology presheaves of any reciprocity sheaf $F \in \mathbf{RSC}_{\mathrm{Nis}}$ satisfy global injectivity; see Corollary 7.7.

If we denote by $i_{\mathbf{RSC}}$ the inclusion $\mathbf{RSC}_{\mathrm{Nis}} \subset \mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}$, we can then identify the functor $\mathcal{L}og$ of (1.4.1) with the composite $\omega^{\mathbf{CI}}_{\log} \circ i_{\mathbf{RSC}}$, where $\omega^{\mathbf{CI}}_{\log}$ is the right adjoint to $\omega^{\log}_{\mathbf{CI}}$. The category **LogRec** seems to share many of the properties of $\mathbf{RSC}_{\mathrm{Nis}}$: in the rest of Section 7 we discuss some of them, in particular in relationship with the monoidal structure. See Proposition 7.11.

Notations and recollections on log geometry

In the whole article we fix a perfect base field k and a commutative unital ring of coefficients Λ. Let S be a Noetherian fine and saturated (fs for short) log scheme. We denote by $\mathbf{lSm}(S)$ the category of fs log smooth log schemes over S. We are typically interested in the case where $S = \mathrm{Spec}(k)$, considered as a log scheme with trivial log structure.

For $X \in \mathbf{lSm}(S)$, we write $\underline{X} \in \mathbf{Sch}(\underline{S})$ for the underlying $\underline{S}$-scheme, where $\underline{S}$ is the scheme underlying S. We also write ∂X for the (closed) subset of $\underline{X}$ where the log structure of X is not trivial. Let $\mathbf{SmlSm}(S)$ be the full subcategory of $\mathbf{lSm}(S)$ having for objects $X \in \mathbf{lSm}(S)$ such that $\underline{X}$ is smooth over $\underline{S}$. By, for example, [7, A.5.10], if $X \in \mathbf{SmlSm}(k)$, then ∂X is a strict normal crossing divisor on $\underline{X}$ and the log scheme X is isomorphic to $(\underline{X}, \partial X)$; that is, to the compactifying log structure associated to the open embedding $(\underline{X} \setminus \partial X) \to \underline{X}$. If $X,Y \in \mathbf{lSm}(S)$, we will write $X \times_S Y$ for the fibre product of X and Y over S computed in the category of fine and saturated log schemes: it exists by [25, Corollary III.2.1.6] and it is again an object of $\mathbf{lSm}(S)$ using [25, Corollary IV.3.1.11]. Unless S has trivial log structure, the underlying scheme $\underline{X \times_S Y}$ does not agree with $\underline{X} \times_{\underline{S}} \underline{Y}$. See [25, §III.2.1] for more details.

We denote by $\mathbf{PSh}^{\log}(S,\Lambda)$ the category of presheaves of Λ modules on $\mathbf{lSm}(S)$. It naturally has the structure of a closed monoidal category. If τ is a Grothendieck topology on $\mathbf{lSm}(S)$ (see Section 2.1), we write $\mathbf{Shv}_\tau^{\log}(S,\Lambda)$ for full subcategory of $\mathbf{PSh}^{\log}(S,\Lambda)$ consisting of τ-sheaves. We typically write a_τ for the τ-sheafification functor.

Let $\widetilde{\mathbf{SmlSm}}(S)$ be the category of fs log smooth S-schemes X that are essentially smooth over S; that is, X is a limit $\varprojlim_{i \in I} X_i$ over a filtered set I, where $X_i \in \mathbf{SmlSm}(S)$ and all transition maps are strict étale (i.e., they are strict maps of log schemes such that the underlying maps $f_{ij} \colon \underline{X}_i \to \underline{X}_j$ are étale)

For $(\underline{X}, \partial X) \in \mathbf{SmlSm}(S)$ and $x \in \underline{X}$, let $\iota\colon \mathrm{Spec}(\mathcal{O}_{X,x}) \to \underline{X}$ be the canonical morphism. Then the local log scheme $(\mathrm{Spec}(\mathcal{O}_{X,x}, \iota^*(\partial X))$ is in $\widetilde{\mathbf{SmlSm}}(S)$.

We frequently allow $F \in \mathbf{PSh}^{\log}(S,\Lambda)$ to take values on objects of $\widetilde{\mathbf{SmlSm}}(S)$ by setting $F(X) := \varinjlim_{i \in I} F(X_i)$ for X as above.

Notations and recollections on reciprocity sheaves

We briefly recall some terminology and notations from the theory of modulus sheaves with transfers; see [14], [15], [18] and [28] for details.

A modulus pair $\mathcal{X} = (\overline{X}, X_\infty)$ consists of a separated k-scheme of finite type $\overline{X}$ and an effective (or empty) Cartier divisor X_∞ such that $X := \overline{X} \setminus |X_\infty|$ is smooth; it is called *proper* if $\overline{X}$ is proper over k. Given two modulus pairs $\mathcal{X} = (\overline{X}, X_\infty)$ and $\mathcal{Y} = (\overline{Y}, Y_\infty)$, with opens $X := \overline{X} \setminus |X_\infty|$ and $Y := \overline{Y} \setminus |Y_\infty|$, an admissible left proper prime correspondence from $\mathcal{X}$ to $\mathcal{Y}$ is given by an integral closed subscheme $Z \subset X \times Y$ that is finite and surjective over a connected component of X, such that the normalisation of its closure $\overline{Z}^N \to \overline{X} \times \overline{Y}$ is proper over $\overline{X}$ and satisfies

$$X_{\infty|\overline{Z}^N} \geq Y_{\infty|\overline{Z}^N},$$

as Weil divisors on $\overline{Z}^N$, where $X_{\infty|\overline{Z}^N}$ (respectively $Y_{\infty|\overline{Z}^N}$) denotes the pullback of X_∞ (respectively Y_∞) to $\overline{Z}^N$. The free abelian group generated by such correspondences is denoted by $\underline{\mathbf{M}}\mathbf{Cor}(\mathcal{X},\mathcal{Y})$. By [14, Propositions 1.2.3, 1.2.6], modulus pairs and left proper admissible correspondences define an additive category that we denote by $\underline{\mathbf{M}}\mathbf{Cor}$. We write $\mathbf{MCor}$ for the full subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ whose objects are proper modulus pairs. We denote by τ the inclusion functor $\tau\colon \mathbf{MCor} \to \underline{\mathbf{M}}\mathbf{Cor}$.

We write **MPST** for the category of additive presheaves on **MCor** and $\underline{\mathbf{M}}$**PST** for the category of additive presheaves on $\underline{\mathbf{M}}$**Cor**.

Let $\mathbf{PSh}^{\mathrm{tr}}(k)$ be Voevodsky's category of presheaves with transfers. Recall from [28, Definition 1.34] that $F \in \mathbf{PSh}^{\mathrm{tr}}(k)$ has reciprocity if for any $X \in \mathbf{Sm}(k)$ and $a \in F(X) = \mathrm{Hom}_{\mathbf{PSh}^{\mathrm{tr}}}(\mathbb{Z}_{\mathrm{tr}}(X), F)$ there exists $\mathcal{X} = (\overline{X}, X_\infty) \in \mathbf{MSm}(X)$ such that the map $\tilde{a}\colon \mathbb{Z}_{\mathrm{tr}}(X) \to F$ corresponding to the section a factors through $h_0(\mathcal{X})$. Here $\mathbf{MSm}(X)$ is the category of objects $\mathcal{X} \in \mathbf{MCor}$ such that $\overline{X} - |X_\infty| = X$, and $h_0(\mathcal{X})$ is the presheaf defined as

$$h_0(\mathcal{X})(Y) = \mathrm{Coker}(\underline{\mathbf{M}}\mathbf{Cor}(Y \otimes \overline{\square}, \mathcal{X}) \xrightarrow{i_0^* - i_1^*} \mathbf{Cor}(Y, X)),$$

where $\overline{\square} = (\mathbf{P}^1, \infty)$ (we will use the same notation for the log scheme in $\mathbf{lSm}(k)$) and the tensor product refers to the monoidal structure in $\underline{\mathbf{M}}\mathbf{Cor}$; see [14]. It is easy to see that **RSC** is an abelian category, closed under subobjects and quotients in $\mathbf{PSh}^{\mathrm{tr}}(k)$. On the other hand, it is a theorem [28, Theorem 0.1] that $\mathbf{RSC}_{\mathrm{Nis}} = \mathbf{RSC} \cap \mathbf{NST}$ is also abelian, where $\mathbf{NST} = \mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k)$ is the category of Nisnevich sheaves with transfers.

2. Preliminaries on logarithmic motives

In this section we review the construction and the basic properties of the categories $\mathbf{logDM}^{\mathrm{eff}}(k, \Lambda)$ and $\mathbf{logDA}^{\mathrm{eff}}(S, \Lambda)$ of motives, with and without transfers, as introduced in [7]. The standard reference for properties of log schemes is [25]. The definitions in this section work for a quite general base log scheme S, but in the rest of the article we will mostly deal with the case $S = \mathrm{Spec}(k)$.

2.1. Topologies on logarithmic schemes

Recall from [7, Definition 3.1.4] that a Cartesian square of fs log schemes

$$Q = \begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

is a *strict Nisnevich distinguished square* if f is strict étale, g is an open immersion and f induces an isomorphism $f^{-1}(\underline{X} - g(\underline{X}')) \xrightarrow{\sim} \underline{X} - g(\underline{X}')$ for the reduced scheme structures. We say that Q is a *dividing distinguished square* (or *elementary dividing square*) if $Y' = X' = \emptyset$ and f is a surjective proper log étale monomorphism. According to [7, Proposition A.11.9], surjective proper log étale monomorphisms are precisely the log modifications, in the sense of F. Kato [19]. We similarly say that Q is a *(strict) Zariski distinguished square* if f and g are (strict) open immersions (note that 'strict' here is redundant, because open immersions in the category of log schemes are automatically strict).

Definition 2.1. The strict Nisnevich cd-structure (respectively the dividing cd-structure) is the cd structure on $\mathbf{lSm}(S)$ associated to the collection of strict Nisnevich distinguished squares (respectively of elementary dividing squares), and the dividing Nisnevich cd structure is the union of the strict Nisnevich and of the dividing cd-structures.

The associated Grothendieck topologies on $\mathbf{lSm}(S)$ are called the *strict Nisnevich* and the *dividing Nisnevich* topologies, respectively. *Mutatis mutandis*, we define the (strict) Zariski and the dividing Zariski topologies on $\mathbf{lSm}(S)$ in a similar fashion.

We write $\mathbf{Shv}_\tau^{\log}(S,\Lambda)$ for the category of τ sheaves of Λ-modules on $\mathbf{lSm}(S)$, where τ is one of the above-defined topologies. The inclusion $\mathbf{Shv}_\tau^{log}(S,\Lambda) \subset \mathbf{PSh}^{\log}(S,\Lambda) = \mathbf{PSh}(\mathbf{lSm}(S),\Lambda)$ has an exact left adjoint, a_τ.

Let S be a Noetherian fs log scheme such that $\underline{S}$ has finite Krull dimension. According to [7, Proposition 3.3.30], the strict Nisnevich and the dividing Nisnevich cd structures on $\mathbf{lSm}(S)$ are complete, regular and quasi-bounded with respect to the dividing density structure [7, Definition 3.3.22]. In particular, any $X \in \mathbf{lSm}(S)$ has finite cohomological dimension. When $S = \mathrm{Spec}(k)$, we can bound the $dNis$ cohomological dimension by the Krull dimension of the underlying scheme, according to the following proposition.

Proposition 2.2 (see [7, Corollary 5.1.4])**.** *Let* $F \in \mathbf{Shv}_{\mathrm{dNis}}^{\log}(k,\Lambda)$ *and let* $X \in \mathbf{lSm}(k)$*. Let* $d = \dim(\underline{X})$*. Then* $\mathbf{H}^i_{\mathrm{dNis}}(X,F_X) = 0$ *for* $i \geq d+1$*.*

Remark 2.3. Because the dividing Nisnevich cd-structure is clearly squareable in the sense of [7, Definition 3.4.2], one can apply [7, Theorem 3.4.6] to get a bound on the $dNis$ cohomological dimension for any $X \in \mathbf{lSm}(S)$ in terms of the dimension of a log scheme computed using the dividing density structure: this is, for a general log scheme X, larger than the Krull dimension of the underlying scheme $\underline{X}$ (see [7, Exercise 3.3.25]). In view of [7, Remark 3.3.27], for $S = \mathrm{Spec}(k)$ and $X \in \mathbf{lSm}(k)$ such dimension agrees with the Krull dimension.

The dividing Nisnevich cohomology groups are, a priori, difficult to compute. The situation looks better for $X \in \mathbf{SmlSm}(k)$ thanks to the following result.

Theorem 2.4 ([7, Theorem 5.1.8])**.** *Let* C *be a bounded below complex of strict Nisnevich sheaves on* $\mathbf{SmlSm}(k)$*. Then for every* $X \in \mathbf{SmlSm}(k)$ *and* $i \in \mathbb{Z}$ *there is an isomorphism*

$$\mathbf{H}^i_{\mathrm{dNis}}(X, a_{\mathrm{dNis}}C) = \varinjlim_{Y \in X^{Sm}_{\mathrm{div}}} \mathbf{H}^i_{\mathrm{sNis}}(Y,C) \tag{2.4.1}$$

where X^{Sm}_{div} *is the category of smooth log modifications* $Y \to X$ *of* X*.*

A formula similar to (2.4.1) holds for $X \in \mathbf{lSm}(S)$ as in the following theorem.

Theorem 2.5 ([7, Theorem 5.1.2])**.** *Let* S *be a Noetherian fs log scheme, and let* C *be a bounded below complex of strict Nisnevich sheaves on* $\mathbf{lSm}(S)$*. Then for every* $X \in \mathbf{lSm}(S)$ *and* $i \in \mathbb{Z}$ *there is an isomorphism*

$$\mathbf{H}^i_{dNis}(X, a_{dNis}C) = \varinjlim_{Y \in X_{\mathrm{div}}} \mathbf{H}^i_{sNis}(X,C)$$

where the colimit runs over the set X_{div} *of log modifications of* X *(not necessarily smooth).*

The following result comes in handy to produce long exact sequences.

Lemma 2.6. *Let $X,Y \in$ **SmlSm**, let $D_X \subseteq \underline{X}$ and $D_Y \subseteq Y$ be Cartier divisors such that $D_X + |\partial X|$ and $D_Y + |\partial Y|$ have simple normal crossings.*

Suppose that

$$\begin{tikzcd}
\underline{X} - D_X \arrow[r] \arrow[d] & \underline{X} \arrow[d] \\
\underline{Y} - D_Y \arrow[r] & \underline{Y}
\end{tikzcd}$$

is a Zar- *(respectively* Nis-*) distinguished square in* **Sm**. *Let ∂X^+ and ∂Y^+ be the log structures induced by the divisors $D_X + |\partial X|$ and $D_X + |\partial Y|$, and let $X^+ := (\underline{X}, \partial X^+)$ and $Y^+ := (\underline{Y}, \partial Y^+)$. Then, for every complex $C \in \mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda)$ that is* sZar- *(respectively* sNis-*) fibrant the following square*

$$\begin{tikzcd}
C(X) \arrow[r] \arrow[d] & C(X^+) \arrow[d] \\
C(Y) \arrow[r] & C(Y^+)
\end{tikzcd}$$

is a homotopy pullback.

Proof. Let τ be either Zar or Nis. Because the log structures on $X - D_X$ (respectively $Y - D_Y$) induced by X and X^+ (respectively Y and Y^+) are the same, the following squares are $s\tau$-distinguished:

$$\begin{tikzcd}
X - D_X \arrow[r] \arrow[d] & X \arrow[d] \\
Y - D_Y \arrow[r] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X - D_X \arrow[r] \arrow[d] & X^+ \arrow[d] \\
Y - D_Y \arrow[r] & Y^+
\end{tikzcd}$$

Moreover, the canonical maps $X^+ \to X$ and $Y^+ \to Y$, whose underlying maps of schemes are the identities of $\underline{X}$ and $\underline{Y}$, make the following diagram commutative:

$$\begin{tikzcd}
C(X) \arrow[r] \arrow[d] & C(X^+) \arrow[r] \arrow[d] & C(X - D_X) \arrow[d] \\
C(Y) \arrow[r] & C(Y^+) \arrow[r] & C(Y - D_X)
\end{tikzcd}$$

Because C is $s\tau$-fibrant, the big rectangle and the square on the right are homotopy pullbacks. Hence, the square on the left is a homotopy pullback. ☐

2.2. log correspondences

Following [7], we denote by $\mathbf{lCor}(k)$ the category of finite log correspondences over k. It is a variant of the Suslin-Voevodsky category of finite correspondences $\mathbf{Cor}(k)$ introduced in [32]; see [22]. It has the same objects as $\mathbf{lSm}(k)$, and morphisms are given by the free abelian subgroup

$$\mathbf{lCor}(X,Y) \subseteq \mathbf{Cor}(X - \partial X, Y - \partial Y)$$

generated by elementary correspondences $V^o \subset (X-\partial X)\times(Y-\partial Y)$ such that the closure $V\subset \underline{X}\times\underline{Y}$ is finite and surjective over (a component of) $\underline{X}$ and such that there exists a morphism of log schemes $V^N \to Y$, where V^N is the fs log scheme whose underlying scheme is the normalisation of V and whose log structure is given by the inverse image log structure along the composition $\underline{V^N}\to\underline{X}\times\underline{Y}\to\underline{X}$. See [7, section 2.1] for more details and for the proof that this definition gives indeed a category.

Additive presheaves (of Λ-modules) on the category $\mathbf{lCor}(k)$ will be called *presheaves (of Λ-modules) with log transfers.* Write $\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda)$ for the resulting category. We have a natural adjunction

$$\begin{tikzcd}
\mathbf{PSh}^{\log}(k,\Lambda) \arrow[r, shift left=2, "\gamma_\sharp"] \arrow[r, shift right=2, "\gamma_*"'] & \mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda) \arrow[l, "\gamma^*" description]
\end{tikzcd}$$

where by convention $\gamma_\sharp$ is left adjoint to γ^*, which is left adjoint to γ_*. Here $\gamma\colon \mathbf{lSm}(k)\to \mathbf{lCor}(k)$ is the graph functor. For a topology τ on $\mathbf{lSm}(k)$, a presheaf with log transfers F is a τ-sheaf if γ^*F is a τ-sheaf. We denote by $\mathbf{Shv}^{\mathrm{ltr}}_\tau(k,\Lambda)\subset\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda)$ the subcategory of τ-sheaves. By [7, Proposition 4.5.4] and [7, Theorem 4.5.7], the strict Nisnevich and the dividing Nisnevich topology on $\mathbf{lSm}(k)$ are compatible with log transfers: this means in particular that the inclusion $\mathbf{Shv}^{\mathrm{ltr}}_\tau(k,\Lambda)\subset\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda)$ admits an exact left adjoint a_τ (see [7, Proposition 4.2.10]), and that the category $\mathbf{Shv}^{\mathrm{ltr}}_\tau(k,\Lambda)$ is a Grothendieck Abelian category [7, Proposition 4.2.12].

2.3. Effective log motives

We fix again a Noetherian fs log scheme S and a field k and let $\mathcal{C}$ be either $\mathbf{lSm}(S)$ or $\mathbf{lCor}(k)$. We start by recalling some standard facts. The category $\mathbf{Cpx}(\mathbf{PSh}(\mathcal{C},\Lambda))$ of unbounded complexes of presheaves is equipped with the usual global (projective) model structure $(\mathbf{W},\mathbf{Cof},\mathbf{Fib})$, where the weak equivalences are the quasi-isomorphisms and the fibrations are the degreewise surjective maps (see, for example, the remark after [12, Theorem 9.3.1] or [4, Proposition 4.4.16]).

Let τ be a topology on $\mathcal{C}$ (and we require that τ is compatible with transfers when $\mathcal{C}=\mathbf{lCor}(k)$). Recall that a morphism of complexes of presheaves $F\to G$ in $\mathbf{Cpx}(\mathbf{PSh}(\mathcal{C},\Lambda))$ is called a τ-local equivalence if it induces isomorphisms $a_\tau H_i(F)\simeq a_\tau H_i(G)$ for every $i\in\mathbb{Z}$, where $H_i(F)$ denotes the ith homology presheaf of F.

The left Bousfield localisation of the global model structure on $\mathbf{Cpx}(\mathbf{PSh}(\mathcal{C},\Lambda))$ with respect to the class of τ-local equivalences exists and the resulting model structure $(\mathbf{W}_\tau,\mathbf{Cof},\mathbf{Fib}_\tau)$ is called the τ-*local* model structure (see, for example, [4, Proposition 4.4.31]). The maps in $\mathbf{W}_\tau$ are precisely the τ-local equivalences. It is well known that the homotopy category of $\mathbf{Cpx}(\mathbf{PSh}(\mathcal{C},\Lambda))$ with respect to the local model structure, denoted $\mathbf{D}_\tau(\mathbf{PSh}(\mathcal{C},\Lambda))$, is equivalent to the unbounded derived category $\mathbf{D}(\mathbf{Shv}_\tau(\mathcal{C},\Lambda))$ of the Grothendieck abelian category of τ-sheaves $\mathbf{Shv}_\tau(\mathcal{C},\Lambda)$.

For any $X\in\mathcal{C}$, we write

$$R\Gamma_\tau(X,-)\colon \mathbf{D}_\tau(\mathbf{PSh}(\mathcal{C},\Lambda))\to\mathbf{D}(\Lambda)$$

for the right derived functor of the global section functor $\Gamma(X,-)$. The τ-(hyper) cohomology of X with values in a complex of presheaves C is then computed as

$$\mathbf{H}^*_\tau(X,a_\tau(C)) = \mathbf{H}^*(R\Gamma_\tau(X,a_\tau C)).$$

Finally, let $\overline{\square}_S := (\mathbf{P}^1_S,\infty_S) \in \mathcal{C}$, with $S = \mathrm{Spec}(k)$ if $\mathcal{C} = \mathbf{lCor}(k)$.

Definition 2.7. The $(\tau,\overline{\square}_S)$-local model structure on $\mathbf{Cpx}(\mathbf{PSh}(\mathcal{C},\Lambda))$ is the (left) Bousfield localisation of the τ-local model structure with respect to the class of maps

$$\Lambda(\overline{\square}_S \times_S X)[n] \to \Lambda(X)[n]$$

for all $X \in \mathcal{C}$ and $n \in \mathbb{Z}$.

General properties of the Bousfield localisation (see, e.g., [4, Définition 4.2.64, Proposition 4.2.66]) imply that a complex of presheaves C is $(\tau,\overline{\square}_S)$-fibrant if and only if it is τ-fibrant (i.e., fibrant for the τ-local model structure) and the morphisms $C(X) \to C(X \times_S \overline{\square}_S)$ induced by the projection are quasi-isomorphisms for every $X \in \mathcal{C}$.

Definition 2.8. (1) A complex of presheaves C, seen as an object of $\mathbf{D}_\tau(\mathbf{PSh}(\mathcal{C},\Lambda))$, is called $\overline{\square}_S$-local if for all $X \in \mathcal{C}$ the map

$$R\Gamma_\tau(X,C) \to R\Gamma_\tau(X \times_S \overline{\square}_S, C)$$

is a quasi isomorphism in $\mathbf{D}(\Lambda)$. Equivalently, C is $\overline{\square}_S$-local if and only any τ-fibrant replacement of C is $(\tau,\overline{\square}_S)$-fibrant.

(2) Let $L\colon \mathbf{D}_\tau(\mathbf{PSh}(\mathcal{C},\Lambda)) \to \mathbf{D}_{(\mathrm{dNis},\overline{\square}_S)}(\mathbf{PSh}(\mathcal{C},\Lambda))$ be the localisation functor. A complex of presheaves K, seen as an object of $\mathbf{D}_\tau(\mathbf{PSh}(\mathcal{C},\Lambda))$, is called $(\tau,\overline{\square}_S)$-locally acyclic if $L(K)$ is τ-locally isomorphic to the zero complex; that is, if $R\Gamma_\tau(X,L(K)) \simeq 0$ for all $X \in \mathcal{C}$.

Definition 2.9. The derived category of effective log motives (with transfers)

$$\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda) = \mathbf{logDM}^{\mathrm{eff}}_{\mathrm{dNis}}(k,\Lambda) = \mathbf{D}_{(\mathrm{dNis},\overline{\square})}(\mathbf{Cpx}(\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda)))$$

is the homotopy category of $\mathbf{Cpx}(\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda))$ with respect to the $(\mathrm{dNis},\overline{\square})$-local model structure. Similarly, if S is an fs Noetherian log scheme of finite Krull dimension, the category of effective log motives without transfers $\mathbf{logDA}^{\mathrm{eff}}(S,\Lambda) = \mathbf{logDA}^{\mathrm{eff}}_{\mathrm{dNis}}(S,\Lambda)$ is the homotopy category of $\mathbf{Cpx}(\mathbf{PSh}^{\log}(S,\Lambda))$ with respect to the $(\mathrm{dNis},\overline{\square}_S)$-local model structure.

The interested reader can verify that Definition 2.9 is equivalent to [7, Definition 5.2.1].

We now collect some well-known facts about the $(\tau,\overline{\square}_S)$-local model structure, for $\tau \in \{\mathrm{sNis},\mathrm{dNis}\}$ that we are going to use later. Recall that $\mathbf{Cpx}(\mathbf{PSh}^{\log}(S,\Lambda))$ is a closed monoidal model category with respect to the global model structure by [4, Lemme 4.4.62]. We write $\underline{\mathrm{Hom}}(-,-)$ for the internal Hom functor.

Lemma 2.10. *Let I be a τ-fibrant object (respectively a $(\tau,\overline{\square}_S)$-fibrant object) of $\mathbf{Cpx}(\mathbf{PSh}^{\log}(S,\Lambda))$. Then, for every $X \in \mathbf{lSm}(S)$, the complex $\underline{\mathrm{Hom}}_S(\Lambda(X),I)$ is τ-fibrant (respectively is $(\tau,\overline{\square}_S)$-fibrant).*

Proof. Every representable presheaf $\Lambda(X)$ is cofibrant for the projective model structure, and $-\otimes\Lambda(X)$ is a left Quillen functor. So, for every $A\to B\in \mathrm{Cof}\cap W_\tau$, we have that $A\otimes\Lambda(X)\to B\otimes\Lambda(X)$ is a trivial τ-local cofibration (see [4, Proposition 4.4.63] and observe that the small site Y_τ is coherent for every $Y\in \mathbf{lSm}(S)$ because $\underline{S}$ is quasi-compact and quasi-separated; hence, it has enough points by [3, Exp. VI, Proposition 9.0] and we can apply [4, Proposition 4.4.63]). In particular, every τ-fibrant object I satisfies the lifting property

$$\begin{tikzcd}
A\otimes\Lambda(X) \arrow[r] \arrow[d] & I \\
B\otimes\Lambda(X) \arrow[ur, dashed] &
\end{tikzcd}$$

We conclude that $-\otimes\Lambda(X)$ is a left Quillen functor for the τ-local model structure; hence, $\underline{\mathrm{Hom}}_S(\Lambda(X),-)$ is a right Quillen functor. In particular, $\underline{\mathrm{Hom}}_S(\Lambda(X),I)$ is τ-fibrant. □ In a similar way, if I is $(\tau,\overline{\square})$-fibrant, we have that $\underline{\mathrm{Hom}}_S(\Lambda(X),I)$ is τ-fibrant and $\overline{\square}$-local, so it is $(\tau,\overline{\square})$-fibrant. □

2.11. Let $X\in\mathbf{lSm}(S)$ and let $\lambda\colon X\to S$ be the structural morphism. We have an induced functor $\lambda^*:\mathbf{PSh}^{\log}(S,\Lambda)\to\mathbf{PSh}^{\log}(X,\Lambda)$ given by precomposition with λ. The functor λ^* and its left Kan extension $\lambda_!$ induce two adjoint functors on the categories of complexes:

$$\lambda_!\colon \mathbf{Cpx}(\mathbf{PSh}^{\log}(X,\Lambda))\leftrightarrows\mathbf{Cpx}(\mathbf{PSh}^{\log}(S,\Lambda)):\lambda^*. \tag{2.11.1}$$

Because λ^* is exact, it preserves by definition global fibrations and global weak equivalences; hence, $\lambda_!$ preserves global cofibrations and (2.11.1) is a Quillen adjunction. In fact, by, for example, [4, Theorem 4.4.51], the same holds for the τ-local model structure where τ is a topology on $\mathbf{lSm}(S)$; in particular, λ^* preserves τ-fibrant objects.

Finally, if $C\in\mathbf{Cpx}(\mathbf{PSh}^{\log}(S,\Lambda))$ is $\overline{\square}_S$-local, then λ^*C is $\overline{\square}_X$-local, because for all $U\in\mathbf{lSm}(X)$

$$\lambda^*C(U\times_X\overline{\square}_X)=C(U\times_X X\times_S\overline{\square}_S)\simeq C(U\times_X X)=\lambda^*C(U).$$

We conclude that λ^* preserves $(\tau,\overline{\square})$-fibrant objects as well.

2.12. We end this section with a computation of the localisation functor

$$L=L_{(\tau,\overline{\square}_S)}\colon \mathbf{Cpx}(\mathbf{PSh}^{\log}(S,\Lambda))\to\mathbf{Cpx}(\mathbf{PSh}^{\log}(S,\Lambda))_{(\tau,\overline{\square}_S)}\subset\mathbf{Cpx}(\mathbf{PSh}^{\log}(S,\Lambda)),$$

where $\mathbf{Cpx}(\mathbf{PSh}^{\log}(S,\Lambda))_{(\tau,\overline{\square}_S)}$ denotes the subcategory of $(\tau,\overline{\square}_S)$-local objects. By general properties of the Bousfield localisation, L comes equipped with a natural transformation $\lambda\colon id\to L$, and the pair (L,λ) is unique up to a unique natural isomorphism.

An explicit description of the localisation functor has been worked out by Ayoub in [5, Section 2] for the $\mathbf{P}^1$-localisation. We spell out the construction for presheaves without transfers and for $\tau\in\{\mathrm{sNis},\mathrm{dNis}\}$.

Construction 2.13 (see [5, Construction 2.6])**.** We fix an endofunctor $(-)_\tau$ that gives a *τ-fibrant replacement*. Let $\Lambda(\overline{\square}_S^{\mathrm{red}})$ be the kernel of the map $\Lambda(\overline{\square}_S) \to \Lambda$. For a complex $C \in \mathbf{Cpx}(\mathbf{PSh}^{\log}(S,\Lambda))$ we put

$$\Phi(C) := \mathrm{Cone}\{\delta : \Lambda(\overline{\square}_S^{\mathrm{red}}) \otimes_\Lambda \underline{\mathrm{Hom}}_S(\Lambda(\overline{\square}_S^{\mathrm{red}}), C_\tau) \to C_\tau\},$$

where δ is the counit of the adjuntion $\Lambda(\overline{\square}_S^{\mathrm{red}}) \otimes_\Lambda \_ \dashv \underline{\mathrm{Hom}}_S(\Lambda(\overline{\square}_S^{\mathrm{red}}), \_)$.

We obtain an endofunctor Φ equipped with a natural transformation $\varphi\colon id \to \Phi$, and we define the endofunctor Φ^∞ by taking the colimit of the following sequence:

$$C \xrightarrow{\varphi_C} \Phi(C) \xrightarrow{\varphi_{\Phi(C)}} \Phi^{\circ 2}(L) \xrightarrow{\varphi_{\Phi^{\circ 2}(C)}} \ldots \to \Phi^{\circ n}(C) \to \ldots$$

By construction, the functor Φ^∞ comes equipped with a natural transformation $\varphi^\infty\colon id \to \Phi^\infty$.

Theorem 2.14 (see [5, Théorème 2.7])**.** *Let $C \in \mathbf{Cpx}(\mathbf{PSh}^{\log}(S,\Lambda))$. Then $\Phi^\infty(C)$ is $(\tau,\overline{\square}_S)$-fibrant and φ^∞ is a $(\tau,\overline{\square}_S)$-local equivalence. In other words, the pair $(\Phi^\infty,\varphi^\infty)$ is naturally isomorphic to the $(\tau,\overline{\square}_S)$-localisation (L,λ).*

Proof. We follow the same pattern of the proof in [5], and we divide the proof into two steps. First, we need to show that for any complex of presheaves C the morphism $C \to \Phi^\infty(C)$ is a $(\tau,\overline{\square}_S)$-local equivalence. After that, we have to prove that $\Phi^\infty(C)$ is fibrant for the $(\tau,\overline{\square}_S)$-local model structure.

We begin by observing that for all $F \in \mathbf{Cpx}(\mathbf{PSh}^{\log}(S,\Lambda))$ the tensor product $\Lambda(\overline{\square}^{\mathrm{red}}) \otimes_\Lambda F$ is $(\tau,\overline{\square}_S)$-locally acyclic (see Definition 2.8). Indeed, the subcategory of $(\tau,\overline{\square}_S)$-locally acyclic complexes is a triangulated subcategory of $\mathbf{D}_\tau(\mathbf{PSh}^{\log}(S,\Lambda))$ that is stable by direct sums, and by construction it contains all of the objects of the form $\Lambda(\overline{\square}_S^{\mathrm{red}}) \otimes_\Lambda \Lambda(X)$ for any $X \in \mathbf{lSm}(S)$.

Next, note that because the homotopy fibre of φ_C is given by

$$\Lambda(\overline{\square}_S^{\mathrm{red}}) \otimes_\Lambda \underline{\mathrm{Hom}}_S(\Lambda(\overline{\square}_S^{\mathrm{red}}), C_\tau),$$

which is then $(\tau,\overline{\square}_S)$-locally acyclic in virtue of what we just observed, φ_C is a $(\tau,\overline{\square}_S)$-local equivalence for all complexes C. Because filtered colimits preserve $(\tau,\overline{\square}_S)$-local equivalences, we conclude that the map $C \to \Phi^\infty(C)$ is a $(\tau,\overline{\square}_S)$-local equivalence.

We move to the second part of the proof. By construction, the map $\Phi^{\circ n}(C) \to \Phi^{\circ n+1}(C)$ factors through $\Phi^{\circ n}(C)_\tau$, which are by construction τ-fibrant. Hence, $\Phi^\infty(C)$ is a filtered colimit of τ-fibrant objects.

By Lemma 2.15, filtered colimits preserve τ-fibrant objects; hence, $\Phi^\infty(C)$ is dNis fibrant.

Finally, we need to show that $\Phi^\infty(C)$ is $\overline{\square}_S$-local, which is equivalent to showing that $\underline{\mathrm{Hom}}_S(\overline{\square}_S^{\mathrm{red}}, \Phi^\infty(C))$ is acyclic. The argument in the proof of part (B) of [5, Theorem 2.7] goes through without changes. We leave the verification to the reader. □

Lemma 2.15. *Let S be a Noetherian scheme of finite Krull dimension and let $(C_i)_{i\in I}$ be a filtered diagram in $\mathbf{Cpx}(\mathbf{PSh}^{\log}(S,\Lambda))$. Assume that each C_i is τ-fibrant, then $\varinjlim C_i$ is τ-fibrant.*

Proof. We argue as in [4, Proposition 4.5.62]. For $\tau = \mathrm{sNis}$, it follows from [31, Tag 0737], using that S is Noetherian of finite Krull dimension. For $\tau = \mathrm{dNis}$, we have that for every $X \in \mathbf{lSm}(S)$ and every filtered system $\{F_i\}_{i\in I} \in \mathbf{Shv}^{\log}_{\mathrm{dNis}}(X,\Lambda)$ there is a chain of isomorphisms

$$\begin{aligned}\mathbf{H}^i_{\mathrm{dNis}}(X, \varinjlim_i F_i) &\cong^{(1)} \varinjlim_{Y\in X} \mathbf{H}^i_{\mathrm{sNis}}(Y, \varinjlim_i F_i)\\ &\cong^{(2)} \varinjlim_i \varinjlim_{Y\in X} \mathbf{H}^i_{\mathrm{sNis}}(Y, F_i)\\ &\cong^{(3)} \varinjlim_i \mathbf{H}^i_{\mathrm{dNis}}(X, F_i),\end{aligned}$$

where (1) and (3) follow from Theorem 2.5 and (2) follows from the fact that each Y is also Noetherian of finite Krull dimension. This implies that filtered colimits preserve dNis-fibrant objects. □

Remark 2.16. The proof works verbatim for $C \in \mathbf{Cpx}(\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda))$, where $\otimes_\Lambda$ is changed with the tensor product $\otimes^{\mathrm{ltr}}$.

3. The connectivity theorem following Ayoub and Morel

In this section we show a $\overline{\square}$-analogue of the $\mathbf{A}^1$-connectivity theorem of Morel [24, Theorem 6.1.8], adapting the argument of Ayoub in [5, Section 4]. As in [5], we exploit the notion of *preconnected* complex (see Definition 3.3) and we reduce the proof of the connectivity Theorem 3.2 to a purity statement, namely Theorem 4.4, whose proof will be given in Section 4. The reader should note that though the results in this section are direct analogues of the results in [5], new ingredients are necessary to prove the purity theorem, and this is where our arguments diverge from [5].

Throughout this section, we fix a ground field k and we work with the categories of presheaves and τ-sheaves on $\mathbf{lSm}(k)$ for $\tau \in \{\mathrm{sZar},\mathrm{sNis},\mathrm{dNis}\}$. Recall from [7, Lemma 4.7.2] that $\mathbf{Shv}^{\log}_{\mathrm{dNis}}(k,\Lambda)$ is equivalent to the category $\mathbf{Shv}_{\mathrm{dNis}}(\mathbf{SmlSm}(k),\Lambda)$ of sheaves defined on the full subcategory $\mathbf{SmlSm}(k) \subset \mathbf{lSm}(k)$. If $X = (\underline{X},\partial X) \in \mathbf{SmlSm}(k)$ and $x \in \underline{X}$ is any point, we consider $\mathrm{Spec}(\mathcal{O}_{X,x}) \in \widetilde{\mathbf{SmlSm}}(k)$ with the logarithmic structure induced by the pullback of ∂X.

Definition 3.1. Let $n \in \mathbb{Z}$ and let C be a complex of presheaves on a site $(\mathcal{C},\tau)$. We say that C is *locally* n*-connected* (for the topology τ) if the homology sheaves $a_\tau H_j(C)$ are zero for $j \leq n$.

The main result of this section is the following.

Theorem 3.2. *Assume that k is a perfect field and let $\tau \in \{\mathrm{sNis},\mathrm{dNis}\}$. Let $C \in \mathbf{Cpx}(\mathbf{PSh}^{\log}(k,\Lambda))$ be locally n-connected for the τ-topology. Then for any $(\tau,\overline{\square})$-fibrant replacement $C \to L$, the complex L is locally n-connected.*

The proof will be given at the end of this section, assuming Theorem 4.4. We need some preliminary definitions; compare with [5, Déf. 4.5].

Definition 3.3. (i) A complex C of presheaves is called *generically n-connected* if for all $X \in \mathbf{SmlSm}(k)$ with $\underline{X}$ connected and generic point η_X the homology groups $H_j(C(\eta_X))$ are zero for $j \leq n$.

(ii) A complex C of presheaves is called *n-preconnected* if for all $X \in \widetilde{\mathbf{SmlSm}}(k)$ the homology groups $H_j(C(X))$ are zero for $j \leq n - \dim(\underline{X})$.

Remark 3.4. (1) Clearly $(ii) \Rightarrow (i)$ because a generic point has dimension 0, but it is evident that $(i) \not\Rightarrow (ii)$.

(2) If $C \in \mathbf{Cpx}(\mathbf{PSh}^{\log}(k,\Lambda))$ is locally n-connected for a topology τ where the cohomological dimension equals the Krull dimension of the underlying scheme, then $\mathbf{H}^i_\tau(X,C) = 0$ for $i \geq \dim(\underline{X}) - n$. Hence, if G is a τ-fibrant replacement of C, G is n-preconnected, because $H_i(G(X)) = \mathbf{H}^{-i}_\tau(X,G)$.

We will prove some technical result that will be needed later. Here we let τ be either sZar, sNis or dNis.

Proposition 3.5 (see [5, Proposition 4.8])**.** *Let C be an n-preconnected complex of presheaves; then for all $X \in \mathbf{SmlSm}(k)$ we have $\mathbf{H}^i_\tau(X,C) = 0$ for $i \geq \dim(X) - n$.*

Proof. Without loss of generality we can suppose $n = -1$ – that is, $H_{-j}(C(X)) = 0$ for $j > \dim(X)$ – and we need to show that $\mathbf{H}^i_\tau(X,C) = 0$ for $i > \dim(X)$. Using the descent spectral sequence $\mathbf{H}^i(X,a_\tau H_{-j}(C)) \Rightarrow H^{i+j}_\tau(X,C)$, it is enough to show $\mathbf{H}^i_\tau(X,a_\tau H_{-j}(C)) = 0$ for $i > \dim(X) - j$.

If $j \leq 0$, this follows Proposition 2.2, so suppose $j > 0$. By -1-preconnectedness, $H_{-j}(C)(\mathrm{Spec}(\mathcal{O}_{X,x})) = 0$ for $\mathrm{codim}(x) < j$, because $\mathrm{codim}(x) = \dim(\mathrm{Spec}(\mathcal{O}_{X,x}))$. Using Lemma 3.6, the statement then follows for $\tau \in \{\mathrm{sZar},\mathrm{sNis}\}$.

The result for $\tau = \mathrm{dNis}$ then can be deduced from the case sNis. Indeed, using Lemma 3.6, we get in particular $\mathbf{H}^i_{\mathrm{sNis}}(Y,a_{\mathrm{sNis}}H_{-j}(C)) = 0$ for all $Y \in X^{Sm}_{\mathrm{div}}$, whence, because by (2.4.1) we have that

$$\mathbf{H}^j_{\mathrm{dNis}}(X,a_{\mathrm{dNis}}H_{-j}(C)) = \varinjlim_{Y \in X^{\mathbf{Sm}}_{\mathrm{div}}} \mathbf{H}^j_{\mathrm{sNis}}(Y,a_{\mathrm{sNis}}H_{-j}(C)),$$

the required vanishing holds for dNis as well. □

Lemma 3.6. *Let $\tau \in \{\mathrm{sZar},\mathrm{sNis}\}$. Let F be a presheaf of Λ-modules on the small site X_τ such that for every τ-cover $X' \to X$ and $x' \in X'$ with $\mathrm{codim}_{X'}(x') < j$, we have $F(\mathrm{Spec}(\mathcal{O}_{X',x'})) = 0$. Then $\mathbf{H}^i_\tau(X,a_\tau F) = 0$ for $i > \dim(X) - j$.*

Proof. This is [5, Lemma 4.9]; we reproduce part of the proof in our setting for completeness and to take care of some subtleties. Observe that the forgetful functor $f\colon \mathbf{SmlSm}(k) \to \mathbf{Sm}(k)$ that sends X to the underlying scheme $\underline{X}$ defines an isomorphism of the small sites $f_X\colon X_{\mathrm{sNis}} \xrightarrow{\simeq} \underline{X}_{\mathrm{Nis}}$ (and similarly for sZar and Zar): the inverse functor sends an étale scheme $g\colon \underline{U} \to \underline{X}$ to the morphism of log schemes $U \to X$, where U is the log scheme having $\underline{U}$ as underlying scheme and log structure given by the inverse image log structure along g (note that this would be false for the dNis-topology). A presheaf F on X_{sNis} (respectively on X_{sZar}) then gives canonically a presheaf $\underline{F}$ on $\underline{X}_{\mathrm{Nis}}$ (respectively

$\underline{X}_{\mathrm{Zar}}$), by setting $\underline{X}_{\mathrm{Nis}} \ni \underline{U} \mapsto F(U)$ (respectively $\underline{X}_{\mathrm{Zar}} \ni \underline{V} \mapsto F(V)$). Clearly, there is a canonical isomorphism $\mathbf{H}^i_{\mathrm{sNis}}(X,F) \cong \mathbf{H}^i_{\mathrm{Nis}}(\underline{X},\underline{F})$, and by abuse of notation we drop the underline and write simply F for both presheaves on X_{sNis} or on $\underline{X}_{\mathrm{Nis}}$ (and the same for the Zariski case).

The rest of the proof of the lemma goes through as in [5, Lemme 4.9]. See [5] for more details. $\square$

Corollary 3.7. *Let $C \in \mathbf{Cpx}(\mathbf{PSh}^{\log}(k,\Lambda))$ and let $C \to L$ be a τ-fibrant replacement for $\tau \in \{\mathrm{dNis},\mathrm{sNis}\}$. If C is n-preconnected, then so is L.*

Proof. Follows from the fact that $H_{-j}(L(X)) = \mathbf{H}^j_\tau(X,C)$ and Proposition 3.5.

We have the following set of elementary properties of n-preconnected complexes. $\square$

Lemma 3.8 (see [5, Lemme 4.11])**.** *Let C be an n-preconnected complex of presheaves on $\mathbf{lSm}(k)$:*

(i) *For all G m-connected, $C \otimes_{\mathbb{Z}} G$ is $(n+m+1)$-preconnected.*

(ii) *For all $X \in \mathbf{lSm}(k)$, $\underline{\mathrm{Hom}}(X,C)$ is $n - \dim(X)$-preconnected.*

(iii) *If $\alpha : G \to C$ is a morphism of complexes of presheaves on $\mathbf{lSm}(k)$ and G is $(n-1)$-preconnected, then $Cone(\alpha)$ is n-preconnected.*

Proposition 3.9 (see [5, Theorem 4.12])**.** *Let $F \in \mathbf{Cpx}(\mathbf{PSh}^{\log}(k,\Lambda))$ n-preconnected and $F \to C$ be a $(\tau,\overline{\square})$-fibrant replacement for $\tau \in \{\mathrm{dNis},\mathrm{sNis}\}$. Then C is n-preconnected.*

Proof. The argument of [5, Theorem 4.12] goes through. We have an explicit description of C given by Theorem 2.14. Let

$$\Phi(F) := \mathrm{Cone}(\overline{\square} \otimes \underline{\mathrm{Hom}}(\overline{\square},F_\tau)) \to F_\tau,$$

where F_τ denotes a τ-fibrant replacement of F, which is n-preconnected by Corollary 3.7. By Lemma 3.8(i)–(ii) $\overline{\square} \otimes \underline{\mathrm{Hom}}(\overline{\square},F_\tau))$ is $n-1$-preconnected; hence, by lemma 3.8(iii) the cone $\Phi(F)$ is n-preconnected. Because $C \simeq \varinjlim_n \Phi^{\circ n}(F)$, we conclude. $\square$

Proof of Theorem 3.2. We give a proof for $\tau = \mathrm{dNis}$, because the case $\tau = \mathrm{sNis}$ is identical. Let $C \in \mathbf{Cpx}(\mathbf{PSh}^{\log}(k,\Lambda))$ be a complex of presheaves, locally n-connected for the dNis topology. Because the Krull dimension of any $X \in \mathbf{lSm}(k)$ agrees with the dNis-cohomological dimension by Proposition 2.2, the fact that C is locally n-connected is equivalent to asking that, for any $X \in \mathbf{SmlSm}(k)$, we have $\mathbf{H}^i_{\mathrm{dNis}}(X,C) = 0$ for $i \geq \dim(X) - n$. If G is a dNis-local fibrant replacement of C, this implies that H is n-preconnected (see Remark 3.4(2)), and by Proposition 3.9, any $(\mathrm{dNis},\overline{\square})$-fibrant replacement L of C is then n-preconnected as well. In particular, it is generically n-connected.

We are left to show that every $(\mathrm{dNis},\overline{\square})$-fibrant complex L that is generically n-connected is also locally n-connected. Consider the canonical map $a_{\mathrm{dNis}} H_i(L)(X) \to H_i(L)(\eta_X, triv)$ for any $X \in \mathbf{SmlSm}(k)$ with $\underline{X}$ connected and generic point η_X. Here we write (η_X,triv) to indicate the essentially smooth log scheme given by the scheme η_X with trivial log structure. By Corollary 4.6 (this is where the assumption that k is perfect is used), this map is injective. This implies that $a_{\mathrm{dNis}}H_i(L)(X) = 0$ for any $X \in \mathbf{SmlSm}(k)$ and $i < n$; that is, the homology sheaves $a_{\mathrm{dNis}}H_i(L)$ are zero for $i < n$, proving the claim. □

4. Purity of logarithmic motives

Throughout this section, we fix a base field k and a $(\mathrm{sNis},\overline{\square})$-fibrant complex of presheaves $C \in \mathbf{Cpx}(\mathbf{PSh}^{\log}(k,\Lambda))$.

Lemma 4.1 (see [5, Sous-Lemme 4.14]). *Let $X \in \mathbf{SmlSm}(k)$, $x \in \underline{X}$ and $a \in H_i(C(X))$ such that there is a dense open $U \subseteq X$ and $a_{|U} = 0$. Then there exists an open neighbourhood V of x such that $a_{|V} = 0$ if either one of the following hypotheses is satisfied:*

(i) *$\partial X = \emptyset$; that is, X has trivial log structure.*

(ii) *$\dim(\underline{X}) = 1$ and $|\partial X|$ is supported on a finite number of k-rational points.*

Proof. Let $Z = \underline{X} - \underline{U}$. If $x \notin Z$, there is nothing to prove; hence, we can suppose $x \in Z$. We can apply Gabber's geometric presentation theorem ([8, Theorem 3.1.1] for k infinite; [11, Theorem 1.1] for k finite): by replacing X with an open neighbourhood of x there exist a k-scheme Y and an étale morphism $e : \underline{X} \to \mathbf{A}^1_Y$ such that

(1) Z maps isomorphically to $e(Z)$; that is, there is a Nisnevich distinguished square of schemes

$$\begin{tikzcd} \underline{X} - Z \arrow[r] \arrow[d] & \underline{X} \arrow[d] \\ \mathbf{A}^1_Y - e(Z) \arrow[r] & \mathbf{A}^1_Y \end{tikzcd}$$

(2) The composition

$$Z \to \underline{X} \to \mathbf{A}^1_Y \to Y$$

is finite.

In particular, $e(Z)$ is closed in $\mathbf{P}^1_Y$ and it is disjoint from ∞_Y. We now divide the proof into two parts.

Case (i): Let us suppose that X has trivial log structure. In this case we have two sNis-distinguished squares

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbf{P}^1_Y - e(Z) \arrow[r] & \mathbf{P}^1_Y, \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ (\mathbf{P}^1_Y - e(Z), \infty_Y) \arrow[r] & (\mathbf{P}^1_Y, \infty_Y) \end{tikzcd}$$

where Y is seen as a log scheme with trivial log structure, and $\overline{\square}_Y = (\mathbf{P}^1_Y, \infty_Y)$ (respectively $(\mathbf{P}^1_Y - e(Z), \infty_Y)$) denotes as usual the scheme $\mathbf{P}^1_Y$ (respectively $\mathbf{P}^1_Y - e(Z)$) with compactifying log structure at $\infty_Y = \{\infty\} \times Y$. Furthermore, the morphisms

$(\mathbf{P}^1_Y,\infty_Y) \to \mathbf{P}^1_Y$ and $(\mathbf{P}^1_Y - e(Z),\infty_Y) \to \mathbf{P}^1_Y - e(Z)$, whose underlying morphisms of schemes are the identities on $\mathbf{P}^1_Y$ and $\mathbf{P}^1_Y - e(Z)$, induce a commutative diagram

$$\begin{tikzcd}
U \arrow[rr] \arrow[dr] \arrow[dd] & & X \arrow[dr] \arrow[dd, bend right] & \\
& (\mathbf{P}^1_Y - e(Z),\infty_Y) \arrow[rr] \arrow[dl] & & \overline{\square}_Y \arrow[dl] \\
\mathbf{P}^1_Y - e(Z) \arrow[rr] & & \mathbf{P}^1_Y &
\end{tikzcd}$$

We define the following objects of $\mathbf{D}(\Lambda)$:

$$\begin{aligned}
C_Z(X) &= \mathrm{hofib}(C(X) \to C(U)),\\
C_Z(\mathbf{P}^1_Y) &= \mathrm{hofib}(C(\mathbf{P}^1_Y) \to C(\mathbf{P}^1_Y - e(Z)))\\
C_Z(\overline{\square}_Y) &= \mathrm{hofib}(C(\overline{\square}_Y) \to C(\mathbf{P}^1_Y - e(Z),\infty_Y)).
\end{aligned}$$

Because C is $(\mathrm{sNis},\overline{\square})$-fibrant, it is in particular sNis-fibrant and therefore the three left vertical arrows of the following diagram

$$\begin{tikzcd}
C_Z(\mathbf{P}^1_Y) \arrow[r, "\delta_{\mathbf{P}^1_Y}"] \arrow[d, "s_{\mathbf{P}^1_Y}"] \arrow[dd, bend right, "t"'] & C(\mathbf{P}^1_Y) \arrow[r] \arrow[d] \arrow[dd, bend left, "r"] & C(\mathbf{P}^1_Y - e(Z)) \arrow[d] \arrow[dd, bend left] \\
C_Z(X) \arrow[r, "\delta"] & C(X) \arrow[r] & C(U) \\
C_Z(\overline{\square}_Y) \arrow[r, "\delta_{\overline{\square}_Y}"] \arrow[u, "s_{\overline{\square}_Y}"] & C(\overline{\square}_Y) \arrow[r] \arrow[u] & C(\mathbf{P}^1_Y - e(Z),\infty_Y) \arrow[u]
\end{tikzcd} \tag{4.1.1}$$

denoted $s_{\mathbf{P}^1_Y}$, $s_{\overline{\square}_Y}$ and t, respectively, are quasi-isomorphisms.

Let now $\alpha \in H_i(C(X))$ such that $\alpha_{|U} = 0$; hence, there exists $\beta \in H_i(C_Z(X))$ such that $\alpha = \delta(\beta)$. By the quasi-isomorphism above, there exists a unique $\beta_{\mathbf{P}^1_Y} \in H_iC_Z(\mathbf{P}^1_Y)$ such that $s_{\mathbf{P}^1_Y}(\beta_{\mathbf{P}^1_Y}) = \beta$. Let $\alpha_{\mathbf{P}^1_Y} = \delta_{\mathbf{P}^1_Y}(\beta_{\mathbf{P}^1_Y})$ and let $r\colon C(\mathbf{P}^1_Y) \to C(\mathbf{P}^1_Y,\infty_Y)$ be as in the diagram above. It is enough to show that $r(\alpha_{\mathbf{P}^1_Y}) = 0$ in $H_i(C(\mathbf{P}^1_Y,\infty_Y))$ to conclude that $\alpha = 0$ in $H_i(C(X))$, using (4.1.1).

Write $C_0(\mathbf{P}^1_Y)$ for the homotopy fibre of $C(\mathbf{P}^1_Y) \xrightarrow{s_\infty} C(\infty_Y)$. Because $e(Z)$ is disjoint from ∞_Y, the map $\delta_{\mathbf{P}^1_Y}$ factors as

$$\begin{tikzcd}
C_Z(\mathbf{P}^1_Y) \arrow[r, "\delta_{\mathbf{P}^1_Y}"] \arrow[d] & C(\mathbf{P}^1_Y) \arrow[r] \arrow[d, equal] & C(\mathbf{P}^1_Y - e(Z)) \arrow[d] \\
C_0(\mathbf{P}^1_Y) \arrow[r, "\delta_0"'] & C(\mathbf{P}^1_Y) \arrow[r, "s_\infty"] & C(\infty_Y)
\end{tikzcd}$$

In particular, there exists $\alpha_0 \in H_i(C_0(\mathbf{P}^1_Y))$ such that $\delta_0(\alpha_0) = \alpha_{\mathbf{P}^1_Y}$. We will conclude by showing that $r\delta_0$ is the zero map.

Because C is $\overline{\square}$-local, the projection $\pi\colon \overline{\square}_Y \to Y$ induces a quasi-isomorphism $\pi^*\colon C(Y) \xrightarrow{\simeq} C(\overline{\square}_Y)$. Because clearly π factors through the natural map $\overline{\square}_Y \to \mathbf{P}^1_Y$, we have a commutative diagram

$$\begin{tikzcd}
C_0(\mathbf{P}^1_Y) \arrow[r, "\delta_0"'] & C(\mathbf{P}^1_Y) \arrow[r, "s_\infty"] \arrow[d, "r"] & C(\infty_Y) \arrow[d, equal, "\mathrm{Id}_Y"] \\
 & C(\overline{\square}_Y) & C(Y) \arrow[ul, "\pi^*"'] \arrow[l, "\simeq", "\pi^*"']
\end{tikzcd}$$

and this immediately shows that $r\delta_0$ factors through an acyclic complex, as required.

Case (ii): Let us now suppose that $\dim(\underline{X}) = 1$ and ∂X is nontrivial, supported on a finite set of k-rational points.

If $x \notin |\partial X|$, then we can suppose $X = (\underline{X} - |\partial X|, \mathrm{triv})$ and conclude as before (this in fact does not use the assumption on the dimension of $\underline{X}$). So let us assume that $x \in |\partial X|$: because $\dim(\underline{X}) = 1$, by replacing X with an open neighbourhood of x we can suppose $|\partial X| = x = Z$.

After replacing X with an open neighbourhood of x we have a sNis distinguished square

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & X \arrow[d] \\
(\mathbf{P}^1_{k(x)} - e(x), \mathrm{triv}) \arrow[r] & (\mathbf{P}^1_{k(x)}, e(x)).
\end{tikzcd}$$

Because x is a k-rational point, we conclude that $k = k(x)$ and $e(x)$ is a k-rational point of $\mathbf{P}^1_k$. We drop the subscript k for simplicity. Write as before:

$$C_{\{x\}}(X) = \mathrm{hofib}(C(X) \to C(U))$$
$$C_{\{e(x)\}}(\overline{\square}^1) = \mathrm{hofib}(C(\mathbf{P}^1, e(x)) \to C(\mathbf{P}^1 - e(x))).$$

Because C is (sNis,$\overline{\square}$)-fibrant, hence sNis fibrant, the left vertical arrow of the following diagram

$$\begin{tikzcd}
C_{\{e(x)\}}(\overline{\square}^1) \arrow[r, "\delta_{\overline{\square}_Y}"] \arrow[d, "s_{\overline{\square}_Y}"'] & C(\mathbf{P}^1, e(x)) \arrow[r] \arrow[d] & C(\mathbf{P}^1 - e(x)) \arrow[d] \\
C_{\{x\}}(X) \arrow[r, "\delta"] & C(X) \arrow[r] & C(U)
\end{tikzcd} \tag{4.1.2}$$

is a quasi-isomorphism. Now, because C is $\overline{\square}$-local, the complex $C(\mathbf{P}^1, e(x))$ is quasi-isomorphic to $C(\mathrm{Spec}(k))$, and by choosing any k-rational point of $\mathbf{P}^1 - e(x)$ splitting the projection $(\mathbf{P}^1 - e(x)) \to \mathrm{Spec}(k)$, we see that the map

$$H_i(C(\mathbf{P}^1, e(x))) \to H_i(C(\mathbf{P}^1 - e(x)))$$

is injective for every $i \in \mathbb{Z}$. This, together with the commutativity of (4.1.2), allows us to conclude. □

Corollary 4.2. *Let τ be either* sZar,sNis *or* dNis.

(i) *Let $X \in \mathbf{Sm}(k)$. Then the following map is injective:*

$$a_\tau H_i(C(X,triv)) \hookrightarrow H_i(C(\eta_X,triv)),$$

where η_X is the generic point of X and $(X,triv)$ denotes the scheme X seen as a log scheme with a trivial log structure.

(ii) *Let $X \in \mathbf{SmlSm}(k)$ such that $\dim(\underline{X}) = 1$ and $|\partial X|$ is supported on a finite number of k-rational points. Then the following map is injective:*

$$a_\tau H_i(C(X)) \hookrightarrow H_i(C(\eta_X,triv)),$$

where η_X is the generic point of X.

Proof. We begin by observing that maps in (i) and (ii) exist because $H_iC(\eta_X,\text{triv}) = a_\tau H_iC(\eta_X,\text{triv})$. We first prove (i). Let $\alpha \in a_\tau H_i(C(X,\text{triv}))$ be a section such that $\alpha_{|\eta_X} = 0$. Let $V \to X$ be a τ-cover such that there exists $\beta \in H_i(C(V,\text{triv}))$ mapping to the image of α in $a_\tau H_iC(V,\text{triv})$. Let $\coprod \eta_V$ be the disjoint union of the generic points of V. The following diagram is clearly commutative:

$$\begin{tikzcd}
 & H_i(C(V,\text{triv})) \arrow[d] \\
a_\tau H_i(C(X,\text{triv})) \arrow[r] \arrow[d] & a_\tau H_i(C(V,\text{triv})) \arrow[d] \\
H_i(C(\eta_X,\text{triv})) \arrow[r] & \bigoplus H_i(C(\eta_V,\text{triv}));
\end{tikzcd}$$

hence, β maps to zero in $\bigoplus H_i(C(\eta_V,\text{triv}))$. By Lemma 4.1(i), for all $x \in V$ there exists an open neighbourhood V_x such that $\beta \mapsto 0$ in $H_i(C(V_x,\text{triv}))$. Because we can cover V by the V_x, and because for every topology τ as in the statement open sieves are covering, we conclude that β maps to zero in $a_\tau H_iC(V,\text{triv})$; hence, $\alpha = 0$, because $(V,\text{triv}) \to (U,\text{triv})$ is a τ-cover. This proves (i). The proof of (ii) is similar, replacing (V,triv) with $(V,\partial X_{|V})$ and using Lemma 4.1(ii). □

In order to prove Theorem 4.4, we need the following technical result, which is well known to experts. Recall that a Henselian k-algebra is said to be *of geometric type* if there exists $X \in \mathbf{Sm}(k)$ and $x \in X$ such that $R \cong \mathcal{O}^h_{X,x}$, the henselisation of the local ring $\mathcal{O}_{X,x}$ at x.

Lemma 4.3. *Let k be a perfect field and R a Henselian k-algebra of geometric type. Let $\mathfrak{p} \subseteq R$ such that $R/\mathfrak{p}$ is essentially smooth over k. Then the map $R_\mathfrak{p} \to k(\mathfrak{p})$ has a section.*

Proof. Let κ be the residue field of R. By the properties of Henselian k-algebras of geometric type (see, for example, [28, Lemma 6.1]), there exists a regular sequence $t_1 \dots t_n \in R$ such that $R \cong \kappa\{t_1 \dots t_n\}$, the henselisation of the local ring of $\mathbf{A}^n_\kappa$ at (0), and $\mathfrak{p} = (t_{r+1}, \dots t_n)$; hence, $R/\mathfrak{p} \cong \kappa\{t_1 \dots t_r\}$.

In particular, the map $\pi : R \to R/\mathfrak{p}$ has an evident section $s : \kappa\{t_1,\dots,t_r\} \to \kappa\{t_1,\dots,t_n\}$. Moreover, it is also evident that $\mathrm{Im}(s) \cap \mathfrak{p} = 0$; thus, there exists a unique map $s' : \mathrm{Frac}(\kappa\{t_1 \dots t_r\}) \to A_{\mathfrak{p}}$ such that the following diagram commutes:

$$\begin{tikzcd}
\kappa\{t_1,\dots,t_r\} \arrow[d, "\subseteq"'] \arrow[r, bend right, "s"'] & \kappa\{t_1,\dots,t_n\} \arrow[l, "\pi"'] \arrow[d] \\
\mathrm{Frac}(\kappa\{t_1 \dots t_r\}) \arrow[r, bend right, "s'"'] & \kappa\{t_1,\dots,t_n\}_{(t_{r+1},\dots,t_n)}. \arrow[l, "\pi'"']
\end{tikzcd}$$

Hence, s' is a section of π'. This, together with the isomorphism $k(\mathfrak{p}) \cong \mathrm{Frac}(\kappa\{t_1 \dots t_r\})$, concludes the proof. $\square$

Theorem 4.4. *Let $X \in \widetilde{\mathbf{SmlSm}}(k)$ such that $\underline{X}$ is a Henselian local scheme. Then the map*

$$H_i(C(X)) \to H_i(C(\eta_X, triv)) \tag{4.4.1}$$

is injective.

Proof. Let $|\partial X| = D_1 + \dots + D_n$. We proceed by double induction on $\dim(\underline{X})$ and n.

If $\dim(\underline{X}) = 1$ and $n = 0$, then (4.4.1) is injective by Corollary 4.2 (i). Assume then that $\dim(\underline{X}) = 1$ but $n > 0$. Then ∂X is supported on the closed point x (note that ∂X is automatically irreducible, because $\underline{X}$ is 1-dimensional and local). By Lemma 4.3, the map $\mathrm{Spec}(k(x)) \to \underline{X}$ has a retraction; hence, $X \in \widetilde{\mathbf{SmlSm}}(k(x))$ and $|\partial X|$ is supported on a $k(x)$-rational point.

Let $\lambda\colon \mathrm{Spec}(k(x)) \to \mathrm{Spec}(k)$. Because C is (sNis,$\overline{\square}$)-fibrant in $\mathbf{Cpx}(\mathbf{PSh}^{\log}(k,\Lambda))$, $\lambda^* C$ is (sNis,$\overline{\square}$)-fibrant in $\mathbf{Cpx}(\mathbf{PSh}^{\log}(k(x),\Lambda))$ (see Remark 2.11); hence, we have

$$H_i C(X) = H_i \lambda^* C(X) \xrightarrow{(*1)} H_i \lambda^* C(\eta_X, \mathrm{triv}) = H_i C(\eta_X, \mathrm{triv})$$

and $(*1)$ is injective by Corollary 4.2 (ii). This proves the case for $\dim(\underline{X}) = 1$.

Suppose now that $\dim(\underline{X}) > 1$ and $n = 0$. Then again (4.4.1) is injective by Corollary 4.2 (i). We now pass to the case $\dim(\underline{X}) > 1$ and $n \geq 1$. For every $1 \leq r \leq n$, let $\eta_{D_r} \in \underline{X}$ be the generic point of D_r and $\iota_{D_r} : D_r \to X$ the inclusion. For $Y \in \mathbf{SmlSm}(k)$, we write $c(Y)$ for the number of irreducible components of the strict normal crossing divisor ∂Y.

We make the following claim. $\square$

Claim 4.5. Assume the induction hypothesis above; that is, suppose that Theorem 4.4 holds for every $Y \in \widetilde{\mathbf{SmlSm}}(k)$ local Henselian such that $\dim(\underline{Y}) \leq n-1$ and $c(Y) \geq 0$ and with $\dim(\underline{Y}) = \dim(\underline{X})$ and $c(Y) \leq n-1$. Then, for every $U \subseteq X$ dense open such that $U \cap D_n \subseteq D_n$ is dense, the restriction map $H_i C(X) \to H_i C(U)$ is injective.

We postpone the proof of Claim 4.5 and complete the proof of the theorem. Because filtered colimits are exact in the category of Λ-modules, we get from Claim 4.5 an injective map:

$$H_i(C(X)) \hookrightarrow \varinjlim_{\substack{U\subseteq X\\ \eta_{D_n}\in U}} H_i(C(U)) = H_i(C(\mathrm{Spec}(\mathcal{O}_{\underline{X},\eta_{D_n}}),\iota^*_{D_n}\partial X)). \tag{4.5.1}$$

Let $\mathcal{O}_{\underline{X},\eta_{D_n}}$ be the local ring of $\underline{X}$ at η_{D_n}: it is a discrete valuation ring with generic point η_X and infinite residue field $k(\eta_{D_n})$. Because $\mathcal{O}_{\underline{X},\eta_{D_n}}$ is the localisation of a Henselian k-algebra at a prime ideal generated by a regular sequence, we can apply Lemma 4.3 to get a map $\mathrm{Spec}(\mathcal{O}_{\underline{X},\eta_{D_n}}) \to \mathrm{Spec}(k(\eta_{D_n}))$ that splits $\eta_{D_n} \to \mathrm{Spec}(\mathcal{O}_{\underline{X},\eta_{D_n}})$; hence,

$$(\mathrm{Spec}(\mathcal{O}_{\underline{X},\eta_{D_n}}),\iota^*_{D_n}\partial X) \in \widetilde{\mathbf{SmlSm}(k}(\eta_{D_n}))$$

and $|\iota^*_{D_n}\partial X|$ is a $k(\eta_{D_n})$-rational point.

Let $\lambda : \mathrm{Spec}(k(\eta_{D_n})) \to \mathrm{Spec}(k)$. We argue as above: because C is $(\mathrm{sNis},\overline{\square})$-fibrant in $\mathbf{Cpx}(\mathbf{PSh}^{\log}(k,\Lambda))$, λ^*C is $(\mathrm{sNis},\overline{\square})$-fibrant in $\mathbf{Cpx}(\mathbf{PSh}^{\log}(k(\eta_{D_n}),\Lambda))$ (see again Remark 2.11); hence, by Corollary 4.2 (ii) we have an injective map:

$$\begin{aligned} H_i(C(\mathrm{Spec}(\mathcal{O}_{\underline{X},\eta_{D_n}})),\iota^*_{D_n}\partial X) &= H_i(\lambda^*C(\mathrm{Spec}(\mathcal{O}_{\underline{X},\eta_{D_n}})),\iota^*_{D_n}\partial X) \\ &\hookrightarrow H_i(\lambda^*C(\eta_X,\mathrm{triv})) \\ &= H_i(C(\eta_X,\mathrm{triv})). \end{aligned} \tag{4.5.2}$$

Combining (4.5.1) with (4.5.2), we get the desired injectivity. This reduces the proof of Theorem 4.4 to the proof of Claim 4.5.

Proof of Claim 4.5.

Let $X^- := (\underline{X},\partial X^-) \in \mathbf{SmlSm}(k)$, where ∂X^- is the strict normal crossing divisor $D_1 + \ldots + D_{n-1}$. Because $c(X^-) = n-1$, by hypothesis (this is the induction assumption on the number of components of ∂X), the map $H_iC(X^-) \to H_iC(\eta_X,\mathrm{triv})$ is injective.

Let $\underline{U}$ be an open dense subset of $\underline{X}$ such that $\underline{U} \cap D_n$ is dense in D_n and $U \cap D_i = \emptyset$ if $i \neq n$, and set $U := (\underline{U},\partial X_{|\underline{U}})$. Write $U^- := (U,\partial X^-_{|U}) = (U,\mathrm{triv})$. Hence, we have a commutative diagram:

$$\begin{tikzcd} H_i(C(X^-)) \arrow[r, hook, "(2)"] \arrow[d, hook, "(1)"] \arrow[dr, hook, "(3)"] & H_i(C(X)) \arrow[d] \\ H_i(C(U^-)) \arrow[r] & H_i(C(U)), \end{tikzcd} \tag{4.5.3}$$

where (1), (2) and (3) are injective because they all factor the injective map $H_iC(X^-) \to H_iC(\eta_X,\mathrm{triv})$.

Because $\underline{X}$ is Henselian local of dimension $r \geq n$ with closed point x, there exists an isomorphism $\varepsilon\colon X \cong \mathrm{Spec}(k(x)\{t_1,\ldots,t_r\})$. Without loss of generality, we can assume that t_r is a local parameter for D_n, so that ε induces an isomorphism $D_n \cong \mathrm{Spec}(k(x)\{t_1,\ldots,t_{r-1}\})$. Hence, the map *henselisation at* 0,

$$k(x)\{t_1,\ldots,t_{r-1}\}[t_r] \to k(x)\{t_1,\ldots,t_r\},$$

induces a pro-Nisnevich square[2] of (usual) schemes:

$$\begin{tikzcd} X - D_n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ D_n \times (\mathbf{A}^1 - \{0\}) \arrow[r] & D_n \times \mathbf{A}^1. \end{tikzcd} \tag{4.5.4}$$

By Lemma 2.6, the square

$$\begin{tikzcd} C(D_n \times (\mathbf{A}^1,\mathrm{triv})) \arrow[r] \arrow[d] & C(D_n \times (\mathbf{A}^1,0)) \arrow[d] \\ C(X^-) \arrow[r] & C(X) \end{tikzcd} \tag{4.5.5}$$

is a filtered colimit of homotopy pullbacks; hence, it is itself a homotopy pullback. Consider the system $\{\underline{V}\}$ of open neighbourhoods of $\eta_{D_n} \times \{0\}$ in $D_n \times \mathbf{A}^1$: the system $\{p^{-1}(\underline{V})\}$ is cofinal in the system of open neighbourhoods of η_{D_n} in X. Given any such $\underline{V}$, let $\underline{W_V}$ be the subset of $D_n \times \mathbf{A}^1$ given as

$$(\pi(D_n \times \{0\} \cap V) \times \mathbf{A}^1) \cap V,$$

where $\pi\colon D_n \times \mathbf{A}^1 \to D_n$ is the projection. It is clear by construction that $\underline{W_V}$ contains $(D_n \times \{0\} \cap V)$ and, in fact,

$$\underline{V} \cap (D_n \times \{0\}) = \underline{W_V} \cap (D_n \times \{0\}).$$

Because $\underline{V}$ is an open neighbourhood of $\eta_{D_n} \times \{0\}$, the projection $\pi(\underline{V} \cap (D_n \times \{0\}))$ is open dense in D_n and thus $\underline{W_V}$ is an open neighbourhood of $\eta_{D_n} \times \{0\}$ and the system $\{\underline{W_V}\}$ is cofinal in the system of open neighbourhoods of $\eta_{D_n} \times \{0\}$ in $D_n \times \mathbf{A}^1$. Because $\{p^{-1}(\underline{W_V})\}$ is then cofinal in the system of open neibghborhoods of η_{D_n} in X, we can conclude that there exists $\underline{W} \subseteq \underline{U}$ such that $\underline{W} \cap D_n$ is dense in D_n and induces a pro-Zariski square of (usual) schemes:

$$\begin{tikzcd} \underline{W} - (D_n \cap \underline{W}) \arrow[r] \arrow[d] & \underline{W} \arrow[d] \\ (D_n \cap \underline{W}) \times (\mathbf{A}^1 - \{0\}) \arrow[r] & (D_n \cap \underline{W}) \times \mathbf{A}^1 \end{tikzcd} \tag{4.5.6}$$

Hence, up to refining $\underline{U}$ we can suppose that $\underline{U}$ itself fits in a pro-Zariski square like (4.5.6), so again using Lemma 2.6 and the fact that a filtered colimit of homotopy pullbacks is itself a homotopy pullback, we get the following homotopy pullback square:

$$\begin{tikzcd} C((D_n \cap U) \times (\mathbf{A}^1,\mathrm{triv})) \arrow[r] \arrow[d] & C((D_n \cap U) \times (\mathbf{A}^1,0)) \arrow[d] \\ C(U^-) \arrow[r] & C(U) \end{tikzcd} \tag{4.5.7}$$

---

[2] That is, a cofiltered limit of Nisnevich squares.

We conclude that for C sNis-fibrant the squares (4.5.5) and (4.5.7) induce the following equivalences:

$$\begin{aligned}\mathrm{Cofib}(C(X^-)\to C(X)) &\cong \mathrm{Cofib}(C(D_n\times(\mathbf{A}^1,\mathrm{triv}))\to C(D_n\times(\mathbf{A}^1,0)))\\ &\cong \mathrm{Hom}^\bullet(MTh(N_{D_n/X^-}),C)\end{aligned}$$

$$\begin{aligned}\mathrm{Cofib}(C(U^-)\to C(U)) &\cong \mathrm{Cofib}(C((D_n\cap U)\times(\mathbf{A}^1,\mathrm{triv}))\to C((D_n\cap U)\times(\mathbf{A}^1,0)))\\ &\cong \mathrm{Hom}^\bullet(MTh(N_{D_n\cap U/U^-}),C),\end{aligned}$$

where the last isomorphisms come from the definition of the motivic Thom space [7, Definition 7.4.3], the fact that X is local and $U\subseteq X$ is an open immersion; hence, $N_{D_n/X^-}\cong D_n\times\mathbf{A}^1$ and $N_{D_n\cap U/U^-}\cong(D_n\cap U)\times\mathbf{A}^1$. Here, $\mathrm{Hom}^\bullet(K,C)\in\mathbf{D}(\Lambda)$ for $K\in\mathbf{Cpx}(\mathbf{PSh}^{\log}(k,\Lambda))$ is the mapping complex. In particular, we get for every $i\in\mathbb{Z}$ the following commutative diagram:

$$\begin{tikzcd}
0\to H_i(C(X^-)) \arrow[r] \arrow[d, hook] \arrow[dr, hook] & H_i(C(X)) \arrow[r] \arrow[d] & \mathrm{Hom}(MTh(N_{D_n/X^-}),C[i-1])\to 0 \arrow[d] \\
H_i(C(U^-)) \arrow[r] & H_i(C(U)) \arrow[r] & \mathrm{Hom}(MTh(N_{D_n\cap U/U^-}),C[i-1]),
\end{tikzcd} \tag{4.5.8}$$

where the top horizontal sequence is exact and the bottom horizontal sequence is exact in the middle. We will now show that for every i, the natural map

$$\mathrm{Hom}(MTh(N_{D_n/X^-}),C[i])\to\mathrm{Hom}(MTh(N_{(D_n-Z)/((X^-)-Z)}),C[i])$$

is injective, where $Z=\underline{X}-\underline{U}$: assuming this, by diagram chase in (4.5.8) we finally conclude that the map $H_i(C(X))\hookrightarrow H_i(C(U))$ is injective for every U as above.

We can use [7, Proposition 7.4.5] (note that the condition that C is $(\mathrm{sNis},\overline{\square})$-fibrant is enough) to compute the motivic Thom spaces: we get a commutative diagram where the rows are split exact sequences

$$\begin{tikzcd}
0\to H_iC(D_n,\partial D_n) \arrow[r] \arrow[d] & H_iC((D_n,\partial D_n)\times\mathbf{P}^1) \arrow[r] \arrow[d] & \mathrm{Hom}(MTh(N_{D_n/X^-}),C[i])\to 0 \arrow[d] \\
0\to H_iC(D_n\cap U,\partial D_n^\circ) \arrow[r] & H_iC((D_n\cap U,\partial D_n^\circ)\times\mathbf{P}^1) \arrow[r] & \mathrm{Hom}(MTh(N_{D_n-Z/X^--Z}),C[i])\to 0.
\end{tikzcd} \tag{4.5.9}$$

We have that

$$H_iC(-\times\mathbf{P}^1)=H_i(\underline{\mathrm{Hom}}((\mathbf{P}^1,\mathrm{triv}),C))(-)$$

and $\underline{\mathrm{Hom}}((\mathbf{P}^1,\mathrm{triv}),C)$ is $(\mathrm{sNis},\overline{\square})$-fibrant because C is (see Lemma 2.10). By induction on dimension we conclude that the middle vertical map of (4.5.9) is injective, and because the rows in (4.5.9) are split-exact sequences, the right vertical map is a retract of the middle one; hence, it is injective. This concludes the proof. $\square$

Corollary 4.6. *Let* $X \in \mathbf{SmlSm}(k)$ *and let* τ *be either* sNis *or* dNis. *Then the following map is injective:*

$$a_\tau H_iC(X) \hookrightarrow H_iC(\eta_X, triv),$$

where η_X *is the generic point of* X.

Proof. The case where $\tau = \mathrm{dNis}$ follows from the case of sNis. Indeed, because filtered colimits are exact in the category of Λ-modules, and because for all $Y \in X_{div}$, the map $\underline{Y} \to \underline{X}$ is birational, so that $\eta_Y = \eta_X$, we get

$$a_{\mathrm{dNis}} H_iC(X) = \varinjlim_{Y \in X_{div}} a_{\mathrm{sNis}} H_iC(Y) \hookrightarrow \varinjlim_{Y \in X_{div}} H_iC(\eta_Y, \mathrm{triv}) = H_iC(\eta_X, \mathrm{triv}).$$

Thus, from now on let $\tau = \mathrm{sNis}$. For all $x \in X$, let X^h_x be the henselisation of X at x with log structure induced by the log structure of X, and let $\eta(X^h_x)$ be its fraction field, which is a field extension of η_X. We have a diagram

$$\begin{tikzcd}
a_\tau H_iC(X) \arrow[r, "(*3)"] \arrow[d, "(*1)"] & H_iC(\eta_X, \mathrm{triv}) \arrow[d] \\
\prod_{x \in X} H_iC(X^h_x) \arrow[r, "(*2)"] & \prod_{x \in X} H_iC(\eta(X^h_x), \mathrm{triv}).
\end{tikzcd}$$

The map $(*1)$ is injective by the sheaf condition and the map $(*2)$ is injective by Theorem 4.4 and the fact that injective morphisms are stable under arbitrary products in Λ-modules. Hence, the map $(*3)$ is injective, which concludes the proof. □

5. The homotopy t-structure on logarithmic motives

The goal of this section is to generalise to the logarithmic setting the results of Morel on the existence of the homotopy t-structure on the category of motives. Having the connectivity Theorem 3.2 at disposal, the proofs are fairly straightforward.

Recall that the triangulated categories

$$\begin{aligned} \mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\log}(\mathbf{lSm}(k), \Lambda)) &\cong \mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}(\mathbf{SmlSm}(k), \Lambda)), \\ \mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\mathrm{ltr}}(\mathbf{lSm}(k), \Lambda)) &\cong \mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\mathrm{ltr}}(\mathbf{SmlSm}(k), \Lambda)) \end{aligned} \tag{5.0.1}$$

are equipped with a natural t-structure. The heart is equivalent to the category of dNis-sheaves (with or without transfers)

$$\begin{aligned} \mathbf{Shv}_{\mathrm{dNis}}(\mathbf{lSm}(k), \Lambda) &\cong \mathbf{Shv}_{\mathrm{dNis}}(\mathbf{SmlSm}(k), \Lambda), \\ \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(\mathbf{lSm}(k), \Lambda) &\cong \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(\mathbf{SmlSm}(k), \Lambda). \end{aligned} \tag{5.0.2}$$

The equivalences follow from [7, Lemma 4.7.2] (without transfers) and [7, Proposition 4.7.5] (with transfers), which hold for the dNis-topology but not for the strict Nisnevich topology. We write $\tau_{\geq n}$ and $\tau_{\leq n}$ for the (homologically graded) truncation functors on $\mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}(\mathbf{lSm}(k), \Lambda))$ and $\tau^{tr}_{\geq n}$ and $\tau^{tr}_{\leq n}$ for the (homologically graded) truncation functors on $\mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\mathrm{ltr}}(\mathbf{lSm}(k), \Lambda))$. In view of (5.0.1) and (5.0.2), we will work with the category of sheaves on $\mathbf{SmlSm}(k)$ without further notice and simply write $\mathbf{Shv}^{\log}_{\mathrm{dNis}}(k,\Lambda)$ (respectively $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\Lambda)$) for the abelian category of sheaves (respectively of sheaves with log transfers). The proof of the following theorem is formally identical to [5, Theorem 4.15].

Theorem 5.1. *Let $C \in \mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}(\mathbf{SmlSm}(k),\Lambda))$ and suppose that C is $\overline{\square}$-local (see Definition 2.8). Then for all $n \in \mathbb{Z}$ the truncated complexes $\tau_{\geq n}C$ and $\tau_{\leq n}C$ are $\overline{\square}$-local.*

Proof. Up to shifting, we can clearly assume that $n = 0$, and by the standard properties of the t-structure, it is enough to show the statement for $\tau_{\geq 0}C$. Because C is $\overline{\square}$-local, the natural map $\tau_{\geq 0}C \to C$ factors through $L(\tau_{\geq 0}C)$ as

$$\begin{tikzcd}
\tau_{\geq 0}C \arrow[r, "e_0"] \arrow[dr, "e"'] & L(\tau_{\geq 0}C) \arrow[d, "\ell"] \\
 & C
\end{tikzcd}$$

where $L(\tau_{\geq 0}C)$ is any (dNis,$\overline{\square}$)-fibrant replacement. We have by Theorem 3.2 that $L(\tau_{\geq 0}C)$ is locally -1-connected, so the map ℓ factors as

$$\begin{tikzcd}
\tau_{\geq 0}C \arrow[r, "e_0"] \arrow[dr, "e"'] & L(\tau_{\geq 0}C) \arrow[r, "\ell_0"] \arrow[d, "\ell"] & \tau_{\geq 0}C \arrow[dl, "e"] \\
 & C. &
\end{tikzcd}$$

By the universal property of $\tau_{\geq 0}$ we get that $\ell_0 e_o = id_{\tau_{\geq 0}C}$. Hence, $\tau_{\geq 0}C$ is a direct summand of $L(\tau_{\geq 0}C)$, so it is $\overline{\square}$-local as required. □

Corollary 5.2. *Let $C \in \mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\mathrm{ltr}}(\mathbf{SmlSm}(k),\Lambda))$ and suppose that C is $\overline{\square}$-local. Then for all $n \in \mathbb{Z}$ the truncated complexes $\tau^{tr}_{\geq n}C$ and $\tau^{tr}_{\leq n}C$ are $\overline{\square}$-local.*

Proof. As in the proof of Theorem 5.1, it is enough to prove the statement for $\tau_{\geq 0}C$. Recall that the graph functor $\gamma\colon \mathbf{SmlSm}(k) \to \mathbf{SmlCor}(k)$, which sends a map $X \to Y$ to the finite correspondence $X \xrightarrow{\gamma(f)} Y$ induced by its graph, is faithful: the category $\mathbf{SmlCor}$ is, by definition, the full subcategory of $\mathbf{lCor}(k)$ consisting of all objects in $\mathbf{SmlSm}(k)$ (it is denoted $lCor_{SmlSm}/k$ in [7]). Presheaves with log transfers on $\mathbf{SmlSm}(k)$ are, by definition, presheaves (of Λ-modules) on $\mathbf{SmlCor}(k)$.

The dNis-topology is compatible with log transfers by [7, Theorem 4.5.7]; hence, γ induces a functor

$$\gamma^*\colon \mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\mathrm{ltr}}(\mathbf{SmlSm}(k),\Lambda)) \to \mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}(\mathbf{SmlSm}(k),\Lambda)).$$

It is immediate so see that γ^* is t-exact and conservative and preserves flasque sheaves; hence, for all $X \in \mathbf{SmlSm}(k)$ and $F \in \mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\mathrm{ltr}}(\mathbf{SmlSm}(k),\Lambda))$, we have

$$R\Gamma(X,\gamma^*F) = R\Gamma(X,F).$$

In particular, F is $\overline{\square}$-local if and only if γ^*F is. To prove the corollary, it is then enough to show that $\gamma^*(\tau^{tr}_{\geq 0}C)$ is $\overline{\square}$-local. But because γ^* is t-exact, we have $\gamma^*(\tau^{tr}_{\geq 0}C) = \tau_{\geq 0}\gamma^*C$, which is $\overline{\square}$-local by Theorem 5.1. □

Definition 5.3 (see [7, Definition 5.2.2])**.** Let $F \in \mathbf{Shv}^{\log}_{\mathrm{dNis}}(k,\Lambda)$ (respectively $F \in \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\Lambda)$). We say that F is *strictly* $\overline{\square}$*-invariant* if the cohomology presheaves $\mathbf{H}^i_{\mathrm{dNis}}(\_,F)$ are $\overline{\square}$-invariant.

Analogous to [29], we denote by $\mathbf{CI}^{\log}_{\mathrm{dNis}}$ (respectively $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$) the full subcategory of $\mathbf{Shv}^{\log}_{\mathrm{dNis}}(k,\Lambda)$ (respectively $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\Lambda)$) of *strictly* $\overline{\square}$-invariant sheaves.

Remark 5.4. Note that the above definition is slightly nonstandard: in the context of reciprocity sheaves we typically write $\mathbf{CI}_{\mathrm{Nis}}$ for the category of $\overline{\square}$-invariant Nisnevich sheaves, without 'strictness' condition; that is, without asking the property that the cohomology presheaves are $\overline{\square}$-invariant. If $F \in \mathbf{CI}_{\mathrm{Nis}}$ is moreover semipure in the sense of [28, Definition 1.28], the fact that the cohomology presheaves are $\overline{\square}$-invariant (at least when restricted to the subcategory $\underline{\mathbf{M}}\mathbf{Cor}_{ls}$ defined in [28]) is indeed a difficult result due to S. Saito [28, Theorem 9.3]. In the $\mathbf{A}^1$-invariant context, the analogous statement is due to Voevodsky [22, §24].

Recall that, in general, a sheaf F seen as an object of $\mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\mathrm{t}}(k,\Lambda))$ for $t \in \{\log,\mathrm{ltr}\}$ is $\overline{\square}$-local if and only if it is strictly $\overline{\square}$-invariant.

Corollary 5.5. *Let* $C \in \mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\mathrm{t}}(k,\Lambda))$ *where* $t \in \{\log,\mathrm{ltr}\}$*. Then the following are equivalent:*

(a) C *is* $\overline{\square}$*-local.*

(b) *The homology sheaves* $a_{\mathrm{dNis}}H_iC$ *are strictly* $\overline{\square}$*-invariant for every* $i \in \mathbb{Z}$*.*

Proof. The implication $(b) \Rightarrow (a)$ holds very generally and comes from a spectral sequence argument. The converse implication $(a) \Rightarrow (b)$ comes from the fact that $a_{\mathrm{dNis}}H_iC[i] = \tau_{\geq i}\tau_{\leq i}C$ and Theorem 5.1. $\square$

The following proposition is an instance of the more general fact that if $C \rightleftarrows D$ is an adjoint pair of triangulated categories equipped with t-structures such that the left adjoint is right t-exact, then the induced functors between the hearts are still adjoint. See, for example, [6, Proposition 1.3.17-(iii)].

Proposition 5.6. *The inclusion* $i\colon \mathbf{CI}^{\log}_{\mathrm{dNis}} \hookrightarrow \mathbf{Shv}^{\log}_{\mathrm{dNis}}(k,\Lambda)$ *(respectively* $i^{tr}\colon \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}} \hookrightarrow \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\Lambda)$*) has a left adjoint*

$$h_0 := a_{\mathrm{dNis}}H_0L(-[0])$$

(respectively $h_0^{ltr} := a_{\mathrm{dNis}}H_0^{tr}L^{tr}(-[0])$*).*

We can finally state the promised result on the existence of the t-structure on the category of motives.

Theorem 5.7. *Consider the inclusions*

$$\mathbf{logDA}^{\mathrm{eff}}(k,\Lambda) \hookrightarrow \mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\log}(k,\Lambda)) \tag{5.7.1}$$

$$\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda) \hookrightarrow \mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda)) \tag{5.7.2}$$

that identify $\mathbf{logDA}^{\mathrm{eff}}(k,\Lambda)$ *(respectively* $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$*) with the subcategory of* $\overline{\square}$*-local complexes. Then the standard* t*-structure of* $\mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\log}(k,\Lambda))$ *(respectively of* $\mathbf{D}_{\mathrm{dNis}}(\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda))$*) restricts to a* t*-structure on the category of motives* $\mathbf{logDA}^{\mathrm{eff}}(k,\Lambda)$ *(respectively* $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$*), called the* homotopy t-structure.

The heart of this t*-structre is naturally equivalent to* $\mathbf{CI}^{log}_{\mathrm{dNis}}$ *(respectively* $\mathbf{CI}^{ltr}_{\mathrm{dNis}}$*), which is then a Grothendieck abelian category.*

Proof. The first assertion follows directly from Theorem 5.1 (respectively Corollary 5.2) and the second from Corollary 5.5 and Proposition 5.6. The fact that the heart of a t-structure is abelian is well known [6].

Next, note that the homotopy t-structure is clearly accessible in the sense of [20, Definition 1.4.4.12].

Moreover, filtered colimits commute with cohomology; hence, if $\{F_\alpha\}$ is a filtered system of (dNis,$\overline{\square}$) fibrant objects, then $\varinjlim F_\alpha$ is (dNis,$\overline{\square}$) fibrant because it is dNis-fibrant (as observed in the proof of Theorem 2.14) and

$$\mathbf{H}^i(X,\varinjlim F_\alpha) = \varinjlim \mathbf{H}^i(X,F_\alpha) \cong \mathbf{H}^i(X\times\overline{\square},\varinjlim F_\alpha).$$

So if $H_i^{\overline{\square}}F_\alpha = 0$ for $i\geq 0$ and all α, then

$$H_i^{\overline{\square}}(\varinjlim F_\alpha) = H_i(\varinjlim F_\alpha) = \varinjlim H_iF_\alpha = 0.$$

Hence, the t-structure is compatible with colimits in the sense of [20, Definition 1.3.5.20].

In particular, as observed in [20, Remark 1.3.5.23], the categories $\mathbf{CI}^{\log}_{\mathrm{dNis}}$ and $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$ are Grothendieck abelian categories. □

Proposition 5.8. *The inclusion* $i\colon \mathbf{CI}^{\log}_{\mathrm{dNis}} \hookrightarrow \mathbf{Shv}^{\log}_{\mathrm{dNis}}(k,\Lambda)$ *(respectively* $i^{tr}\colon \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}} \hookrightarrow \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\Lambda)$*) has a right adjoint* h^0 *(respectively* h^0_{tr}*) such that for* $F\in\mathbf{Shv}^{\log}_{\mathrm{dNis}}$ *(respectively* $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}$*)*

$$ih^0F(X) = \mathrm{Hom}_{\mathbf{Shv}^{\log}_{\mathrm{dNis}}}(h_0(a_{\mathrm{dNis}}(\Lambda(X))),F)$$

(respectively $i^{ltr}h^0_{ltr}F(X) = \mathrm{Hom}_{\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}}(i^{ltr}h_0^{ltr}(a_{\mathrm{dNis}}(\Lambda_{\mathrm{tr}}(-))),F)$*).*

Proof. We prove the assertion for $\mathbf{CI}^{\log}_{\mathrm{dNis}}$, because the statement for $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$ is identical. First, note that if the right adjoint h^0 exists, then for $F\in\mathbf{Shv}_{\mathrm{dNis}}$ and $X\in\mathbf{SmlSm}(k)$, we have

$$ih^0(F)(X) = \mathrm{Hom}_{\mathbf{Shv}_{\mathrm{dNis}}}(a_{\mathrm{dNis}}\Lambda(X),ih^0F) = \mathrm{Hom}_{\mathbf{CI}^{\log}_{\mathrm{dNis}}}(h_0(a_{\mathrm{dNis}}\Lambda(X)),h^0F) = \\ \mathrm{Hom}_{\mathbf{Shv}_{\mathrm{dNis}}}(ih_0(a_{\mathrm{dNis}}\Lambda(X)),F)$$

as required. Hence, we only have to prove that h^0 exists.

By the special adjoint functor theorem (see [21, p. 130]), a functor between two Grothendieck abelian categories has a right adjoint if and only if it preserves all (small) colimits, so we need to show that this holds for $i\colon \mathbf{CI}^{\log}_{\mathrm{dNis}} \to \mathbf{Shv}^{\log}_{\mathrm{dNis}}(k,\Lambda)$; that is, that $\mathbf{CI}^{\log}$ is closed under small colimits in $\mathbf{Shv}^{\log}_{\mathrm{dNis}}(k,\Lambda)$.

As observed in the proof of Theorem 5.7, $\mathbf{CI}^{\log}$ is stable under filtered colimits. Because (small) colimits are filtered colimits of finite colimits, it is enough to show that $\mathbf{CI}^{\log}$ is stable under finite limits. Because it is an abelian subcategory, it is enough to show that it is stable under cokernels.

Let $F,G \in \mathbf{CI}^{\log}_{\mathrm{dNis}}$ and let $F \to G$ be a map in $\mathbf{Shv}_{\mathrm{dNis}}$. Then we have that

$$\mathrm{coker}_{\mathbf{Shv}_{\mathrm{dNis}}}(F \to G) = a_{\mathrm{dNis}} H_0(\mathrm{Cofib}(F_{\mathrm{dNis}} \to G_{\mathrm{dNis}})),$$

where F_{dNis} and G_{dNis} denote the dNis-fibrant replacements. Because F and G are strictly $\overline{\square}$-local, F_{dNis} and G_{dNis} are (dNis,$\overline{\square}$)-fibrant; hence, $\mathrm{Cofib}(F_{\mathrm{dNis}} \to G_{\mathrm{dNis}})$ is also (dNis,$\overline{\square}$)-fibrant.

In particular,

$$\begin{aligned}\mathrm{coker}_{\mathbf{Shv}_{\mathrm{dNis}}}(F \to G) &\simeq a_{\mathrm{dNis}} H_0(\mathrm{Cofib}(F_{\mathrm{dNis}} \to G_{\mathrm{dNis}})) \\ &\simeq a_{\mathrm{dNis}} H_0(L^{\mathrm{tr}}(\mathrm{Cofib}(F_{\mathrm{dNis}} \to G_{\mathrm{dNis}}))) \overset{(*)}{\simeq} \mathrm{coker}_{\mathbf{CI}^{\log}_{\mathrm{dNis}}}(F \to G),\end{aligned}$$

where $(*)$ comes from Proposition 5.6 and the fact that h_0 preserves colimits. □

Corollary 5.9. *Let $G \in \mathbf{CI}^{\log}_{\mathrm{dNis}}$ (respectively $G \in \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$). Then*

$$\underline{\mathrm{Ext}}^i_{\mathbf{Shv}^{\log}_{\mathrm{dNis}}}(F,G) \in \mathbf{CI}^{\log}_{\mathrm{dNis}}$$

(respectively $\underline{\mathrm{Ext}}^i_{\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}}(F,G) \in \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$) for every $F \in \mathbf{Shv}^{\log}_{\mathrm{dNis}}(k,\Lambda)$ (respectively $F \in \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\Lambda)$).

Proof. We only prove it for $\mathbf{Shv}^{\log}_{\mathrm{dNis}}$; the proof for $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}$ is identical. Let $G[0] \to G_{\mathrm{dNis}}$ be a dNis-fibrant replacement; hence,

$$\underline{\mathrm{Ext}}^i_{\mathbf{Shv}^{\log}_{\mathrm{dNis}}}(F,G) = a_{\mathrm{dNis}} H_i(\underline{\mathrm{Hom}}(F,G_{\mathrm{dNis}})).$$

Note that for every $X \in \mathbf{SmlSm}(k)$, we have an isomorphism

$$\underline{\mathrm{Hom}}(\Lambda(X),G_{\mathrm{dNis}}) \cong \underline{\mathrm{Hom}}(\Lambda(X \times \overline{\square}),G_{\mathrm{dNis}}),$$

because by adjunction we have

$$\begin{aligned}\Gamma(Y,\underline{\mathrm{Hom}}(\Lambda(X),G_{\mathrm{dNis}})) &\cong \Gamma(Y \times X,G_{\mathrm{dNis}}) \\ &\cong \Gamma(Y \times X \times \overline{\square},G_{\mathrm{dNis}}) \\ &\cong \Gamma(Y,\underline{\mathrm{Hom}}(\Lambda(X \times \overline{\square}),G_{\mathrm{dNis}})).\end{aligned}$$

From this it easily follows that $\underline{\mathrm{Hom}}(F,G_{\mathrm{dNis}})$ is $\overline{\square}$-local; hence, we conclude by Theorem 5.7. □

Theorem 5.10. *Let $F \in \mathbf{CI}^{log}_{\mathrm{dNis}}$ (respectively $F \in \mathbf{CI}^{ltr}_{\mathrm{dNis}}$). Then for all $X \in \mathbf{SmlSm}(k)$ and $U \subseteq X$ an open dense, the restriction $F(X) \to F(U)$ is injective.*

Proof. As before, we give a proof for the version without transfers. Let $F[0] \to G$ be a dNis-fibrant replacement. Because $F[0]$ is $\overline{\square}$-local, G is (dNis,$\overline{\square}$)-fibrant. Because $F = a_{\mathrm{dNis}} H_0 G$, the result follows from Theorem 4.6. □

5.1. Comparison with Voevodsky's motives

Let $\mathbf{Cor}(k)$ be Voevodsky's category of finite correspondences over k [22, §1]. We have a pair of adjoint functors

$$\lambda\colon \mathbf{Cor}(k) \rightleftarrows \mathbf{lCor}(k)\colon \omega,$$

where $\lambda(X) = (X,\mathrm{triv})$ and $\omega(X,\partial X) = X - |\partial X|$. They induce functors on the categories of complexes of presheaves

$$\begin{tikzcd} \mathbf{Cpx}(\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda)) \arrow[r, shift left=2, "\omega_\sharp"] \arrow[r, shift right=2, "\omega_*"'] & \mathbf{Cpx}(\mathbf{PSh}^{\mathrm{tr}}(k,\Lambda)), \arrow[l, "\omega^*" description] \end{tikzcd} \tag{5.10.1}$$

where ω^* denotes as usual the restriction functor, $\omega_\sharp$ its left Kan extension and ω_* the right Kan extension. Because λ is left adjoint to ω, we have $\lambda^* = \omega_\sharp$. By construction, ω^* and $\omega_\sharp$ are t-exact for the global t-structures.

The adjunction $(\omega_\sharp,\omega^*)$ is a Quillen adjunction with respect to the dNis-local model structure on the left-hand side and the Nis-local model structure on the right-hand side (see [7, (4.3.4)]) and with respect to the $(\mathrm{dNis},\overline{\square})$-local model structure on the left-hand side and the $(\mathrm{Nis},\mathbf{A}^1)$-local model structure on the right-hand side (see [7, (4.3.5)]) and therefore induces the following derived adjunctions:

$$\begin{gathered} L\omega_\sharp\colon \mathbf{D}_{\mathrm{dNis}}(\mathbf{Cpx}(\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda))) \rightleftarrows \mathbf{D}_{\mathrm{Nis}}(\mathbf{Cpx}(\mathbf{PSh}^{\mathrm{tr}}(k,\Lambda))) : R\omega^*. \\ L^{\overline{\square}}\omega_\sharp\colon \mathbf{logDM}^{\mathrm{eff}}(k,\Lambda) \rightleftarrows \mathbf{DM}^{\mathrm{eff}}_{\mathrm{Nis}}(k,\Lambda)) : R^{\overline{\square}}\omega^*. \end{gathered} \tag{5.10.2}$$

Similar adjunctions hold for the categories without transfers.

Proposition 5.11. *Let $F \in \mathbf{Cpx}(\mathbf{PSh}^{\mathrm{tr}}(k,\Lambda))$ (respectively $G \in \mathbf{Cpx}(\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda))$). Then $R\omega^*(F) = (\omega^*F)_{\mathrm{dNis}}$ (respectively $L\omega_\sharp(G) = (\omega_\sharp G)_{\mathrm{Nis}}$) and, in particular, $R\omega^*$ is t-exact.*

Proof. Because ω^* and $\omega_\sharp$ from (5.10.1) are t-exact functors, we have that for every $X \in \mathbf{lSm}(k)$ (respectively $Y \in \mathbf{Sm}(k)$),

$$\begin{aligned} H_n(\omega^*F)_{\mathrm{dNis}}(X) &= \mathbf{H}^{-n}_{\mathrm{dNis}}(X,\omega^*F) \\ &\cong \mathbf{H}^{-n}_{\mathrm{Nis}}(\omega(X),F) \\ &\cong H_n(F_{\mathrm{Nis}}(\omega(X))) \\ &= \omega^*(H_nF_{\mathrm{Nis}})(X) \\ &= H_n(\omega^*F_{\mathrm{Nis}})(X) \\ &= H_n(R\omega^*F)(X) \end{aligned} \qquad \begin{aligned} H_n(\omega_\sharp G)_{\mathrm{Nis}}(Y) &= \mathbf{H}^{-n}_{\mathrm{Nis}}(Y,\omega_\sharp G) \\ &\cong \mathbf{H}^{-n}_{\mathrm{dNis}}(\lambda(Y),G) \\ &\cong H_n(G_{\mathrm{dNis}}(\lambda(X))) \\ &= \omega_\sharp(H_nG_{\mathrm{dNis}})(Y) \\ &= H_n(\omega_\sharp G_{\mathrm{dNis}})(Y) \\ &= H_n(L\omega_\sharp G)(Y). \end{aligned}$$

Finally, by [7, (4.3.4)],

$$a_{\mathrm{dNis}}H_n(R\omega^*(F)) = a_{\mathrm{dNis}}\omega^*H_n(F) = \omega^*a_{\mathrm{Nis}}H_n(F).$$

Because ω^* is fully faithful, we conclude that $a_{\mathrm{Nis}}H_n(F) = 0$ if and only if $a_{\mathrm{dNis}}H_n(R\omega^*(F)) = 0$; hence, ω^* is t-exact for the local t-structure. □

Proposition 5.12. *The functor $R^{\overline{\square}}\omega^*$ is t-exact with respect to Voevodsky's homotopy t-structure on $\mathbf{DM}^{\mathrm{eff}}$ and to the homotopy t-structure on $\mathbf{logDM}^{\mathrm{eff}}$ of Theorem 5.7.*

Proof. If K is $(\mathrm{Nis},\mathbf{A}^1)$-fibrant, it is in particular Nis-fibrant; hence, by Proposition 5.11, ω^*K is dNis-fibrant. Hence, we have for every $n \in \mathbb{Z}$ and $X \in \mathbf{SmlSm}$,

$$\mathbf{H}^{-n}_{\mathrm{dNis}}(X,\omega^*K) \cong \mathbf{H}^{-n}_{\mathrm{Nis}}(\omega(X),K) = \mathrm{Hom}(\Lambda_{\mathrm{tr}}(\omega(X))[n],K). \tag{5.12.1}$$

In particular, because $\omega(X \times \overline{\square}) = \omega(X) \times \mathbf{A}^1$, we have that

$$\begin{aligned}\mathrm{Hom}((\Lambda_{\mathrm{ltr}}(X) \otimes \overline{\square})[n],\omega^*K) &= \mathrm{Hom}(\Lambda_{\mathrm{tr}}(\omega(X) \otimes \mathbf{A}^1)[n],K)\\ &= \mathrm{Hom}(\Lambda_{\mathrm{tr}}(\omega(X))[n],K)\\ &= \mathrm{Hom}(\Lambda_{\mathrm{ltr}}(X)[n],\omega^*K),\end{aligned}$$

so ω^*K is $\overline{\square}$-local if K is $(\mathrm{Nis},\mathbf{A}^1)$-local. It follows that ω^* sends $(\mathrm{Nis},\mathbf{A}^1)$-weak equivalences to $(\mathrm{dNis},\overline{\square})$-weak equivalences, so that the following diagram of triangulated categories commutes:

$$\begin{tikzcd}
\mathbf{D}_{\mathrm{dNis}}(\mathbf{Cpx}(\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda))) & \mathbf{D}_{\mathrm{Nis}}(\mathbf{Cpx}(\mathbf{PSh}^{\mathrm{tr}}(k,\Lambda))) \arrow[l, "R\omega^*"'] \\
\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda) \arrow[u, hook, "\iota_{\mathbf{logDM}}"] & \mathbf{DM}^{\mathrm{eff}}_{\mathrm{Nis}}(k,\Lambda), \arrow[l, "R^{\overline{\square}}\omega^*"] \arrow[u, hook, "\iota_{\mathbf{DM}}"]
\end{tikzcd} \tag{5.12.2}$$

where the vertical fully faithful functors are the right adjoint to the localisations $L^{\overline{\square}}$ and $L^{\mathbf{A}^1}$, respectively. By [32, Proposition 3.1.13] and Theorem 5.7, the t-structure on $\mathbf{D}_{\mathrm{Nis}}(\mathbf{Cpx}(\mathbf{PSh}^{\mathrm{tr}}(k,\Lambda)))$ (respectively on $\mathbf{D}_{\mathrm{dNis}}(\mathbf{Cpx}(\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda)))$) induces a t-structure on $\mathbf{DM}^{\mathrm{eff}}$ (respectively on $\mathbf{logDM}^{\mathrm{eff}}$), so that the inclusions $\iota_{\mathbf{DM}}$ and $\iota_{\mathbf{logDM}}$ are both t-exact.

To conclude, we need to show that $R^{\overline{\square}}\omega^* \circ \tau^{\mathbf{DM}}_{\leq n} \cong \tau^{\mathbf{logDM}}_{\leq n} \circ R^{\overline{\square}}\omega^*$. But because $R\omega^*$ is t-exact and (5.12.2) commutes, we have

$$\begin{aligned}R^{\overline{\square}}\omega^*(\tau^{\mathbf{DM}}_{\leq n}K) &= R\omega^*\iota_{\mathbf{DM}}(\tau^{\mathbf{DM}}_{\leq n}K) = R\omega^*(\tau_{\leq n}\iota_{\mathbf{DM}}(K))\\ &= \tau_{\leq n}R\omega^*\iota_{\mathbf{DM}}(K) = \tau^{\mathbf{logDM}}_{\leq n}R^{\overline{\square}}\omega^*(K).\end{aligned}$$

The same argument applies to the truncation $\tau_{\geq n}$, so that we can conclude. □

Remark 5.13. Assume that k satisfies resolution of singularities. Then the functor $R^{\overline{\square}}\omega^*$ is fully faithful, and its essential image is identified with the subcategory of $\mathbf{A}^1$-local objects in $\mathbf{logDM}^{\mathrm{eff}}$ by [7, Theorem 8.2.16]. It follows from Proposition 5.12 that under $R^{\overline{\square}}\omega^*$, the homotopy t-structure on $\mathbf{DM}^{\mathrm{eff}}$ is induced by the homotopy t-structure on $\mathbf{logDM}^{\mathrm{eff}}$.

Corollary 5.14. *The functor $L^{\overline{\square}}\omega_\sharp$ is right t-exact.*

Proof. This follows immediately from the fact that its right adjoint is t-exact (in particular, left t-exact). □

6. Application to reciprocity sheaves

In this section, we discuss some applications to the theory of reciprocity sheaves. As above, for $X \in \mathbf{SmlSm}(k)$, let $|\partial X|$ be the strict normal crossing divisor supporting the log structure of X. We will call the modulus pair $(\underline{X}, |\partial X|_{\mathrm{red}})$ the *associated reduced modulus pair*. We remark that the assignment $X \mapsto (\underline{X}, |\partial X|_{\mathrm{red}})$ *does not* give rise to a functor from $\mathbf{SmlSm}(k)$ to $\mathbf{MCor}$, because a priori there is no control on the multiplicities of the divisor ∂X in the pullback along a morphism in $\mathbf{SmlSm}(k)$.

However, thanks to [29], there exists a functor

$$\mathcal{L}og\colon \mathbf{RSC}_{\mathrm{Nis}}(k) \to \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k, \mathbb{Z})$$

where for $X = (\underline{X}, \partial X) \in \mathbf{SmlSm}(k)$ we have

$$\mathcal{L}og(F)(X) := F^{\log}(X) = \omega^{\mathbf{CI}} F(\underline{X}, |\partial X|_{red}).$$

Here, $\omega^{\mathbf{CI}}\colon \mathbf{RSC}_{\mathrm{Nis}} \to \mathbf{CI}_{\mathrm{Nis}}$ is the functor defined in [18, Proposition 2.3.7] (see also [18, Theorem 2.4.1-2.4.2] and 6.3) and $\mathbf{CI}_{\mathrm{Nis}}$ is the subcategory of $\overline{\square}$-invariant Nisnevich sheaves on $\mathbf{MCor}$, defined in [14] (not to be confused with $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$ introduced in the present article). By [29, Theorem 0.2], $\mathcal{L}og$ is fully faithful and exact.

Proposition 6.1. *The essential image of $\mathcal{L}og$ is a subcategory of $\mathbf{CI}^{ltr}_{\mathrm{dNis}}$.*

Proof. By [29, Theorem 4.1] we have that for $F \in \mathbf{RSC}_{\mathrm{Nis}}$ $\mathcal{L}og(F)$ is strictly $\overline{\square}$-invariant. □

One can wonder whether the two categories agree; that is, whether $\mathcal{L}og$ is essentially surjective onto $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$. This is not the case, as the following example indicates.

Example 6.2 (See [5, Proposition 3.5]). Let $\mathbf{G}_a \in \mathbf{RSC}_{\mathrm{Nis}}$; then

$$\mathcal{L}og(\mathbf{G}_a)(X) = \Gamma(\underline{X}, \mathcal{O}_{\underline{X}}).$$

By, for example, [7, Corolary 9.2.6], we have that $H^n_{\mathrm{dNis}}(X, \mathcal{L}og(\mathbf{G}_a)) = H^n_{sZar}(X, \mathcal{L}og(\mathbf{G}_a))$. Let $\mathcal{L}og(\mathbf{G}_a) \to I^\bullet$ be an injective resolution of dNis-sheaves. Thus, for all $U \subseteq X$ open affine, then there is a quasi-isomorphism

$$\mathcal{L}og(\mathbf{G}_a)(U) \to I^\bullet(U).$$

It follows that for every set A, the map

$$\prod_A \mathcal{L}og(\mathbf{G}_a)(U) \to \prod_A I^\bullet(U)$$

is a quasi-isomorphism. Thus, $\prod_A \mathcal{L}og(\mathbf{G}_a) \to \prod_A I^\bullet$ is a sZar-local equivalence and hence a sNis-local equivalence, so $\prod_A I^\bullet$ is an injective resolution of $\prod_A \mathcal{L}og(\mathbf{G}_a)$.

We conclude that

$$H^n_{\mathrm{dNis}}(X, \prod_A \mathcal{L}og(\mathbf{G}_a)) = H^n(\prod_A I^\bullet(X)) = \prod_A H^n I^\bullet(X) = \prod_A H^n_{\mathrm{dNis}}(X, \mathcal{L}og(\mathbf{G}_a)).$$

In particular, $\prod_A \mathcal{L}og(\mathbf{G}_a)$ is strictly $\overline{\square}$ invariant. On the other hand, by [17, Remark 6.1.2], if A is infinite, $\prod_A \mathbf{G}_a$ does not belong to $\mathbf{RSC}_{\mathrm{Nis}}$.

6.3. We recall some further constructions from the theory of modulus (pre)sheaves with transfers. For $F \in \mathbf{MPST}$, write $h_0^{\overline{\square}}(F)$ for the presheaf

$$\mathcal{X} \mapsto \operatorname{Coker}(F(\mathcal{X} \otimes \overline{\square}) \xrightarrow{i_0^* - i_1^*} F(\mathcal{X})),$$

where i_0^* and i_1^* are as usual the pullbacks along the zero section and the unit section of $\overline{\square}$, respectively. Clearly, $h_0^{\overline{\square}}(F)$ is $\overline{\square}$-invariant in $\mathbf{MPST}$; that is, $h_0^{\overline{\square}}(F) \in \mathbf{CI}$. By [18, Proposition 2.1.5], $h_0^{\overline{\square}}(-)$ is the left adjoint to the inclusion $\iota^{\overline{\square}} \colon \mathbf{CI} \to \mathbf{MPST}$. Note that $\iota^{\overline{\square}}$ has a right adjoint as well by [18, Lemma 2.1.7], denoted $h^0_{\overline{\square}}(-)$.

Let $\omega_! \colon \mathbf{MPST} \to \mathbf{PSh}^{\mathrm{tr}}(k)$ be the left Kan extension of $\omega \colon \mathbf{MCor} \to \mathbf{Cor}(k)$, sending $\mathcal{X} = (X, X_\infty) \mapsto X - |X_\infty|$.[3] We write $\omega^{\mathbf{CI}} \colon \mathbf{RSC} \to \mathbf{CI}$ for the composition $h^0_{\overline{\square}} \circ \omega^* \circ \iota_{\mathbf{RSC}}$, where $\iota_{\mathbf{RSC}}$ is the inclusion of $\mathbf{RSC}$ in $\mathbf{PSh}^{\mathrm{tr}}(k)$ and $\omega^* \colon \mathbf{PSh}^{\mathrm{tr}}(k) \to \mathbf{MPST}$ is the restriction. If no confusion arises, we will use the same symbols to denote the corresponding functor on the subcategories of Nisnevich sheaves, $\omega^{\mathbf{CI}} \colon \mathbf{RSC}_{\mathrm{Nis}} \to \mathbf{CI}_{\mathrm{Nis}}$ and $\omega^* \colon \mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k) \to \mathbf{MNST}$. By [18, Proposition 2.3.7], $\omega_! \omega^{\mathbf{CI}} F = F$ for every $F \in \mathbf{RSC}_{\mathrm{Nis}}$.

Using the above-defined functors, we can compute the sections of $\mathcal{L}og(F)$ on $X \in \mathbf{SmlSm}(k)$ for $F \in \mathbf{RSC}_{\mathrm{Nis}}$ as follows. Write $X = (\underline{X}, \partial X)$ and $X^o = \underline{X} - |\partial X|$. Choose a normal compactification $j \colon \underline{X} \hookrightarrow Y$ with the property that $X^o \to \underline{X} \to Y$ is open dense and such that the complement $Y - X^o = D + \partial X_Y$ for some effective Cartier divisors D and ∂X_Y on Y satisfying $Y - |D| = j(\underline{X})$ and $\partial X_Y \cap \underline{X} = \partial X$ as reduced Cartier divisors. Such a compactification is called a Cartier compactification of $\underline{X}$, and it always exists (cf. [14, Definition1.7.3]). Then we have

$$\mathcal{L}og(F)(X) = (\omega^{\mathbf{CI}} F)^{\log}(X) = \operatorname*{colim}_n \operatorname{Hom}_{\mathbf{MNST}}(h_0^{\overline{\square}}(\underline{X}, nD + \partial X_Y), \omega^* F), \tag{6.3.1}$$

where $\omega^* F \in \mathbf{MNST}$ if $F \in \mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}$. This follows from [14, Lem. 1.7.4(b)] and the definition of $\omega^{\mathbf{CI}}$.

Proposition 6.4. $\mathbf{RSC}_{\mathrm{Nis}}$ *is closed under colimits in* $\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k)$.

Proof. Recall that if $\{F_i\}_{i \in I}$ is a diagram in $\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k)$, then

$$\operatorname*{colim}_{i \in I} F_i = a^V_{\mathrm{Nis}} \operatorname*{colim}_{i \in I} \iota_{\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}}(F_i), \tag{6.4.1}$$

where $\iota_{\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}} \colon \mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k) \to \mathbf{PSh}^{\mathrm{tr}}(k)$ is the inclusion, the colimit on the left-hand side of (6.4.1) is computed in $\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}$ and the colimit on the right-hand side is computed in $\mathbf{PSh}^{\mathrm{tr}}(k)$. Because a^V_{Nis} respects reciprocity by [29, Theorem 0.1], it is enough to prove that $\mathbf{RSC}$ is closed under colimits in $\mathbf{PSh}^{\mathrm{tr}}(k)$. Consider then a diagram $\{F_i\}_{i \in I}$ in $\mathbf{RSC}$. Because $\omega_!$ is a left adjoint and thus it preserves all colimits, we have

$$\overset{\mathbf{PSh}^{\mathrm{tr}}_{\mathrm{Nis}}(k)}{\operatorname*{colim}_{i \in I}} F_i = \omega_! \overset{\mathbf{MPST}}{\operatorname*{colim}_{i \in I}} \omega^{\mathbf{CI}} F_i.$$

---

[3] We follow the notation in [14], to avoid confusion with $\omega_\sharp$ used before, but note that the functor ω in [14] and the functor ω used in this article are very similar, even though they are defined on different categories.

Because **CI** is closed under colimits and $h_0^{\square}$ and $i^{\square}$ are left adjoints, we conclude $i^{\square}h_0^{\square}\operatorname{colim}^{\mathbf{MPST}} F_i = \operatorname{colim}^{\mathbf{MPST}} i^{\square}h_0^{\square}F_i$, so that the colimit is in **RSC**, as required. $\square$

Remark 6.5. For $X,Y \in \mathbf{SmlSm}(k)$, we have by, for example, [25, Section III.2]

$$X \times Y = (\underline{X} \times \underline{Y}, (pr_X^*\mathcal{M}_X \oplus pr_Y^*\mathcal{M}_Y)^{\mathrm{fs}}).$$

The divisor that supports the sheaf of monoids $pr_X^*\mathcal{M}_X \oplus pr_Y^*\mathcal{M}_Y$ is $D_X \times Y + X \times D_Y$, where the divisors D_X and D_Y support $\mathcal{M}_X$ and $\mathcal{M}_Y$, respectively, and the functor $(-)^{\mathrm{fs}}$ does not change the support. We conclude that the associated reduced modulus pair of $X \times Y$ is $\mathcal{X} \otimes \mathcal{Y}$.

Lemma 6.6. *$\mathcal{L}og$ has a pro-left adjoint $\mathcal{R}sc$, given by the formula*

$$\mathcal{R}sc(G) := \underset{X\downarrow G}{\overset{pro\text{-}\mathbf{RSC}_{\mathrm{Nis}}}{\operatorname{colim}}} \text{“}\lim_{n}\text{”} \omega_! h_0^{\square}(\overline{X}, D + nD'), F),$$

for every $G \in \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\mathbb{Z})$.

Proof. It follows directly from Saito's theorem [29, Theorem 6.3] that $\mathcal{L}og$ preserves finite limits, so the existence of a pro-left adjoint is formal (see, e.g., [2, Proposition I.8.11.4]). In the rest of the proof we characterise the pro-adjoint explicitly: such description will be used later in the computation. Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$ and $G \in \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\mathbb{Z})$. For any $X \in \mathbf{SmlSm}(k)$, let $(\underline{X}, D)$ be the associated reduced modulus pair and choose $\overline{X}$ a Cartier compactification of $\underline{X}$. Set $D' := X' \setminus X$.

Recall that

$$\mathcal{L}og(F)(X) = \omega^{\mathbf{CI}}(F)(X,D) = \varinjlim_{n} \operatorname{Hom}(\omega_! h_0^{\square}(\overline{X}, D + nD'), F).$$

Writing G as colimit of representable sheaves, $G = \operatorname{colim}_{s:X\to G} a_{\mathrm{dNis}}\mathbb{Z}_{\mathrm{ltr}}(X)$, we have

$$\begin{aligned}
\operatorname{Hom}_{\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}}(G, \mathcal{L}og(F)) &= \operatorname{Hom}_{\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}}(\operatorname*{colim}_{X\downarrow G} a_{\mathrm{dNis}}\mathbb{Z}_{\mathrm{ltr}}(X), \mathcal{L}og(F))\\
&= \lim_{X\downarrow G} \operatorname{Hom}_{\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}}(a_{\mathrm{dNis}}\mathbb{Z}_{\mathrm{ltr}}(X), \mathcal{L}og(F))\\
&= \lim_{X\downarrow G} \varinjlim_{n} \operatorname{Hom}_{\mathbf{RSC}}(\omega_! h_0^{\square}(\overline{X}, D + nD'), F)\\
&= \lim_{X\downarrow G} \operatorname{Hom}_{pro\text{-}\mathbf{RSC}}(\text{“}\lim_{n}\text{”} \omega_! h_0^{\square}(\overline{X}, D + nD'), F),
\end{aligned}$$

where the last equality simply follows from the definition of the morphisms in the pro-category *pro*-**RSC**. By Proposition 6.4, $\mathbf{RSC}_{\mathrm{Nis}}$ is cocomplete; hence, *pro*-$\mathbf{RSC}_{\mathrm{Nis}}$ is cocomplete by, for example, [13, Proposition 11.1] and we can pass the limit inside the Hom to get

$$\operatorname{Hom}_{\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}}(G, \mathcal{L}og(F)) = \operatorname{Hom}_{pro\text{-}\mathbf{RSC}}(\underset{X\downarrow G}{\overset{pro\text{-}\mathbf{RSC}_{\mathrm{Nis}}}{\operatorname{colim}}} \text{“}\lim_{n}\text{”} \omega_! h_0^{\square}(\overline{X}, D + nD'), F).$$

Thus, we can identify the pro-left adjoint to $\mathcal{L}og$ with the functor

$$\mathcal{R}sc(G) := \underset{X\downarrow G}{\overset{pro\text{-}\mathbf{RSC}_{\mathrm{Nis}}}{\mathrm{colim}}} \text{“}\lim_{n}\text{”}\omega_! h_0^{\overline{\square}}(\overline{X}, D+nD'),$$

from $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\mathbb{Z})$ to $\mathbf{RSC}_{\mathrm{Nis}}$. □

6.7. The category of reciprocity sheaves is equipped with a *lax* monoidal structure constructed in [27], given by

$$(F,G)_{\mathbf{RSC}_{\mathrm{Nis}}} := \underline{\omega_!}(\omega^{\mathbf{CI}}F \otimes^{\mathrm{Nis}}_{\mathbf{CI}} \omega^{\mathbf{CI}}G), \tag{6.7.1}$$

for $F,G \in \mathbf{RSC}_{\mathrm{Nis}}$. More generally, there are functors for $n \geq 1$,

$$\mathbf{RSC}^{\times n}_{\mathrm{Nis}} \to \mathbf{RSC}_{\mathrm{Nis}}, (F_1,\dots,F_n) \mapsto \big(F_1,F_2,\dots,F_n\big)_{\mathbf{RSC}_{\mathrm{Nis}}},$$

which satisfies only a weak form of associativity, see [27, Corollary 4.18-4.21]. See [27] and [23] for some computations. In particular, a nontrivial argument (see [27, Theorem 5.2]) shows that

$$(F,G)_{\mathbf{RSC}_{\mathrm{Nis}}} = F \otimes_{\mathbf{HI}_{\mathrm{Nis}}} G$$

whenever $F,G \in \mathbf{HI}_{\mathrm{Nis}}$ and $\mathrm{ch}(k) = 0$. We can extend the bifunctor $(-,-)_{\mathbf{RSC}_{\mathrm{Nis}}}$ to the pro-category as follows.

Definition 6.8. Let $F = \text{“}\lim\text{”}F_i, G = \text{“}\lim\text{”}G_j \in pro\text{-}\mathbf{RSC}_{\mathrm{Nis}}$; then we define

$$(F,G)^{pro}_{\mathbf{RSC}} := \text{“}\lim_{i,j}\text{”}(F_i,G_j)_{\mathbf{RSC}_{\mathrm{Nis}}}.$$

Proposition 6.9. $(\_,\_)^{pro}_{\mathbf{RSC}}$ *is well defined and bifunctorial.*

Proof. We first show that the assignment is well defined; that is, that it does not depend on the chosen representation of “lim”F as object in pro-$\mathbf{RSC}_{\mathrm{Nis}}$. Thus, let “lim”$F \cong$ “lim”F' be another representation of the pro-system F. For every “lim”G, “lim”$H \in pro$-$\mathbf{RSC}_{\mathrm{Nis}}$, we have canonical identifications

$$\begin{aligned} &\mathrm{Hom}_{pro\text{-}\mathbf{RSC}_{\mathrm{Nis}}}((\text{“}\lim\text{”}F, \text{“}\lim\text{”}G)^{pro}_{\mathbf{RSC}}, \text{“}\lim\text{”}H) \\ &=^{(1)} \varprojlim_{H}\varinjlim_{F,G}\mathrm{Hom}_{\mathbf{RSC}_{\mathrm{Nis}}}((F,G)_{\mathbf{RSC}},H) \\ &=^{(2)} \varprojlim_{H}\varinjlim_{F,G}\mathrm{Hom}_{\mathbf{RSC}_{\mathrm{Nis}}}(\omega_!(\omega^{\mathbf{CI}}F \otimes^{\mathrm{Nis}}_{\mathbf{CI}} \omega^{\mathbf{CI}}G),\omega_!\omega^{\mathbf{CI}}H) \\ &=^{(3)} \varprojlim_{H}\varinjlim_{F,G}\mathrm{Hom}_{\mathbf{CI}^{\tau}_{\mathrm{Nis}}}(\omega^{\mathbf{CI}}F \otimes^{\mathrm{Nis}}_{\mathbf{CI}} \omega^{\mathbf{CI}}G,\omega^{\mathbf{CI}}H), \\ &=^{(4)} \varprojlim_{H}\varinjlim_{F,G}\mathrm{Hom}_{\mathbf{CI}^{\tau}_{\mathrm{Nis}}}(\omega^{\mathbf{CI}}F, \underline{\mathrm{Hom}}(\omega^{\mathbf{CI}}G,\omega^{\mathbf{CI}}H)), \end{aligned} \tag{6.9.1}$$

where (1) is given by the definition of the morphisms in the pro category, (2) is simply the definition of the monoidal structure, (3) follow from the fact that $\omega_!$ restricts to a functor $\mathbf{CI}_{\mathrm{Nis}} \to \mathbf{RSC}_{\mathrm{Nis}}$ that is left adjoint to the fully faithful functor $\omega^{\mathbf{CI}}$ and (4) is the adjunction for the internal Hom structure in $\mathbf{CI}_{\mathrm{Nis}}$. The functor $\omega^{\mathbf{CI}}$ preserves all limits being a right adjoint; hence, it induces a functor on the pro categories defined level-wise:

$$\textit{pro-}\omega^{\mathbf{CI}}\colon \textit{pro-}\mathbf{RSC}_{\mathrm{Nis}} \to \textit{pro-}\mathbf{CI}^{\tau}_{\mathrm{Nis}} \quad \textit{pro-}\omega^{\mathbf{CI}}(\text{“lim”}F) := \text{“lim”}\omega^{\mathbf{CI}}F.$$

Hence, because “lim” $F \cong$ “lim” F', we have that

$$\textit{pro-}\omega^{\mathbf{CI}}(\text{“lim”}F) \cong \textit{pro-}\omega^{\mathbf{CI}}(\text{“lim”}F').$$

In particular, for fixed G and H in $\mathbf{RSC}_{\mathrm{Nis}}$, we have isomorphisms

$$\begin{aligned}
&\varinjlim_{F} \mathrm{Hom}_{\mathbf{CI}^{\tau}_{\mathrm{Nis}}}(\omega^{\mathbf{CI}}F, \underline{\mathrm{Hom}}(\omega^{\mathbf{CI}}G,\omega^{\mathbf{CI}}H))\\
&\quad = \mathrm{Hom}_{\textit{pro-}\mathbf{CI}^{\tau}_{\mathrm{Nis}}}(\textit{pro-}\omega^{\mathbf{CI}}\text{“lim”}F, \underline{\mathrm{Hom}}(\omega^{\mathbf{CI}}G,\omega^{\mathbf{CI}}H)) \qquad (6.9.2)\\
&\quad \simeq \mathrm{Hom}_{\textit{pro-}\mathbf{CI}^{\tau}_{\mathrm{Nis}}}(\textit{pro-}\omega^{\mathbf{CI}}\text{“lim”}F', \underline{\mathrm{Hom}}(\omega^{\mathbf{CI}}G,\omega^{\mathbf{CI}}H))\\
&\quad \simeq \varinjlim_{F'} \mathrm{Hom}_{\mathbf{CI}^{\tau}_{\mathrm{Nis}}}(\omega^{\mathbf{CI}}F', \underline{\mathrm{Hom}}(\omega^{\mathbf{CI}}G,\omega^{\mathbf{CI}}H)).
\end{aligned}$$

Combining (6.9.1) and (6.9.2) we have that

$$\begin{aligned}
&\varprojlim_{H}\varinjlim_{F,G} \mathrm{Hom}_{\mathbf{CI}^{\tau}_{\mathrm{Nis}}}(\omega^{\mathbf{CI}}F, \underline{\mathrm{Hom}}(\omega^{\mathbf{CI}}G,\omega^{\mathbf{CI}}H))\\
&\qquad = \varprojlim_{H}\varinjlim_{F',G} \mathrm{Hom}_{\mathbf{CI}^{\tau}_{\mathrm{Nis}}}(\omega^{\mathbf{CI}}F', \underline{\mathrm{Hom}}(\omega^{\mathbf{CI}}G,\omega^{\mathbf{CI}}H))\\
&\qquad = \varprojlim_{H}\varinjlim_{F',G} \mathrm{Hom}_{\mathbf{RSC}_{\mathrm{Nis}}}((F',G)_{\mathbf{RSC}},H)\\
&\qquad = \mathrm{Hom}_{\textit{pro-}\mathbf{RSC}_{\mathrm{Nis}}}((\text{“lim”}F',\text{“lim”}G)^{pro}_{\mathbf{RSC}},\text{“lim”}H).
\end{aligned}$$

This shows that $(\_,\_)^{pro}_{\mathbf{RSC}}$ is indeed well defined.

We now prove the functoriality statement. Let $f\colon$ “lim” $F \to$ “lim” G be a morphism in *pro-*$\mathbf{RSC}_{\mathrm{Nis}}$. We can use, for example, [1, Appendix 3.2] to reindex the limit by choosing isomorphisms $a\colon \text{“lim”}F \xrightarrow{\simeq} \text{“lim”}_{\alpha}F_{\alpha}$ and $b\colon \text{“lim”}G \xrightarrow{\simeq} \text{“lim”}_{\alpha}G_{\alpha}$ and level-wise defined morphisms $f_{\alpha} : F_{\alpha} \to G_{\alpha}$ in $\mathbf{RSC}_{\mathrm{Nis}}$ such that $f = b^{-1}\text{“lim”}f_{\alpha}a$. Let $H = \text{“lim”}H_{\beta}$ be another pro-reciprocity sheaf. Then for all α,β we have a map

$$(f_{\alpha},id)_{\mathbf{RSC}_{\mathrm{Nis}}} : (F_{\alpha}H_{\beta})_{\mathbf{RSC}_{\mathrm{Nis}}} \to (G_{\alpha},H_{\beta})_{\mathbf{RSC}_{\mathrm{Nis}}}.$$

The previous computations show that both a and b induce isomorphisms

$$\begin{aligned}
(a,id)\colon (F,H)^{pro}_{\mathbf{RSC}} &\to (\text{“lim”}F_{\alpha},H)^{pro}_{\mathbf{RSC}}\\
(b,id)\colon (G,H)^{pro}_{\mathbf{RSC}} &\to (\text{“lim”}G_{\beta},H)^{pro}_{\mathbf{RSC}},
\end{aligned}$$

which then induce a morphism

$$(F,H)^{pro}_{\mathbf{RSC}} \xrightarrow[\simeq]{(a,id)} (\text{“lim”}F_{\alpha},H)^{pro}_{\mathbf{RSC}} \xrightarrow{\text{“lim”}(f_{\alpha},id)} (G_{\alpha},H)^{pro}_{\mathbf{RSC}} \xrightarrow[\simeq]{(b,id)^{-1}} (G,H)^{pro}_{\mathbf{RSC}}.$$

It is clear that this morphism depends only on f, because if $a' : F \xrightarrow{\sim}$ “lim”F_β and b' : $G \xrightarrow{\sim}$ “lim”G_β, then the diagram below commutes:

$$\begin{tikzcd}
(\text{“lim”} F_\alpha, H)^{pro}_{\mathbf{RSC}} \arrow[r, "\text{“lim”}(f_\alpha, id)"] & (\text{“lim”} G_\alpha, H)^{pro}_{\mathbf{RSC}} \arrow[d, "\simeq"] \\
(F,H)^{pro}_{\mathbf{RSC}} \arrow[u, "\simeq"] \arrow[d, "\simeq"'] & (G,H)^{pro}_{\mathbf{RSC}} \\
(\text{“lim”} F_\beta, H)^{pro}_{\mathbf{RSC}} \arrow[r, "\text{“lim”}(f_\beta, id)"] & (\text{“lim”} G_\beta, H)^{pro}_{\mathbf{RSC}} \arrow[u, "\simeq"']
\end{tikzcd}$$

The composition and the identity are clearly respected, and the same computation gives functoriality for the other component. ☐

Remark 6.10. If $\mathcal{C}$ is a category equipped with a monoidal structure $\otimes$ (in particular, associative), then the category *pro*-$\mathcal{C}$ is equipped with the level-wise monoidal structure $\{X_\alpha\} \otimes \{Y_\beta\} = \{X_\alpha \otimes Y_\beta\}$. See [10, 11]. Because the construction (6.7.1) gives a monoidal structure on $\mathbf{RSC}_{\mathrm{Nis}}$ only in a weak sense (in particular, associativity is not known to hold), we need to verify explicitly that the level-wise assignment 6.8 is indeed well defined. Note that the argument is *ad hoc* and only proves the existence of a bi-functor at the level of pro categories.

The functoriality statement of the previous proposition implies in particular that if $(F_i)_{i\in I}$ and $(G_j)_{j\in J}$ are diagrams in *pro*-$\mathbf{RSC}_{\mathrm{Nis}}$, then there is a natural map

$$\underset{i,j}{\overset{pro\text{-}\mathbf{RSC}_{\mathrm{Nis}}}{\mathrm{colim}}} (F_i, G_j)^{pro}_{\mathbf{RSC}} \to \left(\underset{i}{\overset{pro\text{-}\mathbf{RSC}_{\mathrm{Nis}}}{\mathrm{colim}}} F_i, \underset{j}{\overset{pro\text{-}\mathbf{RSC}_{\mathrm{Nis}}}{\mathrm{colim}}} G_j \right)^{pro}_{\mathbf{RSC}}. \tag{6.10.1}$$

In general, there is no reason to expect that (6.10.1) is an isomorphism (see also [10, Exercise 11.2] for a similar problem). Using the explicit description of the pro-left adjoint to $\mathcal{L}og$, we get then the following result.

Theorem 6.11. *For $F, G \in \mathbf{CI}^{ltr}_{\mathrm{dNis}}$, there exists a natural map*

$$\mathcal{R}sc(F \otimes^{ltr} G) \to (\mathcal{R}sc(F), \mathcal{R}sc(G))^{pro}_{\mathbf{RSC}}.$$

Proof. The tensor product in $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k)$ is given by Day convolution from the monoidal structure on $\mathbf{SmlSm}(k)$. So, if $F = \mathrm{colim}_{X\downarrow F}\, a_{\mathrm{dNis}}\mathbb{Z}_{tr}(X)$ and $G = \mathrm{colim}_{Y\downarrow G}\, a_{\mathrm{dNis}}\mathbb{Z}_{tr}(Y)$, then

$$F \otimes^{ltr} G = \underset{X,Y}{\overset{\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}}{\mathrm{colim}}}\, a_{\mathrm{dNis}}\mathbb{Z}_{tr}(X \times Y).$$

A Cartier compactification of $X \times Y$ is given by $\overline{X} \times \overline{Y}$, where $\overline{X}$ and $\overline{Y}$ are Cartier compactifications of X and Y. Let $D'_X = \overline{X} - X$ and $D'_Y = \overline{Y} - Y$. Using the explicit description of the functor $\mathcal{R}sc$ given in the proof of Lemma 6.6, we get

$$\begin{aligned}&\mathcal{R}sc(F\otimes^{ltr}G)\\&\quad=\overset{pro\text{-}\mathbf{RSC}_{\mathrm{Nis}}}{\operatorname{colim}}\text{“lim”}\omega_!h_0^{\square}(\overline{X}\times\overline{Y},D_X\times Y+X\times D_Y+n(D'_X\times Y+X\times D'_Y))\\&\quad=\overset{pro\text{-}\mathbf{RSC}_{\mathrm{Nis}}}{\operatorname{colim}}\text{“lim”}\omega_!(h_0^{\square}(\overline{X},D_X+nD'_X)\otimes_{\mathbf{CI}}h_0^{\square}(\overline{Y},D_Y+nD'_Y)).\end{aligned}$$

Consider now the natural maps

$$h_0^{\square}(\overline{X},D_X+nD'_X)\to\omega^{\mathbf{CI}}\omega_!(h_0^{\square}(\overline{X},D_X+nD'_X),$$

which give a natural map

$$\begin{aligned}&\text{“lim”}\omega_!(h_0^{\square}(\overline{X},D_X+nD'_X)\otimes_{\mathbf{CI}}h_0^{\square}(\overline{Y},D_Y+nD'_Y))\\&\quad\longrightarrow\text{“lim”}\omega_!(\omega^{\mathbf{CI}}\omega_!h_0^{\square}(\overline{X},D_X+nD'_X)\otimes_{\mathbf{CI}}\omega^{\mathbf{CI}}\omega_!h_0^{\square}(\overline{Y},D_Y+nD'_Y))\\&\quad=\text{“lim”}(\omega_!h_0^{\square}(\overline{X},D_X+nD'_X),\omega_!h_0^{\square}(\overline{Y},D_Y+nD'_Y))_{\mathbf{RSC}}.\end{aligned}$$

By definition the last term is equal to

$$(\text{“lim”}h_0^{\square}(\overline{X},D_X+nD'_X),\text{“lim”}h_0^{\square}(\overline{Y},D_Y+nD'_Y))^{pro}_{\mathbf{RSC}}.$$

Hence, we obtained a natural map

$$\begin{aligned}&\mathcal{R}sc(F\otimes^{ltr}G)\to\\&\quad\overset{pro\text{-}\mathbf{RSC}_{\mathrm{Nis}}}{\operatorname{colim}}(\text{“lim”}h_0^{\square}(\overline{X},D_X+nD'_X),\text{“lim”}h_0^{\square}(\overline{Y},D_Y+nD'_Y))^{pro}_{\mathbf{RSC}}.\end{aligned}\tag{6.11.1}$$

Finally, as observed in (6.10.1) there is a natural map

$$\begin{aligned}&\overset{pro\text{-}\mathbf{RSC}_{\mathrm{Nis}}}{\operatorname{colim}}(\text{“lim”}h_0^{\square}(\overline{X},D_X+nD'_X),\text{“lim”}h_0^{\square}(\overline{Y},D_Y+nD'_Y))^{pro}_{\mathbf{RSC}}\to\\&\quad\left(\overset{pro\text{-}\mathbf{RSC}_{\mathrm{Nis}}}{\operatorname{colim}}\text{“lim”}h_0^{\square}(\overline{X},D_X+nD'_X),\overset{pro\text{-}\mathbf{RSC}_{\mathrm{Nis}}}{\operatorname{colim}}\text{“lim”}h_0^{\square}(\overline{Y},D_Y+nD'_Y)\right)^{pro}_{\mathbf{RSC}}\end{aligned}$$

and the last term is equal to $(\mathcal{R}sc(F),\mathcal{R}sc(G))^{pro}_{\mathbf{RSC}}$. □

Corollary 6.12. *Let $F,G\in\mathbf{RSC}$; then there exists a natural map*

$$\mathcal{L}og(F)\otimes_{\mathbf{CI}^{log}_{\mathrm{dNis}}}\mathcal{L}og(G)\to\mathcal{L}og((F,G)_{\mathbf{RSC}_{\mathrm{Nis}}}).\tag{6.12.1}$$

Proof. Let $(-)^p:\mathbf{RSC}_{\mathrm{Nis}}\to pro\text{-}\mathbf{RSC}_{\mathrm{Nis}}$ be the constant functor $F\mapsto\text{“lim”}F$. Because $\mathcal{L}og$ is fully faithful, we have

$$F^p=\mathcal{R}sc(\mathcal{L}og(F)),\quad G^p=\mathcal{R}sc(\mathcal{L}og(G)).$$

By definition, we have that

$$(F^p,G^p)^{pro}_{\mathbf{RSC}}=((F,G)_{\mathbf{RSC}})^p,$$

so the previous lemma gives a natural map

$$\mathcal{R}sc(\mathcal{L}og(F)\otimes^{ltr}\mathcal{L}og(G))\to((F,G)_{\mathbf{RSC}})^p$$

whose adjoint gives a map

$$\mathcal{L}og(F) \otimes^{ltr} \mathcal{L}og(G) \to \mathcal{L}og((F,G)_{\mathbf{RSC}}).$$

Finally, because $\mathcal{L}og((F,G)_{\mathbf{RSC}}) \in \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$, the previous map factors through the localisation $h_0(\mathcal{L}og(F) \otimes^{ltr} \mathcal{L}og(G)) = \mathcal{L}og(F) \otimes_{\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}} \mathcal{L}og(G)$, giving the desired map. ☐

7. Log reciprocity sheaves

In this final section, we assume that our field satisfies resolution of singularities, (RS) for short (see, e.g., [7, Definition 7.6.3] for a precise definition). We construct a full subcategory **LogRec** of $\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k,\Lambda)$ such that $\mathbf{RSC}_{\mathrm{Nis}} \subseteq \mathbf{LogRec}$.

Our definition generalises the construction of [18] and it is very similar in spirit.

Definition 7.1. We define a pair of adjoint functors

$$\omega^{\log}_{\mathbf{CI}} \colon \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}} \rightleftarrows \mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k,\Lambda) \colon \omega^{\mathbf{CI}}_{\log}$$

where $\omega^{\log}_{\mathbf{CI}} := \omega_\sharp i$ and $\omega^{\mathbf{CI}}_{\log} := h^0_{\mathrm{ltr}}\omega^*$, where h^0_{ltr} is the right adjoint to the inclusion of Proposition 5.8. The counit map $i_{\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}} h^0_{\mathrm{ltr}} \to id$ induces for all $F \in \mathbf{Shv}_{\mathrm{dNis}}(k,\Lambda)$ a canonical map

$$i\omega^{\mathbf{CI}}_{\log} F \to \omega^* F. \tag{7.1.1}$$

Lemma 7.2. *For each $F \in \mathbf{Shv}^{\mathrm{tr}}_{\mathrm{dNis}}(k,\Lambda)$, the map (7.1.1) is injective.*

Proof. Let $X \in \mathbf{SmlSm}(k)$ with $\underline{X}$ connected, and let η_X be the generic point of $\underline{X}$. By Theorem 5.10, we have an injective map

$$i\omega^{\mathbf{CI}}_{\log} F(X) \hookrightarrow i\omega^{\mathbf{CI}}_{\log} F(\eta_X,\mathrm{triv}).$$

Hence, we get the following commutative diagram:

$$\begin{tikzcd}
i\omega^{\mathbf{CI}}_{\log} F(X) \arrow[r] \arrow[d, hook] & \omega^* F(X) \arrow[d] \\
i\omega^{\mathbf{CI}}_{\log} F(\eta_X,\mathrm{triv}) \arrow[r] & \omega^* F(\eta_X,\mathrm{triv}).
\end{tikzcd}$$

Because the left vertical arrow is injective, it is enough to check that the bottom arrow is injective.

We have that

$$i\omega^{\mathbf{CI}}_{\log} F(\eta_X) = \mathrm{Hom}(\omega_\sharp h_0(\eta_X,\mathrm{triv}),F).$$

By [7, Proposition 8.2.2] (this is the point where we use the hypothesis that k satisfies (RS)), we have that

$$\omega_\sharp h_0(\Lambda_{\mathrm{ltr}}(\eta_X,\mathrm{triv})) = \omega_\sharp h_0(\omega^*\Lambda_{\mathrm{tr}}(\eta_X)),$$

and by [7, Proposition 8.2.4],

$$h_0(\omega^*\Lambda_{\mathrm{tr}}(\eta_X)) = \omega^* h_0^{\mathbb{A}^1}\Lambda_{\mathrm{tr}}(\eta_X).$$

Finally, using the Suslin complex, we have a surjective map

$$\Lambda_{\mathrm{tr}}(\eta_X) \to h_0^{\mathbb{A}^1}\Lambda_{\mathrm{tr}}(\eta_X).$$

Putting everything together, we conclude that the map

$$\Lambda_{\mathrm{tr}}(\eta_X) \to \omega_\sharp h_0(\Lambda_{\mathrm{ltr}}(\eta_X,\mathrm{triv}))$$

is surjective; hence, the following map

$$\omega_{\log}^{\mathbf{CI}}F(\eta_X,\mathrm{triv}) = \mathrm{Hom}(\omega_\sharp h_0(\Lambda_{\mathrm{ltr}}(\eta_X,\mathrm{triv}),F) \to \mathrm{Hom}(\Lambda_{\mathrm{tr}}(\eta_X),F) = \omega^* F(\eta_X,\mathrm{triv})$$

is injective, which concludes the proof. □

Proposition 7.3. *Assume that k satisfies (RS). The composition*

$$\mathbf{CI}^{ltr}_{\mathrm{dNis}} \xrightarrow{i^{tr}} \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}} \xrightarrow{\omega_\sharp} \mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}$$

is fully faithful and exact.

Proof. Exactness follows from the exactness of i^{tr} and $\omega_\sharp$.

It is enough to show that for all $F \in \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$, the unit map

$$F \to \omega_{\log}^{\mathbf{CI}}\omega^{\log}_{\mathbf{CI}}(F) \tag{7.3.1}$$

is an isomorphism.

Because $F \in \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$, by Theorem 5.10 we have that for all $X \in \mathbf{SmlSm}(k)$,

$$F(X) \hookrightarrow F(\underline{X} - |\partial X|) = \omega^*\omega_\sharp i^{tr}F;$$

hence, $u : F \to \omega^*\omega^{\log}_{\mathbf{CI}}F$ is injective. Because $F \in \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$, the map u factors through $\omega_{\log}^{\mathbf{CI}}\omega^{\log}_{\mathbf{CI}}F$, showing injectivity of (7.3.1).

Let T be the cokernel of (7.3.1) and let Q be the cokernel of u.

On the other hand, we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "(7.3.1)"] \arrow[d, equal] & \omega_{\log}^{\mathbf{CI}}\omega^{\log}_{\mathbf{CI}}F \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r, "u"] & \omega^*\omega^{\log}_{\mathbf{CI}}F \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

By Lemma 7.2, the middle veritcal arrow is injective, so $T \hookrightarrow Q$ is injective.

Because ω^* is fully faithful and exact, we have that $\omega_\sharp Q = 0$; hence, $\omega_\sharp T = 0$ because $\omega_\sharp$ is exact.

Because $T \in \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$, Theorem 5.10 implies that $T = 0$. This concludes the proof. □

Definition 7.4. Let **LogRec** denote the essential image of $\omega^{\log}_{\mathbf{CI}}$; that is, the category of sheaves $F \in \mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}$ such that there exists $G \in \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$ such that $F = \omega_{\mathbf{CI}}G$.

By definition, $\omega^{\log}_{\mathbf{CI}}$ induces an equivalence between $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$ and **LogRec** with quasi-inverse the restriction of $\omega_{\log}^{\mathbf{CI}}$ to **LogRec**.

Remark 7.5. Let $F \in \mathbf{LogRec}$ and let $G \in \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$ such that $F = \omega_\sharp G$. We deduce some immediate properties:

(1) For all $X \in \mathbf{Sm}$ and $U \subseteq X$ dense open, Theorem 5.10 implies that $F(X) \hookrightarrow F(U)$ is injective.

(2) For all n and all $X \in \mathbf{Sm}$, we have that

$$a_{\mathrm{Nis}}\mathbf{H}^n_{\mathrm{Nis}}(\_ \times X, F) = a_{\mathrm{Nis}} H_n(\underline{\mathrm{Hom}}(X, F_{\mathrm{Nis}})) \overset{(*1)}{=} a_{\mathrm{Nis}} H_n(\underline{\mathrm{Hom}}(X, \omega_\sharp G_{\mathrm{dNis}})) \overset{(*2)}{=}$$
$$a_{\mathrm{Nis}} H_n(\omega_\sharp \underline{\mathrm{Hom}}((X,\mathrm{triv}), G_{\mathrm{dNis}})) \overset{(*3)}{=} \omega_\sharp a_{\mathrm{dNis}} H_n(\underline{\mathrm{Hom}}((X,\mathrm{triv}), G_{\mathrm{dNis}})),$$

where $(*1)$ comes from Proposition 5.11, $(*2)$ comes by definition of $\omega_\sharp$ and $(*3)$ from the fact that $\omega_\sharp$ is t-exact and [7, (4.3.4)]. By Corollary 5.9, $a_{\mathrm{dNis}} H_n(\underline{\mathrm{Hom}}((X,\mathrm{triv}), G_{\mathrm{dNis}})) \in \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$, so the cohomology sheaf $a_{\mathrm{Nis}}\mathbf{H}^n_{\mathrm{Nis}}(\_ \times X, F) \in \mathbf{LogRec}$.

Theorem 7.6. *The category* $\mathbf{RSC}_{\mathrm{Nis}}$ *is a full subcategory of* $\mathbf{LogRec}$. *In particular,*

$$\mathcal{L}og = \omega^{\mathbf{CI}}_{\log} i_{\mathbf{RSC}}. \tag{7.6.1}$$

Proof. Because $\mathbf{RSC}_{\mathrm{Nis}}$ is a full subcategory of $\mathbf{Shv}_{\mathrm{Nis}}(k,\Lambda)$, it is enough to show that for every $F \in \mathbf{RSC}_{\mathrm{Nis}}$ there exists $G \in \mathbf{CI}^{\log}_{\mathrm{dNis}}$ such that $F = \omega_\sharp G$.

By [29, Section 4] we have that

$$\omega_\sharp \mathcal{L}og(F)(X) = \omega^{\mathbf{CI}} F(X, \emptyset) = F(X). \tag{7.6.2}$$

Hence, $\mathbf{RSC}_{\mathrm{Nis}}$ is a full subcategory of $\mathbf{LogRec}$.

Finally, because $\omega_\sharp$ is an equivalence, (7.6.1) follows directly from (7.6.2). □

Corollary 7.7. *Let* $F \in \mathbf{RSC}_{\mathrm{Nis}}$ *and let* $X \in \mathbf{Sm}(k)$. *Then the cohomology of* F *satisfies*

$$\mathbf{H}^n(X \times Y, F) \hookrightarrow \mathbf{H}^n(X \times \eta_Y, F)$$

for every $n \geq 0$ *and* Y *Henselian local essentially smooth* k*-scheme with generic point* η_Y.

Proof. It follows immediately from Theorem 7.6 and Remark 7.5. □

Let $i_{\mathbf{RSC}}$ (respectively $i^{\log}_{\mathbf{RSC}}$) denote the inclusion of $\mathbf{RSC}_{\mathrm{Nis}}$ in $\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k)$ (respectively in $\mathbf{LogRec}$). Recall by [18] that $i_{\mathbf{RSC}}$ has a pro-left adjoint ρ such that for $X \in \mathbf{Sm}(k)$ and $\overline{X}$ a Cartier compactification with $D = \overline{X} - X$, then

$$\rho(\mathbb{Z}_{\mathrm{tr}}(X)) = \text{“lim”} \omega_! h_0^{\overline{\square}}(\overline{X}, nD).$$

Proposition 7.8. *The functor* $i^{\log}_{\mathbf{RSC}}$ *has a pro-left adjoint* $\rho_{\log}$, *which factors* ρ. *In particular,*

$$\mathcal{R}sc = \rho_{\log} \omega^{\log}_{\mathbf{CI}}.$$

Proof. Because $i_{\mathbf{RSC}} = i_{\mathbf{LogRec}} i^{\log}_{\mathbf{RSC}}$ and $i_{\mathbf{LogRec}}$ is fully faithful, for $F \in \mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}$ $G \in \mathbf{RSC}_{\mathrm{Nis}}$ we have that

$$\mathrm{Hom}_{pro\text{-}\mathbf{RSC}}(\rho i_{\mathbf{LogRec}} F, G) = \mathrm{Hom}_{\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}}(i_{\mathbf{LogRec}} F, i_{\mathbf{LogRec}} i^{\log}_{\mathbf{RSC}} G) =$$
$$\mathrm{Hom}_{\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}}(i_{\mathbf{LogRec}} F, i_{\mathbf{LogRec}} i^{\log}_{\mathbf{RSC}} G) = \mathrm{Hom}_{\mathbf{LogRec}}(F, i^{\log}_{\mathbf{RSC}} G).$$

Finally, for $F \in \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$ and $G \in \mathbf{RSC}_{\mathrm{Nis}}$, we have that

$$\begin{aligned}\mathrm{Hom}_{pro\text{-}\mathbf{RSC}}(\mathcal{R}sc(F),G) &= \mathrm{Hom}_{\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}}(F,\mathcal{L}og(G))\\ &= \mathrm{Hom}_{\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}}(F,\omega^{\mathbf{CI}}_{\log} i^{\log}_{\mathbf{RSC}} G)\\ &= \mathrm{Hom}_{\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}}(i_{\mathbf{LogRec}}\omega^{\log}_{\mathbf{CI}} F, i_{\mathbf{RSC}} G)\\ &= \mathrm{Hom}_{pro\text{-}\mathbf{RSC}}(\rho_{\log}\omega_{\mathbf{CI}} F,G).\end{aligned}$$

□

Remark 7.9. Because $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$ is a symmetric monoidal Grothendieck abelian category, **LogRec** is symmetric monoidal with tensor product given by

$$F \otimes_{\mathbf{LogRec}} G := \omega_{\sharp}(h_0(\omega^{\mathbf{CI}}_{\log} F \otimes^{\mathrm{ltr}} \omega^{\mathbf{CI}}_{\log} G)).$$

By 6.12, for all $F,G \in \mathbf{RSC}_{\mathrm{Nis}}$ we have a map

$$F \otimes_{\mathbf{LogRec}} G \to (F,G)_{\mathbf{RSC}}.$$

If $\mathrm{ch}(k) \neq 0$, this map is not an isomorphism (see 7.10 below). We do not know whether we expect it to be an isomotphism when $\mathrm{ch}(k) = 0$: this would prove that $(\_,\_)_{\mathbf{RSC}_{\mathrm{Nis}}}$ defines a monoidal structure on $\mathbf{RSC}_{\mathrm{Nis}}$.

7.10. Let $F,G \in \mathbf{RSC}_{\mathrm{Nis}}$ and let $F' \subseteq \omega^{\mathbf{CI}} F$ such that $\omega_! F' = F$ (in the language of [26], F' corresponds to a semi-continuous conductor of F different from the motivic conductor). By construction, there exists a canonical map

$$\omega_!(F' \otimes^{\mathrm{Nis}}_{\mathbf{CI}} \omega^{\mathbf{CI}} G) \to \omega_!(\omega^{\mathbf{CI}} F \otimes^{\mathrm{Nis}}_{\mathbf{CI}} \omega^{\mathbf{CI}} G) = (F,G)_{\mathbf{RSC}}. \tag{7.10.1}$$

This map is surjective: let Q be the cokernel of the inclusion $F' \to \omega^{\mathbf{CI}} F$ such that $\omega_! Q = 0$. Hence, because $\_ \otimes_{\mathbf{CI}} \omega^{\mathbf{CI}} G$ is right exact, there is a right exact sequence

$$F' \otimes^{\mathrm{Nis}}_{\mathbf{CI}} \omega^{\mathbf{CI}} G \to \omega^{\mathbf{CI}} F \otimes^{\mathrm{Nis}}_{\mathbf{CI}} \omega^{\mathbf{CI}} G \to Q \otimes^{\mathrm{Nis}}_{\mathbf{CI}} \omega^{\mathbf{CI}} G \to 0,$$

and because $Q \otimes^{\mathrm{Nis}}_{\mathbf{CI}} \omega^{\mathbf{CI}} G$ is a quotient of $Q \otimes_{\underline{\mathbf{M}}\mathbf{NST}} \omega^{\mathbf{CI}} G$ and $\omega_!$ is exact and monoidal in $\underline{\mathbf{M}}\mathbf{NST}$, we conclude $\omega_!(Q \otimes_{\mathbf{CI}} \omega^{\mathbf{CI}} G) = 0$, which shows the surjectivity of (7.10.1).

The kernel of (7.10.1) encapsulates the obstruction to the associativity of $(\_,\_)_{\mathbf{RSC}}$, and it seems to be very difficult to compute in general. We know that it is not trivial if $\mathrm{ch}(k) \neq 0$; see [27, Theorem 4.17] and [27, Theorem 5.19] for an explicit computation.

On the other hand, we do not have any counterexamples if $\mathrm{ch}(k) = 0$; hence, we do not know whether to expect that the map above is an isomorphism. In this direction, we have the following result.

Proposition 7.11. *Let $F,G \in \mathbf{RSC}_{\mathrm{Nis}}$. Then for all $F' \subseteq \omega^{\mathbf{CI}} F$ (in $\underline{\mathbf{M}}\mathbf{NST}$) such that $\omega_! F' = F$, the canonical map*

$$F \otimes_{\mathbf{LogRec}} G \to (F,G)_{\mathbf{RSC}}$$

factors through $\omega_!(F' \otimes^{\mathrm{Nis}}_{\mathbf{CI}} \omega^{\mathbf{CI}} G)$.

Proof. Let $(-)^{\log}$ be the functor of [29] and recall that $\mathcal{L}og(F) = (\omega^{\mathbf{CI}}F)^{\log}$. Because $\mathcal{L}og(F) = (F')^{\log}$ by construction, we can look at the diagram

$$\begin{tikzcd}
\mathcal{L}og(F) \otimes_{\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}} \mathcal{L}og(G) \arrow[r] \arrow[d, equal] & (\omega^{\mathbf{CI}}F \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)^{\log} \\
(F')^{\log} \otimes_{\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}} \mathcal{L}og(G) & (F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)^{\log} \arrow[u]
\end{tikzcd}$$

It is enough to show that there is a map

$$(F')^{\log} \otimes_{\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}} \mathcal{L}og(G) \to (F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)^{\log}$$

that makes the diagram above commutative. By adjunction, it is enough to construct a map

$$(F')^{\log} \to \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}}(\mathcal{L}og(G), (\omega^{\mathbf{CI}}F \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)^{\log})$$

that factors the map

$$(F')^{\log} = \mathcal{L}og(F) \to \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}}(\mathcal{L}og(G), (\omega^{\mathbf{CI}}F \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)^{\log}). \tag{7.11.1}$$

Consider the following map given by the closed monoidal structure of $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ (see [23, §3]):

$$F' \to \underline{\mathrm{Hom}}_{\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}}(\omega^{\mathbf{CI}}G, F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G). \tag{7.11.2}$$

Let $X \in \mathbf{SmlSm}(k)$ and let $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$ be the corresponding reduced modulus pair. By construction, we have that

$$\begin{aligned}
(\underline{\mathrm{Hom}}_{\mathbf{CI}}(\omega^{\mathbf{CI}}G, F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G))^{\log}(X) &= \mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(\omega^{\mathbf{CI}}G \otimes \mathbb{Z}_{\mathrm{tr}}(\mathcal{X}), F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G) \\
&= \mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(\omega^{\mathbf{CI}}G, \underline{\mathrm{Hom}}_{\mathbf{CI}}(h_0^{\overline{\square}}(\mathcal{X}), F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)).
\end{aligned} \tag{7.11.3}$$

Then the unit $id \to \omega^{\mathbf{CI}}\omega_!$ induces the following map:

$$\begin{aligned}
&\mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(\omega^{\mathbf{CI}}G, \underline{\mathrm{Hom}}_{\mathbf{CI}}(h_0^{\overline{\square}}(\mathcal{X}), F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)) \\
&\quad \to \mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(\omega^{\mathbf{CI}}G, \omega^{\mathbf{CI}}\omega_! \underline{\mathrm{Hom}}_{\mathbf{CI}}(h_0^{\overline{\square}}(\mathcal{X}), F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)) \\
&\quad \overset{(*1)}{=} \mathrm{Hom}_{\mathbf{RSC}}(G, \omega_! \underline{\mathrm{Hom}}_{\mathbf{CI}}(h_0^{\overline{\square}}(\mathcal{X}), F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)) \\
&\quad \overset{(*2)}{=} \mathrm{Hom}_{\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}}(\mathcal{L}og(G), \mathcal{L}og(\omega_! \underline{\mathrm{Hom}}_{\mathbf{CI}}(h_0^{\overline{\square}}(\mathcal{X}), F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G))) \\
&\quad \overset{(*3)}{=} \mathrm{Hom}_{\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}}(\mathcal{L}og(G), (\underline{\mathrm{Hom}}_{\mathbf{CI}}(h_0^{\overline{\square}}(\mathcal{X}), F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G))^{\log}),
\end{aligned} \tag{7.11.4}$$

where $(*1)$ (respectively $(*2)$, respectively $(*3)$) follows from the full faithfulness of $\omega^{\mathbf{CI}}$ (respectively the full faithfulness of $\mathcal{L}og$, respectively the fact that $\mathcal{L}og(\omega_!) = (\_)^{\log}$; see [29, Corollary 2.6 (3)]).

Finally, fix $Y \in \mathbf{SmlSm}(k)$ and let $\mathcal{Y} \in \underline{\mathbf{M}}\mathbf{Cor}$ be the corresponding reduced modulus pair.

We have that

$$\begin{aligned}(\underline{\mathrm{Hom}}_{\mathbf{CI}}(h_0^{\overline{\square}}(\mathcal{X})F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G))^{\log}(Y) &= \underline{\mathrm{Hom}}_{\mathbf{CI}}(h_0^{\overline{\square}}(\mathcal{X}\otimes\mathcal{Y}), F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)\\ &= (F' \otimes_{\mathbf{CI}}^{\mathrm{Nis}} \omega^{\mathbf{CI}}G)(\mathcal{X}\otimes\mathcal{Y}) \overset{(*)}{=} (F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)^{\log}(X\times Y)\\ &= \underline{\mathrm{Hom}}_{\mathbf{Shv}_{\mathrm{dNis}}^{\mathrm{ltr}}}(\mathbb{Z}_{\mathrm{ltr}}(X), (F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)^{\log})(Y), \qquad (7.11.5)\end{aligned}$$

where $(*)$ is true by the observation in Remark 6.5. We conclude that

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{Shv}_{\mathrm{dNis}}^{\mathrm{ltr}}}(\mathcal{L}og(G), (\underline{\mathrm{Hom}}_{\mathbf{CI}}(h_0^{\overline{\square}}(\mathcal{X}), F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)^{\log})\\ &\quad= \mathrm{Hom}_{\mathbf{Shv}_{\mathrm{dNis}}^{\mathrm{ltr}}}(\mathcal{L}og(G), \underline{\mathrm{Hom}}_{\mathbf{Shv}_{\mathrm{dNis}}^{\mathrm{ltr}}}(\mathbb{Z}_{\mathrm{ltr}}(X), (F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)^{\log}))\\ &\quad= \underline{\mathrm{Hom}}_{\mathbf{Shv}_{\mathrm{dNis}}^{\mathrm{ltr}}}(\mathcal{L}og(G), (F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)^{\log})(X). \qquad (7.11.6)\end{aligned}$$

Putting (7.11.3), (7.11.4), (7.11.5) and (7.11.6) together, we have a map

$$(\underline{\mathrm{Hom}}_{\mathbf{CI}}(\omega^{\mathbf{CI}}G, F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G))^{\log} \to \underline{\mathrm{Hom}}_{\mathbf{Shv}_{\mathrm{dNis}}^{\mathrm{ltr}}}(\mathcal{L}og(G), (F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)^{\log}). \qquad (7.11.7)$$

Hence, by applying $(\_)^{\log}$ to (7.11.2) and composing with (7.11.7), we get the map

$$\begin{aligned}(F')^{\log} \to (\underline{\mathrm{Hom}}_{\mathbf{CI}}(\omega^{\mathbf{CI}}G, F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G))^{\log} &\to \underline{\mathrm{Hom}}_{\mathbf{Shv}_{\mathrm{dNis}}^{\mathrm{ltr}}}(\mathcal{L}og(G), (F' \otimes_{\mathbf{CI}}^{\mathrm{Nis}} \omega^{\mathbf{CI}}G)^{\log})\\ &= \underline{\mathrm{Hom}}_{\mathbf{Shv}_{\mathrm{dNis}}^{\mathrm{ltr}}}(\mathcal{L}og(G), \mathcal{L}og\,\omega_!(F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)). \qquad (7.11.8)\end{aligned}$$

Finally, notice that the map (7.11.4) factors the map

$$\begin{aligned}&\mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(\omega^{\mathbf{CI}}G, \underline{\mathrm{Hom}}_{\mathbf{CI}}(h_0^{\overline{\square}}(\mathcal{X}), F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G))\\ &\qquad\to \mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(\omega^{\mathbf{CI}}G, \omega^{\mathbf{CI}}\omega_!\,\underline{\mathrm{Hom}}_{\mathbf{CI}}(h_0^{\overline{\square}}(\mathcal{X}), \omega^{\mathbf{CI}}\omega_!F' \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)).\end{aligned}$$

So, because $\omega_!F' = F$ and $\omega_!(\omega^{\mathbf{CI}}F \otimes_{\mathbf{CI}}^{\mathrm{Nis},sp} \omega^{\mathbf{CI}}G)) = (F,G)_{\mathbf{RSC}}$, the equalities of 7.11.4, 7.11.5 and 7.11.6 with $\omega^{\mathbf{CI}}F$ instead of F' conclude that (7.11.8) factor (7.11.1). This concludes the proof. ☐

Remark 7.12. For $F \in Shv_{\mathrm{Nis}}$, we denote by $h^0_{\mathbf{A}^1}(F)$ the biggest $\mathbf{A}^1$-local subsheaf as defined in [26, 4.34]: for $U \in \mathbf{Sm}$,

$$h^0_{\mathbf{A}^1}(F)(U) := \mathrm{Hom}(h_0^{\mathbf{A}^1}(U), F).$$

On the other hand, for $U \hookrightarrow \underline{X}$ a Cartier compactification such that $\underline{X}$ is proper and smooth over k and $\underline{X} - U$ is a simple normal crossing divisor, for $X = (\underline{X}, \partial X) \in \mathbf{SmlSm}(k)$ such that ∂X is supported on $\underline{X} - U$, by [7, Proposition 8.2.4] we have that

$$h_0^{\mathbf{A}^1}(X) = \omega_\sharp h_0(X).$$

Hence, if $F \in \mathbf{LogRec}$, then

$$h^0_{\mathbf{A}^1}(F)(U) = \mathrm{Hom}(\omega_\sharp h_0(X), F) = \omega^{\mathbf{CI}}_{\log}F(X).$$

Here we underline that this does not depend on X, as long as $\underline{X}$ is *proper*.

We conclude with this observation: for X as above and $\mathcal{X} \in \mathbf{MCor}$ the associated reduced modulus pair, by [26, Corollary 4.36] if $F \in \mathbf{RSC}_{\mathrm{Nis}}$, we have that

$$\mathrm{Hom}(\omega_! h_0^{\overline{\square}, sp}(\mathcal{X}), F) = h^0_{\mathbf{A}^1} F = \mathrm{Hom}(\omega^{\log}_{\mathbf{CI}} h_0^{\overline{\square}}(X), F).$$

This implies that

$$\omega_! h_0^{\overline{\square}, sp}(\mathcal{X}) \cong \omega^{\log}_{\mathbf{CI}} h_0^{\overline{\square}}(X).$$

In particular, by [29, Corollary 2.6 (3)], we have that

$$\mathcal{L}og(\omega_! h_0^{\overline{\square}, sp}(\mathcal{X})) = h_0^{\overline{\square}, sp}(\mathcal{X})^{\log};$$

hence, by the fact that $\omega^{\log}_{\mathbf{CI}}$ is an equivalence on **LogRec**, we have that

$$h_0^{\overline{\square}, sp}(\mathcal{X})^{\log} \cong h_0(X) \cong \omega^* h_0^{\mathbf{A}^1}(U).$$

Again, we stress that these isomorphisms do not depend on X or $\mathcal{X}$, as long as $\underline{X}$ is *proper*.

Acknowledgements The authors thank Joseph Ayoub for a careful reading of a preliminary version of this article and for suggesting an improvement that allowed them to weaken the original assumptions on Theorem 4.4. We are also grateful to Kay Rülling for useful comments, to Doosung Park for many conversations on the subject of this article and to Shuji Saito for his interest in our work. Finally, we thank the referee for their detailed report. F.B. is supported by the PRIN 'Geometric, Algebraic and Analytic Methods in Arithmetic' and is a member of INdAM group GNSAGA. A.M. is supported by the Swiss National Science Foundation (SNSF), project 200020_178729.

References

[1] M. Artin and B. Mazur, *Etale Homotopy, Lecture Notes in Mathematics*, Vol. **100** (Springer, Berlin, 1986). Reprint of the 1969 original.

[2] Michael Artin, Alexander Grothendieck and J. L. Verdier, *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas.* (SGA 4, Volume 1). *Tome 1: Théorie des topos. Exposés I à IV.* 2e éd., Vol. **269** (Springer, Cham, Switzerland, 1972). Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint-Donat.

[3] Michael Artin, Alexander Grothendieck and J. L. Verdier, *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas.* (SGA 4, Volume 2). Tome 2, Vol. **270** (Springer, Cham, Switzerland, 1972). Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint-Donat.

[4] Joseph Ayoub, Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. II, *Astérisque*, **315** (2007), vi+364.

[5] Joseph Ayoub, $\mathbb{P}^1$-Localisation and an arithmetic Kodaira-Spencer class, *Tunisian J. Math.* **3** (2021), 259–308. http://user.math.uzh.ch/ayoub/PDF-Files/Arith-KS.pdf.

[6] A. A. Beĭlinson, J. Bernstein and P. Deligne, Analysis and topology on singular spaces, I *Astérisque*, **100** (1982), 5–171.

[7] Federico Binda, Doosung Park and Paul Arne Østvær, Triangulated categories of logarithmic motives over a field, Preprint, 2020, https://arxiv.org/abs/2004.12298.

[8] Jean-Louis Colliot-Thélène, Raymond T. Hoobler and Bruno Kahn, The Bloch-Ogus-Gabber theorem, in *Algebraic K-Theory*, *Fields Inst. Commun.*, Vol. **16** (American Mathematical Society, Providence, RI, 1997), pp. 31–94.

[9] Frédéric Déglise, Modules homotopiques, *Doc. Math.*, **16** (2011), 411–455.

[10] Halvard Fausk and Daniel C. Isaksen, T-model structures, *Homology Homotopy Appl.*, **9(1)** (2007), 399–438.

[11] Amit Hogadi and Girish Kulkarni, Gabber's presentation lemma for finite fields, *J. Reine Angew. Math.*, **759** (2020), 265–289.

[12] Mark Hovey, John H. Palmieri and Neil P. Strickland, Axiomatic stable homotopy theory, *Mem. Amer. Math. Soc.*, **128** (1997), x+114.

[13] Daniel C. Isaksen, A model structure on the category of pro-simplicial sets, *Trans. Amer. Math. Soc.*, **353(7)** (2001), 2805–2841.

[14] Bruno Kahn, Hiroyasu Miyazaki, Shuji Saito and Takao Yamazaki, Motives with modulus, I: Modulus sheaves with transfers for non-proper modulus pairs, *Épijournal Géom. Algébrique*, **5** (2021), Art. 1.

[15] Bruno Kahn, Hiroyasu Miyazaki, Shuji Saito and Takao Yamazaki, Motives with modulus, II: Modulus sheaves with transfers for proper modulus pairs, *Épijournal Géom. Algébrique*, **5** (2021), Art. 2.

[16] Bruno Kahn, Hiroyasu Miyazaki, Shuji Saito and Takao Yamazaki, Motives with modulus, III: The categories of motives, Preprint 2020, https://arxiv.org/abs/2011.11859.

[17] Bruno Kahn, Shuji Saito and Takao Yamazaki, Reciprocity sheaves, *Compos. Math.*, **152**(9) (2016), 1851–1898. With two appendices by Kay Rülling.

[18] Bruno Kahn, Shuji Saito and Takao Yamazaki, Reciprocity sheaves, II, Preprint, 2019, https://arxiv.org/abs/1707.07398.

[19] Fumiharo Kato, Exactness, integrality, and log modifications. URL: https://arxiv.org/pdf/math/9907124.pdf.

[20] Jacob Lurie, *Higher Algebra* (2017). URL: https://www.math.ias.edu/~lurie/papers/HA pdf.

[21] Saunders MacLane, *Categories for the Working Mathematician*, *Graduate Texts in Mathematics*, Vol. 5, Springer-Verlag, New York, Inc. (1971).

[22] Carlo Mazza, Vladimir Voevodsky and Charles Weibel, *Lecture Notes on Motivic Cohomology*, Clay Mathematics Monographs, Vol. **2** (American Mathematical Society, Providence, RI, 2006).

[23] Alberto Merici and Shuji Saito, Cancellation theorems for reciprocity sheaves, Preprint, 2020, https://arxiv.org/abs/2001.07902.

[24] Fabien Morel, The stable A^1-connectivity theorems, *K-Theory*, **35(1–2)** (2005), 1–68.

[25] Arthur Ogus, *Lectures on Logarithmic Algebraic Geometry*, *Cambridge Studies in Advanced Mathematics*, Vol. 178 (Cambridge University Press, Cambridge, UK 2018).

[26] Kay Rülling and Shuji Saito, Reciprocity sheaves and their ramification filtration, Preprint, 2019, https://arxiv.org/abs/1812.08716.

[27] Kay Rülling, Takao Yamazaki and Rin Sugiyama, Tensor structures in the theory of modulus presheaves with transfers, Preprint, 2019, https://arxiv.org/abs/1911.05291.

[28] Shuji Saito, Purity of reciprocity sheaves, *Adv. Math.*, **366** (2020), 1–70.

[29] Shuji Saito, *Reciprocity Sheaves and Logarithmic Motives*, Preprint, 2020. http://www.lcv.ne.jp/~smaki/articles/RSClog-v2.pdf.

[30] Jean-Pierre Serre, *Algebraic Groups and Class Fields*, 2nd ed., *Publications de l'Institut Mathématique de l'Université de Nancago* [Publications of the Mathematical Institute of the University of Nancago], Vol. 7 (Hermann, Paris, 1984).

[31] The Stacks Project Authors, *Stacks Project*. URL: https://stacks.math.columbia.edu, 2020.

[32] VLADIMIR VOEVODSKY, Triangulated categories of motives over a field, cycles, transfers, and motivic homology theories, in *Ann. of Math. Stud.*, Vol. **143** (Princeton University Press, Princeton, NJ, 2000), pp. 188–238.